EXERCISES, COMPLEX GEOMETRY, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2015/2016

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Sheet 7

Exercise 1. Let $(V, \langle\, , \rangle, I)$ be a euclidian vector space endowed with a compatible almost complex structure. Let $z_1, \ldots, z_n$ be a $\mathbb{C}$-basis of $V^{1,0}$. Write $z_i = \frac{1}{2}(x_i - iI(x_i))$ with $x_i \in V$. Show the following statements.

(1) The set $x_1, y_1 = I(x_1), \ldots, x_n, y_n = I(x_n)$ is an $\mathbb{R}$-basis of V and $x_1, \ldots, x_n$ is a $\mathbb{C}$-basis of (V, I).
(2) Assume $\langle\, , \rangle_{\mathbb{C}}$ on $V^{1,0}$ is given by a matrix $\frac{1}{2}(h_{ij})$. Then $(x_i, x_j) = h_{ij}$, $(x_i, x_j) = -ih_{ij}$ and $(y_i, y_j) = h_{ij}$.
(3) We have $\omega = \frac{i}{2}\sum_{i,j=1}^n h_{ij} z^i \wedge \overline{z}^j$, where upper indices refer to the dual vectors.
(4) If $x_1, y_1, \ldots, x_n, y_n$ is an orthonormal basis, then $\omega = \frac{i}{2}\sum_{i,j=1}^n z^i \wedge \overline{z}^j = \sum_{i=1}^n x^i \wedge y^i$.

Exercise 2. [2, Ex. 1.3.1] Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open subsets and $f \colon U \longrightarrow V$ be a holomorphic map. Show that the natural pullback $f^* \colon \mathcal{A}^k(V) \longrightarrow \mathcal{A}^k(U)$ induces maps $\mathcal{A}^{p,q}(V) \longrightarrow \mathcal{A}^{p,q}(U)$.

Exercise 3. [2, Ex. 1.3.2] Show that $\overline{\partial\alpha} = \overline{\partial}\overline{\alpha}$. Conclude that a real (p,p)-form α is ∂-closed (exact) if and only if it is $\overline{\partial}$-closed (exact).

References

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