EXERCISES, COMPLEX GEOMETRY, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2015/2016

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Sheet 6

Exercise 1. [2, Ex. 2.2.5] Let L be a holomorphic line bundle on a compact complex manifold X. Show that L is trivial if and only if L and its dual L^* admit non-trivial global sections.

Hint: Use the sections to construct a non-trivial section of $\mathcal{O}_X \simeq L \otimes L^*$.

Exercise 2. [2, Ex. 2.2.6 & 2.2.7] For this exercise we will assume the following fact: If $U \subset \mathbb{C}^n$ is open and $f \colon U \setminus \mathbb{C}^{n-2} \longrightarrow \mathbb{C}$ is holomorphic, then there exists a unique holomorphic extension $\overline{f} \colon U \longrightarrow \mathbb{C}$ of f.

Now prove the following statements.

(1) Let X be a compact complex manifold, $L \in \mathrm{Pic}(X)$ and let $Y \subset X$ be a submanifold of codimension at least two. Show that the restriction map $H^0(X, L) \longrightarrow H^0(X \setminus Y, L)$ is surjective.

(2) Now let L' be a second holomorphic line bundle on X. Assume that L and L' are isomorphic on $X \setminus Y$. Prove that $L \simeq L'$.

Exercise 3. [2, Ex. 2.2.8] Show that any non-trivial homogeneous polynomial $0 \neq p \in \mathbb{C}[z_0, \ldots, z_n]_k$ of degree k can be considered as a non-trivial section of $\mathcal{O}(k)$ on $\mathbb{P}^n$.

Exercise 4. (compare [2, Ex. 2.2.9]**)** Let $s \colon \mathbb{P}^n \longrightarrow \mathcal{O}(-1)$ be the zero section of the tautological line bundle. Show that $\mathcal{O}(-1) \setminus s(\mathbb{P}^n)$ is naturally identified with $\mathbb{C}^{n+1} \setminus \{0\}$.

Exercise 5. (compare [2, Ex. 2.2.13]**)** Show that the holomorphic tangent bundle of a complex torus $X = \mathbb{C}^n/\Gamma$ is isomorphic to the trivial vector bundle of rank n.

References

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