EXERCISES, COMPLEX GEOMETRY, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2015/2016

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Sheet 5

Exercise 1. The following exercise is taken from the book "Introduction to Commutative algebra" by Atiyah-MacDonald.

Let I be a partially ordered set, that is, we have a relation $\leq$ which is reflexive, antisymmetric and transitive. We call I a *directed set* if for each pair $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. Now let $(M_i)_{i \in I}$ be a family of modules over a ring R, indexed by a directed set I. Assume that for each pair $i, j \in I$, there exists a homomorphism $\mu_{ij} \colon M_i \longrightarrow M_j$ such that (1) $\mu_{ii} = \mathrm{id}_{M_i}$ for all i; (2) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$. We call the datum (M_i, μ_{ij}) a *direct system*.

Let C be the direct sum of all the M_i and identify every M_i with its image in C. Let D be the submodule of C generated by elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Set $M := C/D$, let $\mu \colon C \longrightarrow M$ be the projection and let μ_i be the restriction of μ to M_i. The module M with the homomorphisms $\mu_i \colon M_i \longrightarrow M$ is called the *direct limit* of the direct system and written $\varinjlim_i M_i$.

(1) Show that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$.
(2) Show that every element in M can be written as $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.
(3) Prove that if N is any R-module such that for any $i \in I$ there exists a homomorphism $\alpha_i \colon M_i \longrightarrow N$ satisfying $\alpha_i = \alpha_j \circ \mu_{ij}$ for $i \leq j$, then there exists a unique homomorphism $\alpha \colon M \longrightarrow N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$. In particular, the direct limit is unique.

Exercise 2. (compare [2, Ex. 2.2.1]**)** Verify the cocycle descriptions given in Proposition 5.5(1)-(3) in the lecture.

Exercise 3. [2, Ex. 2.2.3] Show that for any holomorphic vector bundle E of rank r there exists a non-degenerate pairing

$$\bigwedge^k E \times \bigwedge^{r-k} E \longrightarrow \det(E).$$

Deduce that there is a natural isomorphism of holomorphic vector bundles $\bigwedge^k E \simeq \bigwedge^{r-k} E^* \otimes \det(E)$.

Exercise 4. [2, Ex. 2.2.4] Show that any homomorphism $f \colon E \longrightarrow F$ of holomorphic vector bundles E and F induces a natural homomorphism $f \otimes \mathrm{id}_G \colon E \otimes G \longrightarrow F \otimes G$ for any holomorphic vector bundle G. Show that if f is injective, then so is $f \otimes \mathrm{id}_G$.

References

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