EXERCISES, COMPLEX GEOMETRY, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2015/2016

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Sheet 4

Exercise 1. Let $\mathcal{F}$ be a presheaf on a topological space X. Show that there is a homomorphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{F}^+$ which is an isomorphism if $\mathcal{F}$ is a sheaf. Show that $(\mathcal{F}^+, \varphi)$ satisfies the following universal property: For any morphism $\psi\colon \mathcal{F} \longrightarrow \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there is a unique sheaf morphism $\theta\colon \mathcal{F}^+ \longrightarrow \mathcal{G}$ such that $\theta \circ \varphi = \psi$.

Exercise 2. (compare [1, Ex. II.1.16]**)**

(1) Show that if $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence of sheaves and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then so is $\mathcal{F}''$.
(2) Show that if $f\colon X \longrightarrow Y$ is a continuous map and $\mathcal{F}$ is a flasque sheaf on X, then $f_*\mathcal{F}$ is a flasque sheaf on Y.

Exercise 3. (compare [1, Ex. II.1.19]**)** Let X be a topological space, Z be a closed subset, let $i\colon Z \longrightarrow X$ be the inclusion, $U = X \setminus Z$ be the open complement and $j\colon U \longrightarrow X$ be its inclusion.

(1) Show that $\mathcal{F}$ is a sheaf on Z, then $(i_*\mathcal{F})_x$ is $\mathcal{F}_x$ if $x \in Z$ and 0 otherwise.
(2) Let $\mathcal{F}$ be a sheaf on U. Prove that the assignment $V \longrightarrow \mathcal{F}(V)$ if $V \subset U$ and $V \longmapsto 0$ otherwise defines a presheaf. Show that its sheafification $j_!\mathcal{F}$ satisfies the following property: $(j_!\mathcal{F})_x = \mathcal{F}_x$ if $x \in U$ and $(j_!\mathcal{F})_x = 0$ if $x \notin U$.

Exercise 4. Let A be a commutative ring with 1. Recall that a *complex* of A-modules is a sequence of modules and A-linear maps $d^i\colon M^i \longrightarrow M^{i+1}$ for $i \in \mathbb{Z}$ such that $d^{i+1} \circ d^i = 0$ for all i. Write a complex as $M^\bullet$. A morphism of complexes $f^\bullet\colon M^\bullet \longrightarrow N^\bullet$ is given by maps $f^i\colon M^i \longrightarrow N^i$ for all $i \in \mathbb{Z}$ such that $d^i_{N^\bullet} \circ f^i = f^{i+1} \circ d^i_{M^\bullet}$ for all i.

The *i-cycles* of a complex are by definition $Z^i(M^\bullet) := \ker(d^i)$ and the *i-boundaries* are $B^i(M^\bullet) = \mathrm{im}(d^{i-1})$. Clearly, $B^i(M^\bullet) \subseteq Z^i(M^\bullet)$ and we define the *i-th cohomology* of $M^\bullet$ to be $H^i(M^\bullet) = Z^i(M^\bullet)/B^i(M^\bullet)$.

Show that any morphism of complexes $f^\bullet$ induces a map $H^i(f^\bullet)\colon H^i(M^\bullet) \longrightarrow H^i(N^\bullet)$ for all $i \in \mathbb{Z}$.

A morphism $f^\bullet$ is called a quasi-isomorphism if $H^i(f^\bullet)$ is an isomorphism for all i. Show that the following conditions are equivalent: (1) $M^\bullet$ is exact (that is, cycles and boundaries coincide) at every M^i, (2) $H^i(M^\bullet) = 0$ for all i, (3) the map $0 \longrightarrow M^\bullet$ is a quasi-isomosphism.

Exercise 5. Let $X = S^1$ be the unit circle and consider the constant sheaf $\mathbb{Z}_X$. Cover S^1 by open subsets U and V, where $U = X \setminus \{-1\}$ and $V = X \setminus \{1\}$. Compute the Čech cohomology of $\mathbb{Z}_X$ with respect to this open covering.

Exercise 6. Compute the Čech cohomology of $\mathcal{O}_X$ for $X = \mathbb{P}^1$ with respect to its standard open cover given in the lecture, that is, $\mathbb{P}^1 = U_0 \cup U_1$, where $U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\} = \{[1 : u] \mid u \in \mathbb{C}\} \simeq \mathbb{C}$ and $U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\} = \{[u : 1] \mid u \in \mathbb{C}\} \simeq \mathbb{C}$.

References

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