EXERCISES, COMPLEX GEOMETRY, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2015/2016

P. SOSNA

Sheet 2

Exercise 1. A topological space X is *connected* if it cannot be written as a disjoint union of two open non-empty subsets.

(1) Let $\{A_n\}$, $n \in \mathbb{N}$, be a sequence of connected subspaces of X with the property that $A_k \cap A_{k+1} \neq \emptyset$ for all $k \in \mathbb{N}$. Show that $Y := \cup_{n\in\mathbb{N}} A_n$ is connected.
(2) Let $f\colon X \longrightarrow Y$ be a continuous map between topological spaces. Let $A \subset X$ be a connected subset. Show that $f(A)$ is connected.
(3) Let $f\colon [0,1] \longrightarrow [0,1]$ be a continuous map, where we consider $[0,1]$ with the subspace topology induced by the usual metric on $\mathbb{R}$. Show that there exists a point $x \in [0,1]$ such that $f(x) = x$.

Exercise 2. Recall that a differentiable map $f\colon M \longrightarrow N$ between differentiable manifolds is an *immersion* if its differential is everywhere injective. Let $(x, y) \in \mathbb{R}^2$. In the following we write $\alpha = \alpha_{x,y}$ for the positive number $x^2 + y^2 + 1 = \|(x, y)\|^2 + 1$. Let $\varphi_\pm\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by $(x, y) \longmapsto \frac{1}{\alpha}(2x, 2y, \pm(\alpha - 2))$. Show that both maps are immersions, whose image is contained in the two-dimensional sphere S^2. Furthermore show the following: If $N^\pm = (0, 0, \pm 1)$, then $\varphi_\pm\colon \mathbb{R}^2 \longrightarrow S^2 \setminus \{N^\pm\}$ are homeomorphisms (in fact, they are even diffeomorphisms) with inverse maps $\psi_\pm(x, y, z) = (\frac{x}{1\mp z}, \frac{y}{1\mp z})$

Exercise 3. [1, Ex. 2.1.1] Show that $\mathbb{P}^1$ is simply connected by proving that it is diffeomorphic to S^2.

Exercise 4. [1, Ex. 2.1.2] Show that $\mathbb{C}^n$ does not have any compact submanifolds of positive dimension.

Exercise 5. [1, Ex. 2.1.4] Show that any holomorphic map from $\mathbb{P}^n$ to a complex torus is constant.

Hint: Use that $\mathbb{P}^n$ is simply connected and the lifting property for continuous maps. You may assume that a lift is holomorphic.

References

[1] D. Huybrechts, *Complex geometry: An introduction*, Springer, Berlin (2005).
[2] M. Kashiwara and P. Shapira, *Sheaves on manifolds*, Springer, Berlin (1994).
[3] C. Schnell, *Complex manifolds*, available at `http://www.math.stonybrook.edu/~cschnell/`.
[4] C. Voisin, *Hodge theory and complex algebraic geometry*, Cambridge University Press, Cambridge (2002).