EXERCISES, COMPLEX GEOMETRY, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2015/2016

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Sheet 12

Exercise 1. (compare [2, Ex. 3.3.2]**)** Let X be the Hopf surface considered in the lecture. Show that the Jacobian of X, that is, the quotient $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ is not a compact torus in a natural way.

For the next two exercises you will need the following fact. On any complex manifold X (in fact, the following statement holds far more generally), there is a bijection between the space of complex line bundles and $H^2(X, \mathbb{Z})$.

Exercise 2. (compare [2, Ex. 3.3.7]**)** Show that any complex line bundle on $\mathbb{P}^n$ can be endowed with a unique holomorphic structure. Give a sufficient condition for a compact Kähler manifold to have a complex line bundle which does not admit any holomorphic structure.

Exercise 3. [2, Ex. 3.3.8] Show that on a complex torus $\mathbb{C}^n/\Gamma$ the trivial complex line bundle admits many non-trivial holomorphic structures. How many are there?

Exercise 4. Show that any map from a projective space to a complex torus is constant.

References

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[4] C. Schnell, *Complex manifolds*, available at `http://www.math.stonybrook.edu/~cschnell/`.
[5] C. Voisin, *Hodge theory and complex algebraic geometry*, Cambridge University Press, Cambridge (2002).