EXERCISES, COMPLEX GEOMETRY, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2015/2016

P. SOSNA

Sheet 11

Exercise 1. (compare [2, Ex. 3.1.8]**)** Using that on an n-dimensional compact Kähler manifold with Kähler form ω we have the equality $\int_X \omega^n = n! \cdot \mathrm{vol}(X)$, show that there is an injective ring homomorphism $\mathbb{R}[x]/x^{n+1} \longrightarrow H^*(X, \mathbb{R})$. Conclude that only the two-dimensional sphere admits a Kähler structure.

Exercise 2. [2, Ex. 3.2.1] Let X be a Kähler manifold. Show that the Kähler form ω is harmonic.

Exercise 3. [2, Ex. 3.2.6] Show that the odd Betti numbers of a Kähler manifold are even.

Exercise 4. [2, Ex. 3.2.10] Show that on a compact hermitian manifold any d-harmonic (p, q)-form is also $\overline{\partial}$-harmonic.

Exercise 5. (compare [2, Ex. 3.2.11]**)** Use the ring isomorphism $H^*(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}[x]/x^{n+1}$ (where x has degree 2) to show that $H^{p,q}(\mathbb{P}^n) = 0$ except for $p = q \leq n$. Furthermore, prove that $H^{p,p}(\mathbb{P}^n)$ is one-dimensional for $p \leq n$ and deduce that $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$.

References

[1] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.
[2] D. Huybrechts, *Complex geometry: An introduction*, Springer, Berlin (2005).
[3] M. Kashiwara and P. Shapira, *Sheaves on manifolds*, Springer, Berlin (1994).
[4] C. Schnell, *Complex manifolds*, available at `http://www.math.stonybrook.edu/~cschnell/`.
[5] C. Voisin, *Hodge theory and complex algebraic geometry*, Cambridge University Press, Cambridge (2002).