EXERCISES, COMPLEX GEOMETRY, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2015/2016

P. SOSNA

Sheet 10

Exercise 1. Let X be a Kähler manifold. Show that the equations $[\overline{\partial}^*, L] = i\partial$ and $[\partial^*, L] = -i\overline{\partial}$ hold.

Exercise 2. [2, Ex. 3.1.10] Prove that the product of two Kähler manifolds is again a Kähler manifold.

Exercise 3. [2, Ex. 3.2.1] Show that on a Kähler manifold the fundamental form ω is harmonic.

The following is meant as a gentle reminder about the topics of the lecture so far. Some of the exercises are formulated in a slightly imprecise manner.

(1) Show that the Cauchy-Riemann equations are equivalent to the equation $\frac{\partial f}{\partial \bar{z}}$.
(2) Show that a function is holomorphic if and only if its differential is complex linear.
(3) Let X be any set. Define a subset of X to be open if it is empty or if its complement is finite. Show that this defines a topology, called the cofinite topology, on X. Also show that with respect to this topology X is a compact space (that is, every open cover has a finite subcover). Prove that $\mathbb{R}$ endowed with the cofinite topology is not Hausdorff.
(4) Check the universal property of $\bigwedge^k V$ stated in the lecture.
(5) Convince yourself that one can indeed define a complex manifold either as a differentiable manifold with an equivalence class of holomorphic atlases or as a topological manifold endowed with a holomorphic atlas.
(6) Give an example of a complex manifold whose vector space of holomorphic functions is infinite-dimensional.
(7) Check that a complex torus of dimension n is indeed diffeomorphic to $(S^1)^{2n}$. Can you compute the Betti numbers of a complex torus?
(8) Show that taking sections of a vector bundle indeed defines a sheaf.
(9) Give an example of a sheaf morphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ such that $\mathrm{coker}'(\varphi)$ is not a sheaf.
(10) Show that if $\varphi\colon \mathcal{F}' \longrightarrow \mathcal{F}$ is an injective morphism of sheaves and $H^1(U, \mathcal{F}') = 0$ for all $U \subset X$ open, then the cokernel presheaf $\mathrm{coker}'(\varphi)$ is already a sheaf.
(11) Check that a morphism of (pre-)sheaves defines morphisms on stalks.
(12) Convince yourself (again) that the sheaf of sections of the trivial holomorphic vector bundle of rank r on a complex manifold X is $\mathcal{O}_X^r$.

(13) Check that taking the quotient of $\coprod_i U_i \times \mathbb{C}$ by the equivalence relation (defined by a cocycle) given in the lecture indeed produces a holomorphic vector bundle.
(14) Show that any homomorphism of holomorphic vector bundles defines a kernel and a cokernel holomorphic vector bundle.
(15) Check that the inverse of a line bundle L in the Picard group is indeed given by L^*.
(16) Show that an almost complex structure induces a natural orientation on a vector space.
(17) Let V be a real vector space. Prove that V is the fixed locus of the conjugation on $V_{\mathbb{C}}$.
(18) Prove Lemma 6.4 in the lecture which states that an almost complex structure on V defines a decomposition of V^*.
(19) Prove Lemma 6.6.
(20) Convince yourself that the descriptions of ∂ and $\overline{\partial}$ given in the lecture hold true.
(21) Show that the pullback of differential forms respects the bidegree decomposition.
(22) Give a proof of Proposition 7.1.
(23) Show that the Laplace operator on a Riemannian manifold is self-adjoint.

References

[1] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.
[2] D. Huybrechts, *Complex geometry: An introduction*, Springer, Berlin (2005).
[3] M. Kashiwara and P. Shapira, *Sheaves on manifolds*, Springer, Berlin (1994).
[4] C. Schnell, *Complex manifolds*, available at `http://www.math.stonybrook.edu/~cschnell/`.
[5] C. Voisin, *Hodge theory and complex algebraic geometry*, Cambridge University Press, Cambridge (2002).