EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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Sheet 9

Exercise 1. [2, Ex. 5.5 & 5.6] Let $A \subseteq B$ be rings. Prove the following statements.

(1) If B is integral over A and $x \in A$ has an inverse in B, then this inverse is already in A.
(2) If B is integral over A, the Jacobson radical of A is the contraction of the Jacobson radical of B.
(3) If $B \setminus A$ is closed under multiplication, then A is integrally closed in B.

Exercise 2. [2, Ex. 5.12] Let G be a finite group of automorphisms of a ring A and let

$$A^G = \{a \in A \mid g(a) = a \ \forall g \in G\}.$$

Prove that A^G is a ring (the so-called ring of invariants) and that A is integral over A^G.

If S is a multiplicatively closed subset of A such that $g(S) \subseteq S$ for all $g \in G$, set $S^G = S \cap A^G$. Show that the action of G extends to an action on $S^{-1}A$ and that $(S^{-1}A)^G \simeq (S^G)^{-1}A^G$.

Exercise 3. [1, Ex. 14.4] Let $A \subseteq B$ be rings, B integral over A and let $\mathfrak{p} \in \operatorname{Spec}(A)$. Assume there is only one $\mathfrak{q} \in \operatorname{Spec}(B)$ such that $A \cap \mathfrak{q} = \mathfrak{p}$. Prove that 1) $\mathfrak{q}B_{\mathfrak{p}}$ is the only maximal ideal in $B_{\mathfrak{p}}$, 2) $B_{\mathfrak{p}} = B_{\mathfrak{q}}$ and 3) $B_{\mathfrak{q}}$ is integral over $A_{\mathfrak{p}}$.

Hint. To prove 2) you might want to establish the following statement. If C is any ring, $S \subset T$ are multiplicative subsets and $T' = g_S(T)$ is the image of T under the localisation map $g_S\colon C \longrightarrow S^{-1}C$, then $T^{-1}C = T'^{-1}(S^{-1}R) = T^{-1}(S^{-1}R)$.

Exercise 4. [1, Ex. 14.5] Let $A \subseteq B$ be an integral extension of domains and let $\mathfrak{p} \in \operatorname{Spec}(A)$. Assume that there are at least two distinct prime ideals $\mathfrak{q}$ and $\mathfrak{q}'$ in B such that $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$. Prove that $B_{\mathfrak{q}}$ is not integral over $A_{\mathfrak{p}}$.

References

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