EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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Sheet 8

Exercise 1. [1, Ex. 17.7 & 17.10]

(1) Let $A = \mathbb{Z}$ and let $M = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Compute $\mathrm{Ass}M$ and find submodules L and N of M such that $L + N = M$ but $\mathrm{Ass}N \cup \mathrm{Ass}L \subsetneq \mathrm{Ass}M$.

(2) Let A be a ring with the property that $A_\mathfrak{p}$ is a domain for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Show that every associated prime ideal is minimal.

Exercise 2. [2, Ex. 4.5] Let $A = k[X, Y, Z]$, $\mathfrak{p}_1 = (X, Y)$, $\mathfrak{p}_2 = (X, Z)$ and $\mathfrak{m} = (X, Y, Z)$. Let $I = \mathfrak{p}_1\mathfrak{p}_2$. Prove that $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of I and determine the isolated and embedded components.

Exercise 3. [2, Ex. 4.7] Let I be an ideal in a ring A, let $B = A[X]$ and let $I[X] = \{p = \sum_{i=0}^{n} a_i X^i \in A[X] \mid a_i \in I\} \subseteq A[X]$. Show the following statements:

(1) $I[X]$ is the extension of I in B.

(2) If $\mathfrak{p} \in \mathrm{Spec}(A)$, then $\mathfrak{p}[X] \in \mathrm{Spec}(B)$.

(3) If $\mathfrak{q}$ is $\mathfrak{p}$-primary in A, then $\mathfrak{q}[X]$ is $\mathfrak{p}[X]$-primary in B.

(4) If $I = \cap_{k=1}^{n} J_k$ is a minimal primary decomposition in A, then $I[X] = \cap_{k=1}^{n} J_k[X]$ is a minimal primary decomposition in B.

(5) If $\mathfrak{p}$ is a minimal prime ideal of I, then $\mathfrak{p}[X]$ is a minimal prime ideal of $I[X]$.

Exercise 4. cf. [2, Ex. 4.20 & 4.21]

(1) Let M be an A-module and let $N \subseteq M$ be a submodule. Define

$$\mathrm{rad}_M(N) = \{a \mid \exists k > 0 : \ a^k M \subseteq N\}.$$

Prove that $\mathrm{rad}_M(N) = \mathrm{rad}(N : M) = \mathrm{rad}(\mathrm{Ann}(M/N))$.

(2) If M' is any A-module, recall that $a \in A$ is a zero-divisor on M if $am = 0$ for some $m \neq 0$; a is nilpotent on M if $a^k M = 0$ for some $k > 0$.

We say that a submodule Q of M is primary if $Q \neq M$ and every zero-divisor for M/Q is nilpotent. Show that if Q is primary, then $(Q : M)$ is a primary ideal. Conclude that $\mathrm{rad}_M(Q)$ is a prime ideal.

References

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[3] P. L. Clark, *Commutative algebra*, http://www.math.uga.edu/ pete/integral.pdf.
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