EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

P. SOSNA

Sheet 7

Exercise 1. cf. [2, Ex. 4.2 & 4.4]

a) Let A be a ring and I let be an ideal which is equal to its radical. Show that I has a (possibly infinite) primary decomposition without embedded primes.

b) If $f\colon A \longrightarrow B$ is a ring homomorphism and I is $\mathfrak{p}$-primary in B, then I^c is $\mathfrak{p}^c$-primary in A. Show that the converse holds if f is surjective.

c) Let $A = \mathbb{Z}[t]$. Show that $\mathfrak{m} = (2, t)$ is a maximal ideal, that $J = (4, t)$ is $\mathfrak{m}$-primary but J is not a power of $\mathfrak{m}$.

Exercise 2. [3, Ex. 10.5] Let $A = \mathbb{Z}[t]/(t^2 + 3)$ and $I = (2) \subseteq A$.

a) Show that there exists a unique maximal ideal $\mathfrak{m}$ such that $A/\mathfrak{m} \simeq \mathbb{Z}/2\mathbb{Z}$.

b) Prove that $\mathrm{rad} I = \mathfrak{m}$ and deduce that I is $\mathfrak{m}$-primary.

c) Show that I is not a product of prime ideals.

Exercise 3. [1, Ex. 18.7] Let k be a field, $A = k[X, Y]$ and $I = (X^2, XY)$. Show that $\mathrm{rad} I$ is prime and that I is not primary. Prove that if $fg \in I$, then either $f^2 \in I$ or $g^2 \in I$.

Exercise 4. [1, Ex. 18.16] Let k be a field, $A = k[X, Y, Z]$ and $I = (XY, X - YZ)$. Show that

$$I = (X, Z) \cap (Y^2, X - YZ)$$

and that this is a minimal primary decomposition of I.

References

[1] A. Altman and S. Kleiman, *A term of commutative algebra*, http://web.mit.edu/18.705/www/12Nts-2up.pdf.

[2] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading Mass.-London-Don Mills, 1969.

[3] P. L. Clark, *Commutative algebra*, http://www.math.uga.edu/ pete/integral.pdf.

[4] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford University Press, Oxford, 2002.

[5] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1989.