EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

P. SOSNA

Sheet 6

Exercise 1. Recall that a nonzero module M is called simple if its only submodules are 0 and M. Prove that the following statements are equivalent.

(1) The module M is a direct sum of simple modules.
(2) Every submodule N of M is a direct summand, that is, there exists a submodule N' such that $N \oplus N' = M$.
(3) The module M is a sum of simple submodules.

Exercise 2. cf. [1, Ex. 19.2] Show that an A-module M is simple if and only if $M \simeq A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ and if this holds, then $\mathfrak{m} = \mathrm{Ann}(M)$. Furthermore, prove that a module of finite length is finitely generated.

Exercise 3. [2, Ex. 6.8] A topological space is called Noetherian if the open subsets satisfy the ascending chain condition (alternatively, the maximal condition) or, equivalently, the closed subsets satisfy the descending chain condition (alternatively, the minimal condition).

If A is a Noetherian ring, show that $X = \mathrm{Spec}(A)$ is a Noetherian topological space. Give an example where $\mathrm{Spec}(A)$ is Noetherian but A is not.

Exercise 4. [2, Ex. 6.5] A topological space is called quasi-compact if whenever $X = \cup_i U_i$ is a cover of X by open subsets U_i, then finitely many of the U_i already cover X.

Show that if X is a Noetherian topological space, then every subspace of X is also Noetherian and that X is quasi-compact.

References

[1] A. Altman and S. Kleiman, *A term of commutative algebra*, http://web.mit.edu/18.705/www/12Nts-2up.pdf.
[2] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading Mass.-London-Don Mills, 1969.
[3] P. L. Clark, *Commutative algebra*, http://www.math.uga.edu/ pete/integral.pdf.
[4] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford University Press, Oxford, 2002.
[5] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1989.