EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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Sheet 5

Exercise 1. (cf. [1]) Let A be a ring, S a multiplicatively closed subset of A and I an ideal of A. The *saturation* of I is the set

$$I^S = \{a \mid \exists s \in S : as \in I\}.$$

We call I saturated if $I = I^S$.

Prove the following statements: i) $\ker(A \longrightarrow S^{-1}A) = (0)^S$, ii) $I \subseteq I^S$, iii) I^S is an ideal, iv) if $I \subseteq J$ are ideals, then $I^S \subseteq J^S$, v) $(I^S)^S = I^S$ and (vi) $(I^S J^S)^S = (IJ)^S$.

Show that if M is an A-module, then the kernel of the map $M \longrightarrow S^{-1}M$ is the set of elements $m \in M$ satisfying $\mathrm{Ann}(m) \cap S \neq \emptyset$. In particular, if $\mathrm{Ann}(M) \cap S \neq \emptyset$, then $S^{-1}M = 0$.

Exercise 2. (cf. [1]) Let M be an A-module and $M_1 \subseteq M_2$ be submodules of M. Then $M_1 = M_2$ if and only if $M_1 \cap N = M_2 \cap N$ and $(M_1 + N)/N = (M_2 + N)/N$ for all submodules N of M.

Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence of modules. If M_1 and M_2 are submodules of M such that $g(M_1) = g(M_2)$ and $f^{-1}(M_1) = f^{-1}(M_2)$, is it true that $M_1 = M_2$?

Exercise 3. [2, Ex. 6.3] Let M be an A-module and N_1, N_2 be submodules of M. Assume that M/N_1 and M/N_2 are Noetherian (resp. Artinian). Prove that then $M/(N_1 \cap N_2)$ is Noetherian (resp. Artinian).

Exercise 4. [5, Ex. 3.1 & 3.2], also cf. [1] Let A be a ring and $I_1, \dots, I_n$ be ideals such that every ring A/I_k is Noetherian. Show that $M = \oplus_k A/I_k$ is a Noetherian R-module. Prove that if $\cap_k I_k = 0$, then R is a Noetherian ring. Use this to show the following statement: If A and B are Noetherian rings and $f\colon A \longrightarrow C$ and $g\colon B \longrightarrow C$ are surjective ring homomorphisms, then the *fibre product* $A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$ is a Noetherian ring.

Let now k be a field and A be a k-algebra. Prove that if A is finite-dimensional as a k-vector space, then A is Noetherian and Artinian.

References

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