# EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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## Sheet 4

**Exercise 1.** [1, Ex. 3.5] Let $A$ be a ring. Prove that $A$ has no nilpotent elements if $A_{\mathfrak{p}}$ does not have any nilpotent elements for any prime ideal $\mathfrak{p}$. Find an example of a ring $A$ such that $A_{\mathfrak{p}}$ is an integral domain for any prime ideal $\mathfrak{p}$ but $A$ is not an integral domain.

**Exercise 2.** (cf. [1, Ex. 3.12 & 3.13]) Let $A$ be an integral domain and $M$ be an $A$-module. Recall that $\mathrm{tors}(M)$ is the submodule of torsion elements of $M$, that is, elements $m$ such that $am = 0$ for some $0 \neq a \in A$. Prove that $M/\mathrm{tors}M$ is a torsion-free module.

Now let $S$ be a multiplicatively closed subset of $A$. Show that $S^{-1}(\mathrm{tors}M) = \mathrm{tors}(S^{-1}M)$. Conclude that the following statements are equivalent: i) $M$ is torsion-free, ii) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals $\mathfrak{p}$ and iii) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

**Exercise 3.** (cf. [1, Ex. 3.19]) Let $A$ be a ring and $M$ be an $A$-module. We define the *support* of $M$ to be the set of all prime ideals $\mathfrak{p}$ of $A$ such that $M_{\mathfrak{p}} \neq 0$. This set is denoted by $\mathrm{Supp}(M)$. Show the following statements.

(1) $M \neq 0 \iff \mathrm{Supp}(M) \neq 0$.
(2) $V(I) = \mathrm{Supp}(A/I)$ for any ideal $I$ of $A$.
(3) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence, then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.
(4) If $M = \sum_\alpha M_\alpha$, then $\mathrm{Supp}(M) = \cup_\alpha \mathrm{Supp}(M_\alpha)$.
(5) If $M$ is finitely generated, then $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$.
(6) If $M$ and $N$ are finitely generated, then $\mathrm{Supp}(M \otimes N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.
(7) If $I$ is an ideal and $M$ is finitely generated, then $\mathrm{Supp}(M/IM) = V(I + \mathrm{Ann}(M))$.

**Exercise 4.** (cf. [1, Ex. 3.20]) Let $f\colon A \longrightarrow B$ be a ring homomorphism, et $I$ be an ideal of $A$ and $J$ be an ideal of $B$. Recall that $I^e$ is the ideal generated by $f(I)$ in $B$ and $J^c$ is $f^{-1}(J)$. Show that $I \subseteq I^{ec}$, $J^{ce} \subseteq J$ and use this to prove that $I^e = I^{ece}$ and $J^c = J^{cec}$. Now prove the following statements concerning the map $f^*\colon \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$.

(1) The map $f^*$ is surjective if and only if every prime ideal of $A$ is a contracted ideal.
(2) If every prime ideal of $B$ is an extended ideal, then $f^*$ is injective.

## References

[1] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading Mass.-London-Don Mills, 1969.

[2] P. L. Clark, *Commutative algebra*, http://www.math.uga.edu/ pete/integral.pdf.
[3] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford University Press, Oxford, 2002.
[4] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1989.