EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

P. SOSNA

Sheet 2

Exercise 1. [1, Ex. 1.12] Prove that if A is a local ring and e an idempotent, that is, $e^2 = e$, then $e = 0$ or $e = 1$.

Exercise 2. [1, Ex. 1.17] Let A be a ring, $f \in A$, $X = \mathrm{Spec}(A)$ and $X_f = X \setminus V(f)$. Prove the following statements.

(1) $X_f \cap X_g = X_{fg}$,
(2) $X_f = \emptyset \iff f$ is nilpotent,
(3) $X_f = X \iff f$ is a unit,
(4) $X_f = X_g \iff \mathrm{rad}(f) = \mathrm{rad}(g)$.

Exercise 3. [1, Ex. 2.9] Show that if $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ is a short exact sequence of A-modules and M', M'' are finitely generated, then so is M.

Exercise 4. [2, Ex. 3.18] Let A be an integral domain. An element $m \in M$ is said to be torsion if $\mathrm{Ann}(m) \neq 0$. Denote by $\mathrm{tors}M$ the set of all torsion elements of M. We will say that M is torsionfree if $\mathrm{tors}M = 0$ and that M is torsion if $M = \mathrm{tors}M$.

Let

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence. Show the following statements or answer the questions, respectively.

(1) $\mathrm{tors}M$ is a submodule of M.
(2) If M is torsion, then the same holds for M' and M''.
(3) If M is torsionfree, then M' is torsionfree, but M'' need not be.
(4) If M' and M'' are torsion, is then M is also torsion?
(5) If M' and M'' are both torsionfree, does the same hold for M?

References

[1] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading Mass.-London-Don Mills, 1969.
[2] P. L. Clark, *Commutative algebra*, http://www.math.uga.edu/ pete/integral.pdf.
[3] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1989.