EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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Sheet 13

Exercise 1. [6, Ex. 4.4.1] Let A be a regular local ring and $x_1, \dots, x_d \in \mathfrak{m}$ map to a basis of $\mathfrak{m}/\mathfrak{m}^2$. Prove that every quotient ring $A/(x_1, \dots, x_i)A$ is regular local of dimension $d - i$.

Solution. Clearly, $A/(x_1, \dots, x_i)A$ is local and we have seen that $x_1, \dots, x_d$ is an A-sequence, hence $\dim(A/(x_1, \dots, x_i)A) = \dim(A) - i$. On the other hand, by assumption, $x_{i+1}, \dots, x_d$ map to a basis of $\mathfrak{m}'/\mathfrak{m}'^2$, where $\mathfrak{m}'$ is the maximal ideal of $A/(x_1, \dots, x_i)A$, that is, the image of $\mathfrak{m}$ under the quotient map. Therefore, the claim follows.

Exercise 2.

(1) Let A be a local ring and $0 \to N' \to N \to N'' \to 0$ be an exact sequence of finitely generated A-modules. Show that $\mathrm{depth}(N) \geq \min\{\mathrm{depth}(N'), \mathrm{depth}(N'')\}$.
(2) Let A be as local ring and $M \neq 0$ be a finitely generated A-module. We call M maximal Cohen-Macaulay (MCM) if $\mathrm{depth}(M) = \dim(A)$. Show that if in an exact sequence $0 \to M' \to M \to M'' \to 0$, the modules M' and M'' are MCM, then the same holds for M.
Hint. Use that $\mathrm{depth}(M) \leq \dim(M)$, where $\dim(M) = \dim(\mathrm{Supp}(M))$. The proof of this fact is similar to that of the statement $\mathrm{depth}(A) \leq \dim(A)$ established in the lecture.
(3) Prove that if M is MCM and admits a direct sum decomposition $M = M_1 \oplus M_2$, then M_1 and M_2 are MCM.

Solution. To prove the first claim, we recall that $\mathrm{depth}(M) = \min\{l \mid \mathrm{Ext}^l(A/\mathfrak{m}, M) \neq 0\}$ and consider the long exact Ext-sequence associated to $\mathrm{Hom}(A/\mathfrak{m}, -)$ from which the claim immediately follows.

The second claim follows from the first and the fact that $\mathrm{depth}(M) \leq \dim(M) \leq \dim(A)$ (we used that $\mathrm{Supp}(M)) = V(\mathrm{Ann}(M)) = \mathrm{Spec}(A/V(\mathrm{Ann}(M))$ is a closed subset).

Finally, since $\mathrm{Ext}^l(A/\mathfrak{m}, M) = \mathrm{Ext}^l(A/\mathfrak{m}, M_1 \oplus M_2) = \mathrm{Ext}^l(A/\mathfrak{m}, M_1) \oplus \mathrm{Ext}^l(A/\mathfrak{m}, M_2)$, the last claim follows from the first and the hint.

Exercise 3. Let A be a regular local ring and M be an MCM module. Show that M is free.

Solution. First note that $\mathrm{pdim}(M) < \infty$, since $\mathrm{pdim}(M) \leq \mathrm{gdim}(A) < \infty$ by assumption. Hence, the Auslander-Buchsbaum formula gives $\dim(A) = \mathrm{depth}(A) = \mathrm{depth}(M) + \mathrm{pdim}(M)$, so $\mathrm{pdim}(M) = 0$, thus M is projective, hence free.

References

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