EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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Sheet 13

Exercise 1. [6, Ex. 4.4.1] Let A be a regular local ring and $x_1, \dots, x_d \in \mathfrak{m}$ map to a basis of $\mathfrak{m}/\mathfrak{m}^2$. Prove that every quotient ring $A/(x_1, \dots, x_i)A$ is regular local of dimension $d - i$.

Exercise 2.

(1) Let A be a local ring and $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ be an exact sequence of finitely generated A-modules. Show that $\mathrm{depth}(N) \geq \min\{\mathrm{depth}(N'), \mathrm{depth}(N'')\}$.

(2) Let A be as local ring and $M \neq 0$ be a finitely generated A-module. We call M maximal Cohen-Macaulay (MCM) if $\mathrm{depth}(M) = \dim(A)$. Show that if in an exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, the modules M' and M'' are MCM, then the same holds for M.
Hint. Use that $\mathrm{depth}(M) \leq \dim(M)$, where $\dim(M) = \dim(\mathrm{Supp}(M))$. The proof of this fact is similar to that of the statement $\mathrm{depth}(A) \leq \dim(A)$ established in the lecture.

(3) Prove that if M is MCM and admits a direct sum decomposition $M = M_1 \oplus M_2$, then M_1 and M_2 are MCM.

Exercise 3. Let A be a regular local ring and M be an MCM module. Show that M is free.

References

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