EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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Sheet 12

Exercise 1. Recall that a complex of A-modules is a sequence of modules and A-linear maps $d^i \colon M^i \longrightarrow M^{i+1}$ for $i \in \mathbb{Z}$ such that $d^{i+1} \circ d^i = 0$ for all i. Write a complex as $M^\bullet$. The i-cycles of a complex is by definition $Z^i(M^\bullet) := \ker(d^i)$ and the i-boundaries are $B^i(M^\bullet) = \mathrm{im}(d^{i-1})$. Clearly, $B^i(M^\bullet) \subseteq Z^i(M^\bullet)$ and we define the i-th cohomology of $M^\bullet$ to be $H^i(M^\bullet) = Z^i(M^\bullet)/B^i(M^\bullet)$. A morphism of complexes $f^\bullet \colon M^\bullet \longrightarrow N^\bullet$ is given by maps $f^i \colon M^i \longrightarrow N^i$ for all $i \in \mathbb{Z}$ such that $d^i_{N^\bullet} \circ f^i = f^{i+1} \circ d^i_{M^\bullet}$ for all i.

Show that any morphism of complexes $f^\bullet$ induces a map $H^i(f^\bullet) \colon H^i(M^\bullet) \longrightarrow H^i(N^\bullet)$ for all $i \in \mathbb{Z}$.

A morphism $f^\bullet$ is called a quasi-isomorphism if $H^i(f^\bullet)$ is an isomorphism for all i. Show that the following conditions are equivalent: (1) $M^\bullet$ is exact at every M^i, (2) $H^i(M^\bullet) = 0$ for all i, (3) the map $0 \longrightarrow M^\bullet$ is a quasi-isomosphism.

Exercise 2. cf. [6, Ex. 2.4.3] Let $F \colon \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor between abelian categories (for instance, the categories of modules over some rings). If $\mathcal{A}$ has enough injectives, the i-th right derived functor R^iF of F is constructed as follows. For any $A \in \mathcal{A}$, take an injective resolution $A \longrightarrow E^\bullet$ and define $R^iF(A) = H^i(F(E^\bullet))$. This definition does not depend on the choice of injective resolution and if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact, then there is a long exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow R^1F(A') \longrightarrow R^1F(A) \longrightarrow R^1F(A'') \longrightarrow \dots$$
$$\dots \longrightarrow R^iF(A') \longrightarrow R^iF(A) \longrightarrow R^iF(A'') \longrightarrow R^{i+1}F(A') \longrightarrow \dots$$

If $0 \longrightarrow A \longrightarrow E \longrightarrow M \longrightarrow 0$ is exact and E is injective, show that $R^iF(A) \simeq R^{i-1}F(M)$ for $i \geq 2$ and that $R^1F(A) = \mathrm{coker}(F(E) \longrightarrow F(M))$. More generally, show that if

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow \dots \longrightarrow E^m \longrightarrow M \longrightarrow 0$$

is exact and all E^i are injective, then $R^iF(A) \simeq R^{i-m-1}F(M)$ for $i \geq m+2$ and $R^{m+1}F(A) = \mathrm{coker}(F(E^m) \longrightarrow F(M))$.

Write down the corresponding "dimension shifting" statement for left derived functors of a right exact functor F which are constructed using projective resolutions and convince yourself that a similar proof works in this case as well.

Exercise 3. cf. [6, Example 3.1.7 & Ex. 3.2.1] Let M be an A-module. Consider the endofunctor $\mathrm{Mod}A \longrightarrow \mathrm{Mod}A$ defined by $N \longmapsto N \otimes M$ and $f \longmapsto f \otimes \mathrm{id}_M$. This functor is right exact and $\mathrm{Mod}A$ has enough projectives, so there exist left derived functors defined by $\mathrm{Tor}_i(M, N) = H^i(P_\bullet \otimes M)$, where $P_\bullet$ is any projective resolution of N. It is a fact that $\mathrm{Tor}_i(M,N) = \mathrm{Tor}_i(N,M) = H^i(P'_\bullet \otimes N)$, where $P'_\bullet$ is any projective resolution of M. Furthermore, if $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ is exact, we get a long exact sequence

$$\ldots \longrightarrow \mathrm{Tor}_1(N',M) \longrightarrow \mathrm{Tor}_1(N,M) \longrightarrow \mathrm{Tor}_1(N'',M) \longrightarrow N' \otimes M \longrightarrow N \otimes M \longrightarrow N'' \otimes M \longrightarrow 0.$$

Suppose that $a \in A$ is not a zero-divisor. Show that $\mathrm{Tor}_0(A/a,M) \simeq M/aM$, $\mathrm{Tor}_1(A/a,M) \simeq \{m \in M \mid am = 0\}$ and $\mathrm{Tor}_n(A/a,M) = 0$ for all $n \geq 2$.

Show that the following conditions are equivalent: (1) N is flat, (2) $\mathrm{Tor}_n(M,N) = 0$ for all $n \geq 1$ and all modules M, (3) $\mathrm{Tor}_1(M,N) = 0$ for all modules M.

Exercise 4. [2, Ex. 2.25 & 2.26]

(1) Let A be any ring and let $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ be an exact sequence of A-modules with N'' flat. Show that N is flat if and only if N' is flat.

(2) Show that an A-module N is flat if and only $\mathrm{Tor}_1(A/I,N) = 0$ for all finitely generated ideals $I \subseteq A$.

References

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