EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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Sheet 11

Exercise 1. [6, Ex. 14.2] Let $(A, \mathfrak{m})$ be a Noetherian local ring and let $G = G_{\mathfrak{m}}(A)$. For $a \in A$, suppose that $a \in \mathfrak{m}^i$, but $a \notin \mathfrak{m}^{i+1}$ and write a^* for the image of a in $\mathfrak{m}^i/\mathfrak{m}^{i+1} \subseteq G$. Set $0^* = 0$. Prove the following statements.

(1) If $a^*b^* \neq 0$, then $a^*b^* = (ab)^*$.
(2) If a^* and b^* have the same degree and $a^* + b^* \neq 0$, then $a^* + b^* = (a+b)^*$.
(3) Let $I \subseteq \mathfrak{m}$ be an ideal. Write $I^* \subseteq G$ be the ideal generated by the elements i^* for $i \in I$. Setting $B = A/I$ and $\mathfrak{n} = \mathfrak{m}/I$, we have $G_{\mathfrak{n}}(B) = G/I^*$.

Exercise 2. [2, Ex. 11.1] Let k be an algebraically closed field and let $f \in k[x_1, \dots, x_n]$ be an irreducible polynomial. We call a point $p \in Z(f)$ non-singular if not all the partial derivatives $\partial f/\partial x_i$ vanish at p.

Let $A = k[x_1, \dots, x_n]/(f)$ and let $\mathfrak{m}$ be the maximal ideal of A corresponding to p (if $p = (a_1, \dots, a_n)$, then $\mathfrak{m}$ is the image in A of $\mathfrak{m}_p = (x_1 - a_1, \dots, x_n - a_n)$; here we use the Nullstellensatz). Show that p is non-singular if and only if $A_{\mathfrak{m}}$ is a regular local ring.

Exercise 3. [2, Ex. 11.6] Let A be a ring. Show that

$$\dim(A) + 1 \leq \dim(A[X]) \leq 1 + 2\dim(A).$$

Hint. Use the following fact:

If $f\colon B \longrightarrow B'$ is a ring homomorphism and $f^*\colon \mathrm{Spec}(B') \longrightarrow \mathrm{Spec}(B)$ the induced map on the spectra, then for any $\mathfrak{p} \in \mathrm{Spec}(B)$ the fibre $f^{*-1}(\mathfrak{p})$ is homeomorphic to $\mathrm{Spec}(B'_{\mathfrak{p}}/\mathfrak{p}B'_{\mathfrak{p}}) = \mathrm{Spec}(\kappa(\mathfrak{p}) \otimes_B B')$ where $\kappa(\mathfrak{p})$ is the residue field of the local ring $B_{\mathfrak{p}}$.

Exercise 4. [3, Ex. 11.10] Let A be a Noetherian ring. Show that A is reduced if and only if i) the localisation of A at any prime of height 0 is regular and ii) every prime associated with 0 is of height 0.

References

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