EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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Sheet 10

Exercise 1. [1, Ex. 14.6] let k be a field, X an indeterminate, $Y = X^2$, $A = k[Y]$ and $B = k[X]$. Define $\mathfrak{p} = (Y-1)A$ and $\mathfrak{p}' = (X-1)B$. Investigate whether $B_{\mathfrak{p}'}$ is always integral over $A_{\mathfrak{p}}$.

Exercise 2.

(1) (cf. [3, Ex. 14.2]) Let $A \subseteq B$ be rings and let C be the integral closure of A in B. Now assume that A and B are fields and that B is algebraically closed. Prove that C is algebraically closed.

(2) Let $A \subseteq B$ be rings and assume that B is integral over A. Show that the Krull dimensions of A and B are equal.

Exercise 3. [2, Ex. 5.28] Let A be an integral domain and K its field of fractions. We say that A is a *valuation ring* of K if for any $0 \neq x \in K$ either $x \in B$ or $x^{-1} \in B$. Show that the following conditions are equivalent.

(1) A is a valuation ring of K.

(2) For any ideals I, J of A we either have $I \subseteq J$ or $J \subseteq I$.

Deduce that if A is a valuation ring and $\mathfrak{p}$ is a prime ideal of A, then $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are valuation rings of their fields of fractions.

Exercise 4. [2, Ex. 5.30] Let A be a valuation ring of a field K. The group U of units of A is a subgroup of the multiplicative group K^* of K. Set $\Gamma = K^*/U$ and note that Γ is a commutative group.

Given $\alpha, \beta \in \Gamma$, pick representatives $x, y \in K^*$ and define

$$\alpha \geq \beta \Longleftrightarrow xy^{-1} \in A.$$

Show that this is a well-defined total ordering (a transitive, antisymmetric and total relation) on Γ which is compatible with the group structure, that is, $\alpha \geq \beta$ implies that $\alpha\gamma \geq \beta\gamma$ for all $\gamma \in \Gamma$. The totally ordered abelian group Γ is called the *value group* of A.

Let $v\colon K^* \longrightarrow \Gamma$ be the canonical homomorphism. Prove that $v(x+y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K^*, x \neq -y$.

References

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