EXERCISES, COMMUTATIVE ALGEBRA, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2014/2015

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Sheet 1

Exercise 1. [1, Ex. I.2 b)+a)]

b) Prove that $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ is nilpotent if and only if all the a_i are.

a) Show that $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ is a unit if and only if a_0 is a unit in A and a_i is nilpotent for $i \geq 1$.

Exercise 2. [1, Ex. 1.4] Show that in $A[X]$ the Jacobson radical and the nilradical are equal.

Exercise 3. [1, Ex. 1.10] Show that the following conditions are equivalent: i) a ring A has only one prime ideal, ii) every element in A is either nilpotent or a unit, iii) $A/\mathrm{rad}A$ is a field.

Exercise 4. (cf. [3, Ex. 1.1.3]) Let $f\colon A \twoheadrightarrow B$ be a ring homomorphism. Show that $f(\mathrm{rad}A) \subseteq \mathrm{rad}B$. Give an example of a *surjective* f such that the inclusion is strict.

References

[1] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading Mass.-London-Don Mills, 1969.

[2] P. L. Clark, *Commutative algebra*, http://www.math.uga.edu/ pete/integral.pdf.

[3] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1989.