SEMINAR WS 2010/11: DERIVED CATEGORIES

Time/Place: Fr, 12:15–13:45, Geom 430

Description: The topic of this seminar is derived categories in algebraic geometry. The central text is [FM]. Caldararu's survey [Ca] can be recommended as a summarizing source. It contains many insightful remarks and discusses also some topics that will not be covered in the seminar. A cheerful, short introduction into derived categories is [Th].

Program

(1) Schemes [AG, II.2 (II.3,II.4)] (22.10. Bernd Siebert)
(2) Coherent sheaves [AG, II.5,II.6] (29.10. Matthias Bargmann)
(3) Ampleness [AG, II.7] (5.11. Jan Christian Rohde)
(4) Triangulated categories I [FM, §1.1,1.2] (12.11. Hung-Ming Tsoi)
(5) Triangulated categories II [FM, §1.3,1.4] (26.11. Max Pumperla)
(6) Derived categories [FM, §2.1] (3.12. Henrik Bachmann)
(7) Derived functors [FM, §2.2] (10.12. Carsten Liese)
(8) Spectral sequences [FM, §2.3] (17.12. Mara Sommerfeld)
(9) Derived categories of coherent sheaves [FM, §3.1,3.2] (7.1. Ana Ros Camacho)
(10) Derived functors in algebraic geometry [FM, §3.3] (14.1. Benjamin Wieneck)
(11) The theorem of Bondal and Orlov [FM, §4.1,4.2] (21.1. Chen Xin)
(12) Fourier-Mukai transforms [FM, §5.1] (28.1. Sjuvon Chung)

References

[AG] R. Hartshorne: *Algebraic geometry*, Springer 1977.

[Ca] A. Căldăraru: *Derived categories: A skimming*, in: Snowbird lectures in algebraic geometry, 43–75, Contemp. Math. **388**, Amer. Math. Soc. 2005, `http://arxiv.org/pdf/math/0501094v1`.

[FM] D. Huybrechts: *Fourier-Mukai transforms in algebraic geometry*, Clarendon Press, Oxford 2006.

[Th] R. Thomas: *Derived categories for the working mathematician*, in: Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), 349–361, AMS/IP Stud. Adv. Math. **23**, Amer. Math. Soc. 2001, `http://arxiv.org/pdf/math/0001045v2`.

Description of talks

(1) **Schemes** [AG, II.2–II.4]
The first three talks shall provide the most essential prerequisites from algebraic geometry that go beyond Chapter 1 of [AG]. We don't need to get too fancy and can restrict to the category of schemes of finite type over an algebraically closed field of characteristic 0, whenever this simplifies the treatment. You should motivate the introduction of non-closed points and then explain how to work with ringed spaces and the most basic notions concerning schemes (affine and projective schemes, finite type, affine and finite morphisms, reduced and integral schemes, quasi-compactness, separatedness and properness, open and closed subschemes, fibred products). The emphasis should be on getting the picture across via examples, rather than proofs.

(2) **Coherent sheaves** [AG, II.5,II.6]
Coherent sheaves are the central object of this seminar. You can follow the text [AG] here rather closely, but omit the most technical proofs such as Lemma 5.3. This is also the place to remind the participants of some basic constructions with sheaves (push-forward, pull-back, direct sum, tensor product etc.). Emphasize Corollary 5.5. The homogeneous version of the tilde construction is not so esssential. Rather introduce the basic line bundle $\mathcal{O}_X(1)$ of a projective scheme $X \subset \mathbb{P}^n$ as restriction of $\mathcal{O}_{\mathbb{P}^n}(H)$ ($= \mathscr{L}(H)$ in the notation of [AG]) for a hyperplane $H \subset \mathbb{P}^n$, once you have discussed the basic correspondence between (Cartier) divisors and invertible sheaves. To save time you might have to restrict to schemes that are regular in codimension one and view a Cartier divisor as a locally principal Weil divisor.

(3) **Ampleness** [AG, II.7] Discuss the basic correspondence between subspaces of sections of an invertible sheaf and morphisms to projective space. Define ample invertible sheaves. Introduce the relative Proj-construction and general blowups.

(4),(5) **Triangulated categories I and II** [FM, §1.1-1.4] This talk is almost purely categorical. The one exception is the discussion of Serre functors, where you should include a digression on Serre duality of a nonsingular projective variety. Rather than going through the proofs line by line try to find illustrating examples.

Take the homotopy category of an abelian category as a running example.

(6) **Derived categories** [FM, §2.1] You should follow the text closely here.

(7) **Derived functors** [FM, §2.2] Again you should follow the text closely, but add background material from the more traditional presentation in [AG, Ch.III, §§1,2,6,8] whenever appropriate.

(8) **Spectral sequences** [FM, §2.3] This section is about the Grothendieck spectral sequence computing the composition of derived functors. Explain the basic principle behind spectral sequences, for example following the discussion in Serge Lang's book on Algebra. Feel free to illustrate spectral sequences by other examples, e.g. from topology.

(9) **Derived categories of coherent sheaves** [FM, §3.1,3.2]. We can follow the text closely here. You might have to help the next speaker by providing background material.

(10) **Derived functors in algebraic geometry** [FM, §3.3] This talk presents the standard derived functors in algebraic geometry and their relationships. The presentation in [FM] seems quite appropriate, except we should try to add some explicit examples.

(11) **The theorem of Bondal and Orlov** [FM, §4.1,4.2] This theorem from 2001 says that a smooth projective variety with ample canonical or ample anticanonical bundle is characterized uniquely by its bounded derived category. We can follow the text closely here.

(12) **Fourier-Mukai transforms** [FM, §5.1] Given two projective varieties X and Y, an object of $D^b(X \times Y)$ induces an exact functor $F : D^b(X) \to D^b(Y)$, called a Fourier-Mukai transform. A deep theorem of Orlov says that if X, Y are smooth then any exact and fully faithful functor F is obtained in this fashion. While the proof is beyond the scope of this seminar you should peek ahead (and consult [Ca]) and indicate some results from the later parts of the book illustrating the use of the theorem for relating $D^b(X)$ to the geometry of X.