Complex Geometry
WS 16/17

Prof. Dr. Bernd Siebert

Exercises 4

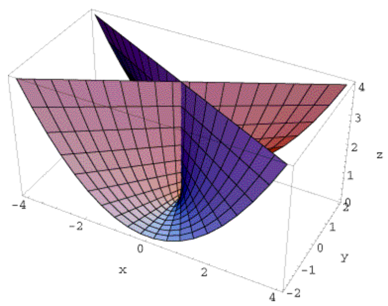
1. For $V \subset \mathbb{C}^n$ open, the common zero locus of a subset $S \subset \mathcal{O}(U)$,

$$Z(S) := \big\{ z \in V \,\big|\, \forall f \in S : f(z) = 0 \big\} \;=\; \bigcap_{f \in S} f^{-1}(0)$$

is called an *analytic subset of* V.

Show: If $f : U \to V$ is a holomorphic map between open subsets $U \subset \mathbb{C}^m$, $V \subset \mathbb{C}^n$ and $X \subset V$ is an analytic subset, then $f^{-1}(X)$ is an analytic subset of V.

2. For $X := Z(zu^2 - v^2) \subset \mathbb{C}^3$ determine the largest set of points $x \in X$ where X is a submanifold. Determine also the subset of points where X is *locally reducible*, that is, the union of two local analytic subsets.



3. Let $f_1, \ldots, f_r, g_1, \ldots, g_s \in \mathcal{O}_{\mathbb{C}^n,0}$ and $(f_1, \ldots, f_r) = (g_1, \ldots, g_s)$. Show:

$$Z(f_1) \cap \ldots \cap Z(f_r) = Z(g_1) \cap \ldots \cap Z(g_s).$$

(This implies that an ideal $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ has a well-defined associated germ of an analytic set $Z(I) \subset (\mathbb{C}^n, 0)$.)

4. Let R be a ring and $S \subset R$ a subring. An element $x \in R$ is called *integral* over S if there exists $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in S$ with

$$x^k + a_1 x^{k-1} + \ldots + a_k = 0,$$

that is, if there exists a normed polynomial $F \in S[T]$ with $F(x) = 0$.

Show: $x \in \mathbb{Q}$ is integral over $\mathbb{Z} \iff x \in \mathbb{Z}$.