Complex Geometry
WS 16/17

Prof. Dr. Bernd Siebert

Exercises 11

1. Let X be a complex manifold and $Y \subset X$ a submanifold of codimension k with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Any $f \in \mathcal{I}(U)$, $U \subset X$ open, defines a morphism

$$N_{Y/X}|_U \longmapsto \underline{\mathbb{C}}, \qquad \sum_i a_i \partial_{z_i} \longmapsto \sum_i a_i \partial_{z_i} f$$

from the restriction to U of the normal bundle of Y in X into the trivial bundle. Show that this map is indeed well-defined and that it induces an isomorphism

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}(N^*_{Y/X})$$

of sheaves of $\mathcal{O}_Y$-modules.

2. Let V be a real vector space and $I \in \mathrm{End}(V)$ a complex structure on V (that is, $I^2 = -\mathrm{id}$. Check in detail the claims from the class on the equivalence of (a) Hermitian products on the complex vector space (V, I) (b) I-invariant scalar products g on V and (c) I-invariant symplectic forms on V.

3. Let V be a real vector space of dimension $2n$, I a complex structure on V and g an I-invariant scalar product. Show:

(1) There are $u_1, \ldots, u_n \in V$ such that $u_1, v_1 = I(u_1), u_2, v_2 = I(u_2), \ldots, u_n, v_n = I(u_n)$ are an orthonormal basis of V.
(2) Denote by $\Lambda : \bigwedge^2 V \to \mathbb{C}$ the dual Lefschetz operator in degree 2 (see §8.3 in class). Then for any $\alpha \in \bigwedge^2 V^*$ it holds $\Lambda\alpha = \sum_{i=1}^n \alpha(u_i, v_i)$.