Weierstraß polynomials and plane pseudo-holomorphic curves

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Abstract

For an almost complex structure J on $U \subset \mathbb{R}^4$ pseudo-holomorphically fibered over $\mathbb{C}$ a J-holomorphic curve $C \subset U$ can be described by a Weierstrass polynomial. The J-holomorphicity equation descends to a perturbed $\bar{\partial}$-operator on the coefficients; the operator is typically $(0, 2/m)$-Hölder continuous if m is the local degree of C over $\mathbb{C}$. This sheds some light on the problem of parametrizing pseudo-holomorphic deformations of J-holomorphic curve singularities.

Introduction

Many of the elementary properties of plane holomorphic curves have been established also for pseudo-holomorphic curves. These include isolatedness of critical points and of points of intersection, positivity of intersection indices, removable singularities, existence of (singular) limits under a volume bound, cf. [AuLa] and references therein. Maybe even more strikingly, singularities of plane pseudo-holomorphic curves topologically look quite the same as holomorphic curve singularities. In fact, there is a local C^1-diffeomorphism of the ambient space mapping the pseudo-holomorphic curve singularity to a holomorphic one ([MiWh], Theorem 6.2).

Surprisingly the situation is unclear when it comes to *deformations* of plane pseudo-holomorphic curve singularities. In the holomorphic world there is the notion of *semi-universal deformation*. It consists of the germ of a holomorphic deformation over some parameter space $(S, 0)$, its *base*. Its characterizing property is that up to isomorphism any deformation of $(C, 0)$, with parameter space $(T, 0)$ say, is obtained by pull-back via a holomorphic "classifying" map $(T, 0) \to (S, 0)$. The classifying map is unique only up to an isomorphism fixing the map on the tangent spaces $T_{T,0} \to T_{S,0}$. Explicitely, let $(C, 0)$ be the germ of a plane holomorphic curve given by $F \in \mathcal{O}_{\mathbb{C}^2,0} \simeq \mathbb{C}\{z, w\}$. Then $(S, 0) = (\mathbb{C}^\tau, 0)$ is a *smooth* space of dimension equal to the Tyurina number

$$\tau = \dim_{\mathbb{C}} \mathbb{C}\{z, w\}/(F, \partial_z F, \partial_w F)\,.$$

The interest in a similar result for pseudo-holomorphic curves for us comes from a possible analytic treatment of the *isotopy problem* for symplectic submanifolds of $\mathbb{CP}^2$ or the two S^2-bundles over S^2 [SiTi1]. There are some indications that on these spaces symplectic submanifolds are isotopic iff they are homologous. One crucial obstacle in proving this statement by the technique of J-holomorphic curves is the lack of understanding that we have for deformations of singular J-holomorphic curves. One question we should answer and which is related to the holomorphic deformation theory discussed above runs as follows.

Question I. *Let J be an almost complex structure on the unit ball $B \subset \mathbb{C}^2$ and $C \subset B$ a J-holomorphic curve with $0 \in C$. Does there exist an open neighbourhood $U \subset B$ of 0 and an open subset $\mathcal{M}$ in a Banach space parameterizing J-holomorphic curves in $\operatorname{cl} U$ that are sufficiently close to $C \cap \operatorname{cl} U$ in the Hausdorff topology?*

This is of course true holomorphically. For example, taking appropriate linear coordinates z, w on $\mathbb{C}^2$ the defining equation of C can be taken in Weierstraß form

$$F(z, w) = w^d - a_1(z)w^{d-1} + \ldots + (-1)^d a_d(z)$$

for (z, w) in a polycylinder $\Delta \times \Delta$ contained in B. Here d is the intersection multiplicity of the line $z = 0$ with C, and a_i are holomorphic functions on $\operatorname{cl} \Delta$. Obviously, deformations of C are in one-to-one correspondence with deformations of the coefficients a_i. Introducing an appropriate Banach space completion of $\mathcal{O}(\operatorname{cl} \Delta)$ answers the holomorphic analogue of Question I affirmatively.

A related question that is both relevant to the isotopy problem and interesting in its own right is the *local isotopy problem* for plane pseudo-holomorphic curves. Let $U \subset \mathbb{C}^2$ be an open set with piecewise smooth boundary. We call two submanifolds with boundary $(\Sigma, \partial\Sigma)$, $(\Sigma', \partial\Sigma')$ in $(\operatorname{cl} U, \partial U)$ *isotopic* if there is a continuous family of submanifolds $(\Sigma_t, \partial\Sigma_t) \subset (\operatorname{cl} U, \partial U)$, $t \in [0, 1]$, connecting Σ and Σ' ($\Sigma = \Sigma_0$, $\Sigma' = \Sigma_1$). Note that $\partial\Sigma_t$ is then a tame isotopy of the links $\partial\Sigma$, $\partial\Sigma' \subset \partial U$. In case Σ, Σ' are symplectic (or pseudo-holomorphic, J-holomorphic respectively) then the isotopy will be called *symplectic (pseudo-holomorphic, J-holomorphic)* if Σ_t can be chosen symplectic (pseudo-holomorphic, J-holomorphic) for all t. Here "pseudo-holomorphic" means J-holomorphic for *some* J.

Question II. *Let $C \subset B$ be a J-holomorphic curve with singular locus $C_{\text{sing}} = \{0\}$. If $\{\Sigma_n\}$ and $\{\Sigma'_n\}$ are two sequences of J-holomorphic curves in B with Hausdorff limit C, then are Σ_n and Σ'_n (symplectically, pseudo-holomorphically, J-holomorphically) isotopic for n sufficiently large?*

In the holomorphic category this again has a positive answer, for the set of tuples $(a_1, \ldots, a_d) \in \mathcal{M}$ parametrizing singular holomorphic curves in Weierstraß form does not disconnect $\mathcal{M}$. On a technical level this follows by a straightforward application of the Sard-Smale theorem on an appropriate space of paths in $\mathcal{M}$.

In the almost complex setting it is still possible to bring C and all small deformations of C into Weierstraß form. To do this we may assume by a real, linear change of coordinates that $J_{|0}$ is the standard complex structure on $\mathbb{C}^2$. Let (z, w) be the standard linear coordinates on $\mathbb{C}^2$. Possibly after another (now complex-) linear change of coordinates we may assume the tangent lines of smooth irreducible components of C at 0 to be disjoint from $z = 0$, and that the closed polycylinder $|z| \le 1$, $|w| \le 1$ maps to the domain of definition of J and C. In [Tb], Lemma 5.4, it is shown that possibly after shrinking the polycylinder there is a local diffeomorphism of the form

$$\Theta : (z, w) \longmapsto (z, w + \varphi(z, w))$$

such that $w \mapsto \Theta(z, w)$ is an embedded J-holomorphic disk with $\Theta(z, 0) = (z, 0)$ for every z. Moreover, $\nabla\varphi$ can be made arbitrarily small by considering a sufficiently small polycylinder, that is by rescaling z and w. Changing coordinates by Θ we may therefore assume that for every $z \in \operatorname{cl} \Delta$ the disk $\{z\} \times \Delta$ is J-holomorphic and not contained in C. The antiholomorphic tangent space may now be written

$$T^{0,1}_{\mathbb{C}^2, J} = \langle \partial_{\bar{w}}, \partial_{\bar{z}} - a\partial_z - b\partial_w \rangle \tag{1}$$

for complex valued functions a and b. The point here is of course that ∂_w is contained in a J-holomorphic disk and hence lies in the holomorphic tangent space. Now let $d > 0$ be the intersection index of the disk $z = 0$ with the disjoint J-holomorphic curve C at 0. Then possibly after rescaling w and z, for every $z \in \Delta$ there are exactly d points of intersection of $\{z\} \times \Delta$ with C, counted with multiplicities. We obtain a map from the domain of z to the d-fold symmetric product $S^d\Delta$ of w, which is an open subset of $S^d\mathbb{C} \simeq \mathbb{C}^d$. Explicitely, to a zero cycle $\sum_{i=1}^d \lambda_i$ we associate the complex polynomial

$$(w - \lambda_1) \cdot \ldots \cdot (w - \lambda_d) = w^d - a_1 w^{d-1} + \ldots + (-1)^d a_d$$

with $a_i = \sigma_i(\lambda_1, \ldots, \lambda_d)$ the i-th elementary symmetric polynomial. This yields d complex functions $a_1(z), \ldots, a_d(z)$ with

$$C = \{(z, w) \in \Delta \times \Delta \mid w^d - a_1(z)w^{d-1} + \ldots + (-1)^d a_d(z) = 0\}\,.$$

The same argument holds for J-holomorphic curves that are sufficiently close to C in the Hausdorff topology. Moreover, using local representatives as given in [MiWh] it is not hard to check that the a_i are continuous. In turn, the C^0-topology on the a_i induces the Hausdorff topology on the space of J-holomorphic curves.

The crucial point is then to characterize those tuples $a_1, \ldots, a_d$ actually corresponding to J-holomorphic curves. We will find that this can be done by a nonlinear $\bar\partial$-equation provided the projection $(z, w) \mapsto z$ is holomorphic, in the almost complex sense. This is the case iff $a \equiv 0$ in (1). In other words we require that J is given by only one complex function b instead of two:

$$T^{0,1}_{\mathbb{C}^2, J} = \langle \partial_{\bar w}, \partial_{\bar z} - b\partial_w \rangle\,. \tag{2}$$

For the precise statement we assume that b is extended to all of $\Delta \times \mathbb{C}$ with *uniformly bounded* C^1-norm. From b we will construct d complex functions $b_1, \ldots, b_d$ on $\Delta \times S^d\mathbb{C}$. Let $D \subset S^d\mathbb{C}$ be the discriminant locus. The b_r are smooth away from $\Delta \times D$ and Hölder of some exponent $0 < \alpha \le 1$ depending only on d (Lemma 3). We are now ready to state our first theorem.

Theorem I. *There is a one-to-one correspondence between the sets*

$$\Big\{ J\text{-holomorphic curves } C \subset \Delta \times \mathbb{C} \text{ that are proper of degree } d \text{ over } \Delta \Big\}$$

and

$$\Big\{ \boldsymbol{a} = (a_1, \ldots, a_d) \in W^{1,p}_{\mathrm{loc}}(\Delta; \mathbb{C}^d) \,\Big|\, \delta(\boldsymbol{a}) \not\equiv 0\,,\ \partial_{\bar z} a_r = b_r(z, \boldsymbol{a})\,,\ r = 1, \ldots, d \Big\}$$

(any finite $p > 2$). The J-holomorphic curve belonging to $(a_1, \ldots, a_d)$ is

$$\{(z, w) \in \Delta \times \mathbb{C} \mid w^d - a_1(z)w^{d-1} + \ldots + (-1)^d a_d(z) = 0\}\,.$$

Moreover, the a_r are even of class $C^{1,\alpha}(\Delta; \mathbb{C})$ for some $\alpha = \alpha(d) > 0$.

Remark 1 By dropping the requirement $\delta(\boldsymbol{a}) \not\equiv 0$, the correspondence extends to any J-holomorphic cycles. Too see this we first observe that the map from the first set to the second is still well-defined: Given a J-holomorphic cycle $C = \sum_i m_i C_i$, where

C_i is a branched cover over Δ of degree d_i, let f_i be the Weierstrass polynomial of C_i constructed in Theorem I. Put $f = \prod_i f_i^{m_i}$. Its coefficients $\boldsymbol{a}_j$ define a map to $W^{1,p}_{\mathrm{loc}}(\Delta;\mathbb{C}^d)$. This map can be also regarded as a pseudo-holomorphic section of the d-fold relative symmetric product of $\Delta\times\mathbb{C}$ over Δ. This latter symmetric product has a pseudo-holomorphic stratification according to partitions of d. The section which arises from $C = \sum_i m_i C_i$ belongs to the stratum associated to $d = \sum_i m_i d_i$. In order to prove that the extension gives rise to a one-to-one correspondence, we need to show that a pseudo-holomorphic section $\boldsymbol{a}$ stays in one stratum except at finitely many points. This is true but a bit delicate. It follows from a unique continuation theorem for pseudo-holomorphic sections of the above type, that is, if the interesection of a pseudo-holomorphic curve with the closure of a stratum has an accumulation point, then it lies in the closure of the stratum. By induction, one can reduce it to the case that the stratum is a hypersurface. If the stratum is the hypersurface corresponding to $d = 2+1+\cdots+1$, then it amounts to check that the discriminant either is identically zero or has only finitely many zeroes. In general, one can have a function which plays the role of the discriminant.

If the b_r are Lipschitz we can easily proceed to parametrize solutions $(a_1,\ldots,a_r)$ of the nonlinear PDE by a Banach space of d holomorphic functions on Δ, possibly after shrinking the domain of z. Unfortunately this is not generally true. If one stratifies the discriminant locus D according to partitions of d, then at a point of a stratum indexed by $d = d_1+\ldots+d_l$, one expects b_r to be generally not better than C^α with $\alpha = 2/\max\{d_1,\ldots,d_l\}$. The exception is if b is indeed holomorphic in w, or for $d=2$.

Theorem II. *Let an almost complex structure J on $\Delta\times\mathbb{C}$ be given of the form (2). Let $C\subset\Delta\times\mathbb{C}$ be a J-holomorphic curve mapping properly to Δ. Put $d=\deg(C\to\Delta)$. Assume that*

a) either $d\le 2$

b) or $\bar\partial_w b\equiv 0$.

Then for sufficiently small $\varepsilon>0$ the space

$$\mathcal{M}_\varepsilon \;=\; \{C'\subset B_\varepsilon(0)\times\mathbb{C}\;\; J\text{-holomorphic curve}\}$$

is a Banach manifold at $C\cap(B_\varepsilon(0)\times\mathbb{C})$. It is modelled on the Banach space

$$\mathcal{O}^{1,p}(\Delta;\mathbb{C}^d) \;:=\; W^{1,p}(\Delta;\mathbb{C}^d)\cap\mathcal{O}(\Delta;\mathbb{C}^d)$$

endowed with the $W^{1,p}$-norm, where d is the degree of the projection $C\to\Delta$.

While this is only a partial result we would like to point out that it includes singularities of arbitrarily high Milnor number. Moreover, by computing the Nijenhuis tensor one can check that the assumption $\bar\partial_w b\equiv 0$ is equivalent to integrability of the almost complex structure. As we remarked before the parametrization by a Banach manifold is no surprise in this case. However, this description is useful for the global parametrization problem for pseudo-holomorphic curves on S^2-bundles, see [SiTi2].

To describe our result in the general case, that is without the restrictive assumptions (a) or (b), we remind the reader of the decomposition $W^{1,p}(\Delta,\mathbb{C}^d) = \mathcal{O}^{1,p}(\Delta,\mathbb{C}^d)\oplus$

$L^p(\Delta, \mathbb{C}^d)$ provided by a right-inverse T, see (6) and the discussion following it below. A pair $(\boldsymbol{h}, \boldsymbol{\xi})$ on the right yields the function $\boldsymbol{h} + T\boldsymbol{\xi}$ of Sobolev class $(1,p)$. Using a Leray-Schauder fixed point theorem we obtain the following result.

Theorem III. *Let an almost complex structure J on $\Delta \times \mathbb{C}$ be given of the form (2) and let $d > 0$. For every $\boldsymbol{h} \in \mathcal{O}^{1,p}(\Delta, \mathbb{C}^d)$ there exists an $\boldsymbol{\xi} \in L^p(\Delta, \mathbb{C}^d)$ with $\boldsymbol{a} = \boldsymbol{h} + T\boldsymbol{\xi}$ corresponding to a (possibly non-reduced) J-holomorphic curve in the Weierstrass picture. In other words, the projection map*

$$\mathrm{Id} - T \circ \partial_{\bar{z}} : \; \{\boldsymbol{a} \in W^{1,p}(\Delta, \mathbb{C}^d) \,|\, \partial_{\bar{z}} a_r = b_r(z, \boldsymbol{a})\} \longrightarrow \mathcal{O}^{1,p}(\Delta, \mathbb{C}^d)$$

is surjective.

Note that there is no assumption on the smallness of b, so this last theorem is in fact a global result.

The simplicity of the describing PDE also clearly exhibits the analytical difficulty that parametrizing deformations of C poses. The Weierstraß picture provides a uniform formulation for all deformations of C with only the non-linear, zero order term being semsitive to the change of topology.

One note on conventions: Throughout the text, a J-holomorphic curve in an almost complex manifold (M, J) is always understood as a closed subset of M. Viewed as a 2-cycle we therefore assume all components to have multiplicity one.

1 Proof of Theorem I

In this section we derive the PDE and prove Theorem I. For the equation we observe first that in view of (2) the graph of a function $\lambda : \Delta \to \mathbb{C}$ is a pseudo-holomorphic curve with respect to $J = J(b)$ iff

$$\partial_{\bar{z}} \lambda \;=\; b(z, \lambda(z)) \,. \tag{3}$$

Now if $C \subset \Delta \times \mathbb{C}$ is a J-holomorphic curve that maps properly to Δ the special form of J implies that the projection $C \to \Delta$ is a finite *holomorphic* map, hence a branched covering, of covering degree d say. Away from the discrete critical set $D_C \subset \Delta$, locally C is the union of the graphs of d functions $\lambda_1, \ldots \lambda_d$. As noted in the introduction C is then given in Weierstraß form $w^d - a_1(z)w^{d-1} + \ldots + (-1)^d a_d(z) = 0$ with

$$a_r = \sigma_r(\lambda_1, \ldots, \lambda_d) \,.$$

Taking $\partial_{\bar{z}}$ of a_r yields

$$\partial_{\bar{z}} a_r \;=\; \partial_{\bar{z}} \sum_{\{i_1, \ldots, i_r\} \subset \{1, \ldots, d\}} \lambda_{i_1} \ldots \lambda_{i_r} \;=\; \sum_{\nu=1}^{d} \sigma_{r-1}(\lambda_1, \ldots, \hat{\lambda}_\nu, \ldots, \lambda_d) \cdot b(z, \lambda_\nu) \,,$$

where the entry with a hat is to be omitted. The right-hand side of this equation is a function that is invariant under the action of the symmetric group on the branches, and hence can be expressed as function $b_r(z; a_1, \ldots, a_d)$ in z and the a_r. Conversely, the union of the graphs of $\lambda_1, \ldots, \lambda_d$ with $\lambda_i(z) \neq \lambda_j(z)$ for $i \neq j$ and all z is J-holomorphic iff the functions $a_r := \sigma_r(\lambda_1, \ldots, \lambda_d)$ fulfill

$$\partial_{\bar{z}} a_r \;=\; b_r(z; a_1, \ldots, a_d) \,, \quad r = 1, \ldots, d \,. \tag{4}$$

This follows by applying the linear map

$$A: \ (v_r)_{r=1,\ldots,d} \ \longmapsto \ \Big(\sum_{\nu=1}^{d} \sigma_{r-1}(\lambda_1,\ldots,\hat{\lambda}_\nu,\ldots,\lambda_d)\cdot v_\nu\Big)_{r=1,\ldots,d}$$

to $v_r = \partial_{\bar z}\lambda_r - b(z,\lambda_r)$ and noting the following elementary fact.

Lemma 2 *If $\lambda_i \neq \lambda_j$ for all $i \neq j$ then A is invertible.*

Proof. An explicit inverse can be seen by writing

$$\begin{aligned}
&(\lambda_r-\lambda_1)\ldots(\widehat{\lambda_r-\lambda_r})\ldots(\lambda_r-\lambda_d)\cdot v_r\\
&\quad= \Big(\sum_\nu (w-\lambda_1)\ldots(\widehat{w-\lambda_\nu})\ldots(w-\lambda_d)v_\nu\Big)\Big|_{w=\lambda_r}\\
&\quad= \sum_{\nu,\mu}(-1)^\mu\sigma_\mu(\lambda_1,\ldots,\hat{\lambda}_\nu,\ldots,\lambda_d)\cdot v_\nu\cdot\lambda_r^{d-1-\mu} \ = \ \sum_\mu(-1)^\mu\lambda_r^{d-1-\mu}\,(A\cdot v)_\mu\,.
\end{aligned}$$

◇

To extend over the critical set $D_C \subset \Delta$ we interpret b_r as functions on $\Delta\times S^d\mathbb{C}\simeq\Delta\times\mathbb{C}^d$. Recall that the isomorphism $S^d\mathbb{C}\simeq\mathbb{C}^d$ is given by the elementary symmetric functions $\sigma_1,\ldots,\sigma_d$, which in fact provide global holomorphic coordinates on $S^d\mathbb{C}$. We henceforth endow $S^d\mathbb{C}$ with the differentiable structure thus inherited.

Lemma 3 *The functions on $\Delta\times S^d\mathbb{C}$ induced by*

$$b_r: \ \Delta\times\mathbb{C}^d \ \longrightarrow \ \mathbb{C}\,, \quad (z,\lambda_1,..,\lambda_d) \ \longmapsto \ \sum_\nu \sigma_r(\lambda_1,\ldots,\hat{\lambda}_\nu,\ldots,\lambda_d)\cdot b(z,\lambda_\nu)$$

are of Hölder class C^α for some $\alpha=\alpha(d)>0$. They are smooth away from Δ times the discriminant locus $D\subset S^d\mathbb{C}$. Moreover, if b depends holomorphically on w, or if $d=2$, then the b_r are Lipschitz. The Lipschitz constant tends to zero with $\|\nabla b\|_\infty$.

Proof. It is clear by the definition of the topology on $S^d\mathbb{C}=\mathbb{C}^d/S_d$ that b_r and all partial derivatives in z are continuous. For the Hölder property we consider $\sum_{\nu=1}^d \sigma_{r-1}(\lambda_1,\ldots,\hat{\lambda}_\nu,\ldots,\lambda_d)\cdot b(z,\lambda_\nu)$ as smooth function in $(z;\lambda_1,\ldots,\lambda_d)\in\Delta\times\mathbb{C}^d$. Now b_r equals $1/d!$ times the trace of this function under the branched cover $\Delta\times\mathbb{C}^d\to\Delta\times S^d\mathbb{C}$. Functions of this type have been studied by Barlet. Among other things he proved that after blowing up the base $\Delta\times S^d\mathbb{C}$ to make the branch divisor simple normal crossing, traces of smooth functions are locally of class $C^{\tilde\alpha}$ with $\tilde\alpha=2/\beta$ with β the maximal ramification index, see [Ba1], Theorem 3 together with Lemma 4, p.158 and [Ba2]. Now quite generally, if f is a function on some $U\subset\mathbb{C}^d$ and the pull-back σ^*f under some blowing-up σ is Hölder, then f is also Hölder, but possibly of smaller exponent.

The smoothness statement follows since D is the branch locus of the covering $\mathbb{C}^d\to S^d\mathbb{C}$.

If b is holomorphic in w then also the b_r are holomorphic in w by the Riemann Extension Theorem. Hence in this case the b_r are even smooth. Finally, for $d=2$ the discriminant locus is smooth and the Lipschitz property follows from Barlet's results. Alternatively, one can do a simple explicit computation. ◇

We are now ready to prove Theorem I.

Proof of Theorem I. Given a tuple of functions $\boldsymbol{a} = (a_1, \ldots, a_d)$ with $\partial_{\bar z} a_r = b_r(\boldsymbol{a})$ and discriminant $\delta(\boldsymbol{a}) \not\equiv 0$ let $C \subset \Delta \times \mathbb{C}$ be the associated curve. We first discuss regularity. The Sobolev embedding $W^{1,p}(\Delta) \subset C^{1-\frac{2}{p}}(\Delta)$ shows local boundedness of the a_r. Then under our hypothesis on b, by the preceding lemma $b_r(\boldsymbol{a})$ is Hölder of some exponent $\alpha > 0$ too. Now for any smooth function ρ on Δ with compact support we can reconstruct ρa_r from

$$\partial_{\bar z}(\rho a_r) \;=\; \partial_{\bar z}\rho \cdot a_r + \rho \cdot b_r(\boldsymbol{a}) \;\in\; C^{\alpha}(\Delta)$$

by application of the Cauchy integral operator. From the standard estimates for the latter [Ve] we obtain $\rho a_r \in C^{1,\alpha}(\Delta)$ and hence $a_r \in C^{1,\alpha}_{\mathrm{loc}}(\Delta)$.

Now let us assume we are given $\boldsymbol{a}$ fulfilling (4). Near any $P \in \Delta \setminus D_C$ there exist $\lambda_1, \ldots, \lambda_d$ with $a_r = \sigma_r(\lambda_1, \ldots, \lambda_d)$. From our discussion following Equation 4 it follows that C is indeed J-holomorphic over a neighbourhood of P. To investigate J-holomorphicity near D_C we need a lemma.

Lemma 4 *The zero set of the discrimant $\delta(\boldsymbol{a})$ is discrete.*

Proof. In a neighbourhood of some $P \in \Delta \setminus D_C$ we may write $a_r = \sigma_r(\lambda_1, \ldots, \lambda_d)$. Using the equation $\partial_{\bar z}\lambda_i = b(z, \lambda_i)$ we compute

$$\begin{aligned}
\partial_{\bar z}\delta(\boldsymbol{a}) \;&=\; \partial_{\bar z}\prod_{i<j}(\lambda_i - \lambda_j)^2 \;=\; 2\sum_{i<j}(\lambda_i - \lambda_j)(\partial_{\bar z}\lambda_i - \partial_{\bar z}\lambda_j) \prod_{\substack{k<l \\ (k,l)\neq(i,j)}} (\lambda_k - \lambda_l)^2 \\
&=\; 2\sum_{i<j}(\lambda_i - \lambda_j)(b(z,\lambda_i) - b(z,\lambda_j)) \prod_{\substack{k<l \\ (k,l)\neq(i,j)}} (\lambda_k - \lambda_l)^2 .
\end{aligned}$$

Therefore, the discriminant fulfills the linear $\bar\partial$-equation $\partial_{\bar z}\delta(\boldsymbol{a}) = f \cdot \delta(\boldsymbol{a})$ with coefficient

$$f = 2\sum_{i<j}\frac{b(z,\lambda_i) - b(z,\lambda_j)}{\lambda_i - \lambda_j}.$$

Because $\nabla_w b$ is uniformly bounded $f \in L^\infty$. A standard trick now reduces to the case of holomorphic functions [Ve]: Let $g \in W^{1,p}(\Delta)$ solve $\partial_{\bar z} g = -f$; then $e^g \cdot \delta(\boldsymbol{a})$ is holomorphic. Hence the claim. $\diamond$

For any $P \in \Delta$ we may thus choose a domain $U \subset \Delta$ with $U \cap D_C = \{P\}$. By J-holomorphicity of the projection $p : \Delta \times \mathbb{C} \to \Delta$, the map $p : C \cap p^{-1}(U \setminus \{P\}) \to U \setminus \{P\}$ is a holomorphic, finite, unbranched cover. The curve C thus decomposes over $U \setminus \{P\}$ into a finite disjoint union of pointed disks Δ^* such that in appropriate local holomorphic coordinates $p|_{\Delta^* \subset C} : t \mapsto z = t^m$. On this branch C is thus the image of a map of the form

$$\Delta^* \;\longrightarrow\; C\,, \quad t \;\longmapsto\; (t^m, \lambda(t))$$

for some smooth function λ. Pseudo-holomorphicity is expressed in terms of λ by

$$\partial_{\bar t}\lambda(t) \;=\; \partial_{\bar z}\lambda \cdot \partial_{\bar t}\bar z \;=\; m\bar t^{m-1} b(t^m, \lambda(t))\,. \tag{5}$$

This shows that $|\partial_{\bar t}\lambda|$ is uniformly bounded. Moreover, properness of the map $\mathbb{C}^d \to S^d\mathbb{C}$ plus continuity of the a_r show that λ has a continuous extension to Δ. It is a well-known fact that this implies $\lambda \in W^{1,2}(\Delta)$, cf. e.g. [Sk, Lemma 2.4.2]. By elliptic bootstrapping λ is smooth and hence $t \mapsto (t^k, \lambda(t))$ is indeed J-holomorphic.

Conversely, starting from a J-holomorphic curve C we checked at the beginning of this section that on $\Delta \setminus D_C$ there are uniquely defined smooth functions a_r fulfilling equation (4). To extend over the branch points let $\varphi : \Sigma \to \Delta \times \mathbb{C}$ be the J-holomorphic map with image C. So Σ is a union of Riemann surfaces with boundary, and the composition of φ with the projection $\Delta \times \mathbb{C} \to \Delta$ exhibits Σ as a branched cover of the unit disk. Properness of the projection $C \to \Delta$ implies boundedness of the a_r. In particular, $\partial_{\bar{z}} a_r = b_r(z, \boldsymbol{a})$ is bounded in L^p. Now elliptic regularity as above shows $a_r \in W^{1,p}_{\mathrm{loc}}(\Delta)$ for every $2 < p < \infty$. $\diamond$

Remark 5 While this was not a stimulus for this paper we would like to point out that the use of symmetric polynomials in the study of pseudo-holomorphic curves is not entirely new. It has been used by Taubes in a static picture (for just one curve) to investigate the singularities of an almost everywhere pseudo-holomorphic current ([Tb], proof of Lemma 6.13).

2 Proof of Theorems II and III

In this section we give sufficient conditions under which the solution space to Equation 4 is a Banach manifold. As this PDE becomes singular near multiple points we certainly want to restrict to tuples $\boldsymbol{a}$ with discriminant $\delta(\boldsymbol{a})$ not vanishing identically. Note that rescaling $z \to \varepsilon^{-1} z$ leads merely to a change $b \to \varepsilon b$ in the describing equation (3). Since we are only interested in the local behaviour we may thus work over the unit disk and assume that the C^1-norm of b is as small as we want. Notice also that rescaling w does not have any effect in that regard. The cases we can treat are singularities of multiplicity 2 ($d = 2$) and the more artificial case that b is holomorphic fiberwise (in the w-direction).

Proof of Theorem II. We view (4) as nonlinear map

$$\Phi : \mathcal{B} \longrightarrow \mathcal{E}, \quad \boldsymbol{a} = (a_1, \ldots, a_d) \longmapsto (\partial_{\bar{z}} a_r - b_r(z, \boldsymbol{a}))_{r=1,\ldots,d}$$

between function spaces

$$\mathcal{B} := \{\boldsymbol{a} \in W^{1,p}(\Delta, \mathbb{C}^d) \,|\, \delta(\boldsymbol{a}) \not\equiv 0\}, \quad \mathcal{E} := L^p(\Delta, \mathbb{C}^d)$$

where we choose some $2 < p < \infty$. Let $\boldsymbol{a}$ fulfill $\Phi(\boldsymbol{a}) = 0$, $\delta(\boldsymbol{a}) \not\equiv 0$. We want to apply a fixed point method to find a bijection between small holomorphic perturbations of $\boldsymbol{a}$ and solutions of $\Phi = 0$. To this end we need an approximate right-inverse to the linearization of Φ. The latter can easily been checked to exist and to be a zero order perturbation of the $\partial_{\bar{z}}$-operator. We thus simply take the right-inverse T to $\partial_{\bar{z}} : W^{1,p}(\Delta; \mathbb{C})^d \to L^p(\Delta; \mathbb{C})^d$ provided by the Cauchy integral operator for our approximate right inverse:

$$T : L^p(\Delta; \mathbb{C})^d \longrightarrow W^{1,p}(\Delta; \mathbb{C})^d, \quad (T\boldsymbol{\xi})(z) = \Big(\frac{1}{2\pi i} \int_\Delta \frac{\xi_r(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}\Big)_r. \qquad (6)$$

From T we obtain the decomposition $W^{1,p}(\Delta)^d = \mathcal{O}^{1,p}(\Delta)^d \oplus L^p(\Delta)^d$ with $\mathcal{O}^{1,p}(\Delta)$ the space of holomorphic functions on the unit disk of Sobolev class $(1, p)$. The correspondence is

$$\mathcal{O}^{1,p}(\Delta)^d \oplus L^p(\Delta)^d \ni (\boldsymbol{h}, \boldsymbol{\xi}) \mapsto \boldsymbol{h} + T\boldsymbol{\xi} \in W^{1,p}(\Delta)^d,$$

with inverse $f \mapsto (f - T\partial_{\bar{z}} f, \partial_{\bar{z}} f)$. We therefore want to find solutions of Equation 4 of the form

$$\Phi(\boldsymbol{a} + \boldsymbol{h} + T\boldsymbol{\xi}(\boldsymbol{h})) \;=\; 0\,.$$

For any $\boldsymbol{h}$ we claim that the map

$$K_{\boldsymbol{h}}(\boldsymbol{\xi}) \;=\; \boldsymbol{\xi} - \Phi(\boldsymbol{a} + \boldsymbol{h} + T\boldsymbol{\xi})$$

is contractive. Expanding and using $\Phi(\boldsymbol{a}) = 0$ we obtain

$$K_{\boldsymbol{h}}(\boldsymbol{\xi}) \;=\; b(z, \boldsymbol{a} + \boldsymbol{h} + T\boldsymbol{\xi}) - b(z, \boldsymbol{a})\,.$$

We therefore have to estimate

$$\|K_{\boldsymbol{h}}(\boldsymbol{\xi}) - K_{\boldsymbol{h}}(\boldsymbol{\zeta})\|_p \;=\; \|\boldsymbol{b}(z, \boldsymbol{a} + \boldsymbol{h} + T\boldsymbol{\xi}) - \boldsymbol{b}(z, \boldsymbol{a} + \boldsymbol{h} + T\boldsymbol{\zeta})\|_p$$

by the L^p-distance of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. Under our hypothesis $\boldsymbol{b}$ is Lipschitz, with arbitrarily small Lipschitz constant $q = q(\|\nabla b\|_\infty)$ (Lemma 3). We obtain

$$\|K_{\boldsymbol{h}}(\boldsymbol{\xi}) - K_{\boldsymbol{h}}(\boldsymbol{\zeta})\|_p \;\leq\; a\|T\| \cdot \|\boldsymbol{\xi} - \boldsymbol{\zeta}\|_p\,.$$

Assuming $q < 1/\|T\|^{-1}$ we thus see that K is indeed contractive. Therefore, for any $\boldsymbol{h}$ the equation $K_{\boldsymbol{h}}(\boldsymbol{\xi}) = \boldsymbol{\xi}$ has a unique solution $\boldsymbol{\xi}(\boldsymbol{h})$. Moreover, the norm of $\boldsymbol{\xi}$ tends to zero with $\boldsymbol{h}$. $\diamond$

Proof of Theorem III. Without bounds on the linearization of the equation we need to use stronger functional analytic methods. For our existence problem the following version of the Leray-Schauder fixed point theorem is custom made.

Theorem 6 *[Tl, Thm.14.B.5] Let $\mathcal{B}$ be a Banach space and let $F : [0,1] \times \mathcal{B} \to \mathcal{B}$ be a continuous and compact map. Putting $F_\sigma(\boldsymbol{\xi}) = F(\sigma, \boldsymbol{\xi})$ for $\sigma \in [0,1]$, we assume that $F_0 \equiv \boldsymbol{b}_0$ for some $\boldsymbol{b}_0 \in \mathcal{B}$, and that the fixed points of F_σ are uniformly bounded for all $\sigma \in [0,1]$:*

$$F_\sigma(\boldsymbol{\xi}) = \boldsymbol{\xi} \;\Rightarrow\; \|\boldsymbol{\xi}\| < M\,.$$

Then F_1 has a fixed point.

With the notations of the proof of Theorem II above, we apply this theorem with $\mathcal{B} = L^p(\Delta)^d$ to

$$F_\sigma(\boldsymbol{\xi}) := \sigma \boldsymbol{b}(z, \boldsymbol{h} + T\boldsymbol{\xi})\,.$$

A point $\boldsymbol{\xi} \in \mathcal{B}$ is a fixed point of F_1 iff $\boldsymbol{\xi} = \partial_{\bar{z}}(\boldsymbol{h} + T\boldsymbol{\xi})$ equals $\boldsymbol{b}(z, \boldsymbol{h} + T\boldsymbol{\xi})$, so these are in one-to-one correspondence with solutions of (3) of the form $\boldsymbol{h} + T\boldsymbol{\xi}$. Moreover, $F_0 \equiv 0$, and F is a compact map as composition of the compact operator $\mathrm{Id} \times T : [0,1] \times L^p(\Delta)^d \to [0,1] \times C^0(\Delta)^d$ with the continuous map

$$[0,1] \times C^0(\Delta)^d \longrightarrow L^p(\Delta)^d\,, \quad (\sigma, \boldsymbol{v}) \longmapsto \sigma \boldsymbol{b}(z, \boldsymbol{h} + \boldsymbol{v})\,.$$

Finally, the uniform estimate for fixed points $\boldsymbol{\xi} = F_\sigma(\boldsymbol{\xi})$:

$$\|\boldsymbol{\xi}\|_p \;=\; \|\sigma \boldsymbol{b}(z, \boldsymbol{h} + T\boldsymbol{\xi})\|_p \;\leq\; \pi^{1/p}\|\boldsymbol{b}\|_\infty \;=:\; M\,.$$

$\diamond$

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