Counterexample to the equivariance of versal deformations

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Moduli theory poses the following question of equivariance to deformation theory: Let X_0 be an algebraic variety, say complete over a field k, and let $\mathfrak{X} \to S$ be a (formally) semiuniversal deformation of X_0. Then does the action of $\mathrm{Aut}\,(X_0)$ extend to an action on $\mathfrak{X} \to S$? Obviously, by versality the action of any single $g \in \mathrm{Aut}\,(X_0)$ extends to $\mathfrak{X}$; but as the extension need not be unique the problem is to select appropriate extensions to make it an action. By this argument the answer is of course positive in case $\mathfrak{X} \to S$ is in fact universal.

Otherwise the only general result known to the author is by D.S. Rim saying that the vanishing of group cohomologies $H^1(G, \_) = H^2(G, \_) = 0$ with entries certain G-modules depending on the deformation, is sufficient for a G-action on X_0 to extend [Ri]. The condition is fulfilled for reductive groups G.

On the other hand, no example seems to be known of an action that would not extend. The purpose of this note is to provide such an example. For X_0 we take the reducible plane quadric $C = V(xy) \subset \mathbb{P}^2_k$ over a field k of characteristic $\neq 2$, and G will be the component of the identity in $\mathrm{Aut}\,(C)$.

Recall that the semiuniversal deformation of C can be constructed as follows: Consider the projection $p : \mathbb{P}^1_{\mathbb{A}^1_k} \to \mathbb{A}^1_k$ and let $X \to \mathbb{P}^1_{\mathbb{A}^1_k}$ be the blowing up in a k-rational point in the fiber over $0 \in \mathbb{A}^1_k$. Then the restriction $\mathfrak{X} \to S := \mathrm{Spec}\, k[\![t]\!]$ of the composition $X \to \mathbb{P}^1_{\mathbb{A}^1_k} \to \mathbb{A}^1_k$ is the formally semiuniversal deformation of C:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\
\mathrm{Spec}\, k \arrow[r] & S = \mathrm{Spec}\, k[\![t]\!]\,.
\end{tikzcd}$$

The automorphism group of C is a split extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathrm{Aut}\,(\mathbb{A}^1_k)^2$. It can be explicitely realized as stabilizer of C under the action of $PGL(2,k)$ on $\mathbb{P}^2_k$. The finite group acts by swapping the two irreducible components of C, while each copy of $\mathrm{Aut}\,(\mathbb{A}^1_k)$ acts on one component linearly and fixing the other.

Theorem 1 *The action of $G = \mathrm{Aut}\,(\mathbb{A}^1_k)^2$ on C does not extend to the semiuniversal deformation $\mathfrak{X}$.*

The *proof* is by contradiction. Assume that the G-action does extend to $\mathfrak{X}$. Recall that the Lie algebra of $\mathrm{Aut}\,(\mathbb{A}^1_k)$ can be generated by two elements A, B with single non-trivial relation $[A, B] = B$. They can be taken in such a way that the action on the two branches of C map A, B to the vector fields X_0, Y_0 and Z_0, W_0 on C respectively with

$$X_0 = x\partial_x\,, \quad Y_0 = x^2\partial_x\,, \quad Y_0 = y\partial_y\,, \quad Z_0 = y^2\partial_y\,,$$

in affine coordinates x, y on $\mathbb{P}^2$. The extension of the action then provides 4 vector fields X, Y, Z, W on $\mathfrak{X}$ with commutators

$$[X,Y] = Y\,, \quad [Z,W] = W\,, \quad [X,Z] = [X,W] = [Y,Z] = [Y,W] = 0$$

and restricting to X_0, Y_0, Z_0, W_0 on C.

The computations simplify considerably on the blown down surface $P = \mathbb{P}^1_S$ instead of on $\mathfrak{X}$. By normality of P vector fields on $\mathfrak{X}$ push forward to vector fields on P by extension across the blowing up point. Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ be the restriction of the vector fields thus obtained to the first infinitesimal neighbourhood $\bar{P} = \mathbb{P}^1_{S^{[1]}}$, $S^{[1]} = \operatorname{Spec} k[t]/(t^2)$, of the central fiber P_0 of $P \to S$. Taking the blowing down to contract the line $x = 0$, in appropriate affine coordinates t, u on P the blowing down is given by

$$t = xy \quad u = x\,.$$

The restrictions of $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ to P_0 are then

$$\bar{X}_0 = u\partial_u \quad \bar{Y}_0 = u^2\partial_u \quad \bar{Z}_0 = 0 \quad \bar{W}_0 = 0\,.$$

Lemma 2 *Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ be nonzero vector fields on $\bar{P} = \mathbb{P}^1_{S^{[1]}}$ restricting to*

$$\bar{X}_0 = u\partial_u \quad \bar{Y}_0 = u^2\partial_u \quad \bar{Z}_0 = 0 \quad \bar{W}_0 = 0$$

in a linear coordinate u on $P_0 = \mathbb{P}^1_k$ and obeying the commutation relations

$$[\bar{X},\bar{Y}] = \bar{Y}\,, \quad [\bar{Z},\bar{W}] = \bar{W}\,, \quad [\bar{X},\bar{Z}] = [\bar{X},\bar{W}] = [\bar{Y},\bar{Z}] = [\bar{Y},\bar{W}] = 0\,.$$

Then

$$\bar{X} \;=\; (u + t\alpha)\partial_u - t\partial_t\,,$$

for some $\alpha \in k[u]$.

Proof. The computation will be done on the affine part $\operatorname{Spec} k[u,t]/t^2 \subset \bar{P}$. The tangent sheaf $\Theta_{\bar{P}}$ splits into the pull-backs of $\Theta_{\mathbb{P}^1_k} \simeq \mathcal{O}_{\mathbb{P}^1_k}(2)$ and of $\Theta_{S^{[1]}}$ under the projections $\bar{P} \to \mathbb{P}^1_k$ and $\bar{p} : \bar{P} \to S^{[1]}$. By properness of $\bar{p}$ it holds $\bar{p}_*\Theta_{S^{[1]}} \simeq \mathcal{O}_{S^{[1]}}$. We may thus write

$$\begin{aligned} \bar{X} &= (u + t\alpha)\partial_u + \kappa t\partial_t\,, & \bar{Y} &= (u^2 + t\beta)\partial_u + \lambda t\partial_t \\ \bar{Z} &= t\gamma\partial_u + \mu t\partial_t\,, & \bar{W} &= t\delta\partial_u + \nu t\partial_t \end{aligned}$$

with $\kappa, \lambda, \mu, \nu \in k$ and $\alpha, \beta, \gamma, \delta \in k[u]$ of degree ≤ 2.
From

$$[\bar{Z},\bar{W}] = t(\mu\delta - \nu\gamma)\partial_u \overset{!}{=} \bar{W}$$

we deduce first $\nu = 0$ and in turn $\mu = 1$ for $\bar{W} \neq 0$. Similarly,

$$\begin{aligned} [\bar{X},\bar{Y}] &= \Big(u^2 + t((2u - \lambda)\alpha + (\kappa - 1)\beta + u\partial_u\beta - u^2\partial_u\alpha)\Big)\partial_u \\ &\overset{!}{=} (u^2 + t\beta)\partial_u + t\lambda\partial_t \end{aligned}$$

implies $\lambda = 0$. Next,

$$[\bar{Y},\bar{W}] = t(u^2\partial_u\delta + \lambda\delta - 2u\delta)\partial_u \overset{!}{=} 0$$

shows $\delta = \delta_2 u^2$. Plugging this into

$$[\bar{X}, \bar{W}] = t(u\partial_u \delta + (\kappa - 1)\delta) \overset{!}{=} 0$$

then yields $\kappa = -1$. ◇

To finish the proof of the theorem we simply observe that a vector field on $\bar{P}$ of the form

$$\bar{X} = (u + t\alpha)\partial_u - t\partial_t$$

can not come from a vector field on $\mathfrak{X}$ restricting to $x\partial_x$ on $C \subset \mathfrak{X}$. In fact, the restriction of X to the first infinitesimal neighbourhood of C in $\mathfrak{X}$ (i.e. modulo t^2 with $t = xy$) has the form

$$(x + t(f(x) + g(y)))\partial_x + t(h(x) + k(y))\partial_y\,. \tag{1}$$

Pushing forward to $\bar{P}$ maps x to u, y to t/u and

$$\partial_x \longmapsto \frac{t}{u}\partial_t + \partial_u\,, \quad \partial_y \longmapsto u\partial_t\,.$$

Applied to (1) we obtain

$$\Big(u + t(f(u) + g(\tfrac{t}{u}))\Big)(\tfrac{t}{u}\partial_t + \partial_u) + t(h(u) + k(\tfrac{t}{u}))u\partial_t\,.$$

Modulo t^2 this reduces to

$$(u + tf(u))\partial_u + t(1 + uh(u))\partial_t\,.$$

The contradiction comes from the sign of the term $t\partial_t$ which does not match the one in Lemma 2. ◇

References

[Ri] D.S. Rim: *Equivariant G-structure on versal deformations*, Trans. Amer. Math. Soc. **257** (1980) 217–226

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