Monoidal Structures, Operads and Stable Categories II

Last time: An ∞-operad consists of an ∞-category $\mathcal{O}^{\otimes}$ and a functor $p: \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$, s.t.

(i) Every inert map $\alpha: p(\underline{X}) \to \langle m \rangle$ in Fin_* has a p-cocart. lift $\bar{\alpha}: \underline{X} \to \underline{Y}$

(ii) The lifts of ρ_i exhibit $\mathcal{O}^{\otimes}_{\langle n \rangle} \xrightarrow[(\rho_i)_!]{\simeq} (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^{\times n} =: \mathcal{O}^{\times n}$

(iii) $\mathrm{Map}^{\alpha}([X_1, \dots, X_n], [Y_1, \dots, Y_m]) \simeq \prod_{1 \le j \le m} \mathrm{Map}^{\rho_j \circ \alpha}([X_1, \dots, X_n], Y_j)$

Ex/
- Every symmetric colored operad
- "Same thing" as Kan-enriched operads
- Symm. monoidal ∞-cats, since as $\mathcal{C}^{\otimes}_{\langle n \rangle} = (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^{\times n}$ via ρ_i.

$$\mathrm{Map}^{\alpha}_{\mathcal{C}^{\otimes}}(\underline{X}, \underline{Y}) \xrightarrow{-\circ(\rho_j)_!} \prod_{j=1}^{m} \mathrm{Map}^{\rho_j \circ \alpha}(\underline{X}, Y_j)$$

$$\mathrm{Map}_{\mathcal{C}^{\otimes}_{\langle m \rangle}}(\alpha_! \underline{X}, \underline{Y}) \xrightarrow{\simeq} \prod_{j=1}^{m} \mathrm{Map}^{\rho_j}_{\mathcal{C}^{\otimes}}(\alpha_! \underline{X}, Y_j) \xleftrightarrow{(ii)} = \prod \mathrm{Map}_{\mathcal{C}}(\rho_{j!}\alpha_! \underline{X}, \rho_{j!} \underline{Y})$$

- $\mathbb{E}_\infty^{\otimes}$ commutative operad $\mathrm{Id}: \mathrm{Fin}_* \to \mathrm{Fin}_*$ with one color a and

$$\mathrm{Mul}_{\mathbb{E}_\infty}(\overbrace{a, \dots, a}^{n}; a) := \mathrm{Map}^{t}_{\mathbb{E}_\infty^{\otimes}}([a, \dots, a], [a]) = \{t\}$$

with $t: \langle n \rangle \to \langle 1 \rangle$, $1, \dots, n \mapsto 1$

- $\mathbb{E}_1^{\otimes}$ associative operad with one color a and

$$\mathrm{Mul}_{\mathbb{E}_1}(a, \dots, a; a) = \{\text{total orders on } \{1, \dots, n\}\} = \Sigma_n$$

- $\mathrm{LM}^{\otimes}$ with two colors a, ℓ and

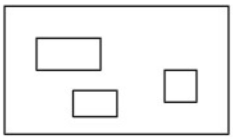
$$\mathrm{Mul}_{\mathrm{LM}}(a, \dots, a; a) = \Sigma_n = \mathrm{Mul}(a, \dots, a, \ell; \ell) \text{ and rest empty}$$

- $\mathrm{LM}^{\otimes}_{pt}$ with $\mathrm{Mul}(a, \dots, a; \ell) = \Sigma_n$ as well ~ pointing for $n = 0$.
- $\mathbb{E}_k^{\otimes}$ with a single color a, and

$$\square^k := [-1, 1]^{\times k} \subseteq \mathbb{R}^k$$

$$\mathrm{Mul}(a, \dots, a; a) = \Pi_\infty \mathrm{RectEmb}(\square^k \times \{1, \dots, n\}, \square^k)$$

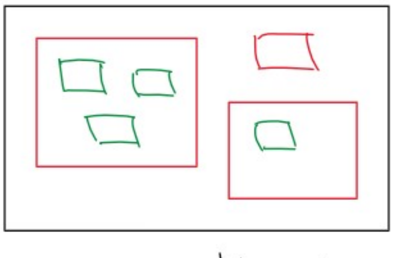
where RectEmb is the topol. space of rectilinear embeddings



$\mathrm{RectEmb}(\square^k \times \{1, \dots, n\}, \square^k)$ is a closed subset of the real VS $\mathbb{R}^{n \cdot 2k} = \mathbb{R}\langle a_\ell^{(i)}, b_\ell^{(i)} \rangle$, which we equip with the std. topology

$$(x_1, \dots, x_k) \in \square^k \times \{i\} \longmapsto (a_1^{(i)} x_1 + b_1^{(i)}, \dots, a_k^{(i)} x_k + b_k^{(i)})$$

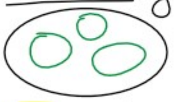
Composition via iterative embeddings:



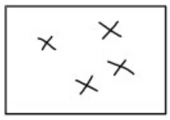
Unit given by the empty embedding

$$\emptyset \longrightarrow \square^k$$

Alternatively, $\mathrm{RectEmb}(\square^k \times \{1, \dots, n\}, \square^k) \simeq$



$$\simeq \mathrm{Emb}^{fr}(D^k \times \{1, \dots, n\}, D^k) \simeq \mathrm{Conf}_n(\mathbb{R}^k)$$



Rem:
- $\mathbb{E}_1^{\otimes} =$  is the associative operad
- $\mathbb{E}_n^{\otimes} \to \mathbb{E}_{n+1}^{\otimes}$ (map of operads ~ later) via  $\longmapsto$ 

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map of operads ⇝ later

• $\mathbb{E}_\infty^{\otimes} \simeq \operatorname{colim}_n \mathbb{E}_n^{\otimes}$

Can define $\mathbb{E}_M$ for M any topol. mfd., st. $\mathbb{E}_{\mathbb{R}^n} = \mathbb{E}_n$ ⇝ Factorization algebras!

Idea: $\mathbb{E}_n^{\otimes}$ describes the algebraic structure on the n-th homotopy group.

e.g. in $\mathbb{E}_2^{\otimes}$ one can commute , leading to braiding.

Algebras and Monoidal Structures

Ex/ $\mathbb{E}_1^{\otimes} \hookrightarrow LM^{\otimes}$

$\mathbb{E}_1^{\otimes} \hookrightarrow LM^{\otimes}_{pt}$ via a

Def A map of ∞-operads $f: \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a functor

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "f"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{P}^{\otimes} \arrow[dl, "q"] \\ & Fin_* & \end{tikzcd}$$

that sends p-coCart. lifts of inert morph. to ''.

Obtain $Alg_{\mathcal{O}}(\mathcal{P}) \subseteq Fun_{/Fin_*}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}) = Fun(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}) \times_{Fun(\mathcal{O}^{\otimes}, Fin_*)} \{p\}$

Def Given symm monoidal ∞-categories $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes} \to Fin_*$, a

→ lax monoidal functor is a map of operads $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$

→ monoidal functor is a functor

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\ & Fin_* & \end{tikzcd}$$

sending all p-coCart. morphisms to q-coCart. morph.

$\overset{\text{Groth. constr.}}{\longleftrightarrow}$

$$\begin{tikzcd} Fin_* \arrow[r, bend left, "\mathcal{C}"] \arrow[r, bend right, "\mathcal{D}"'] \arrow[r, phantom, "\Downarrow"] & Cat_\infty \end{tikzcd}$$

(Rem: Factorization algebras are something "in between".)

Def Given an ∞-operad $\mathcal{O}^{\otimes}$ and an ∞-category $\mathcal{C}$ with products, an $\mathcal{O}$-monoid in $\mathcal{C}$ is a functor $M: \mathcal{O}^{\otimes} \to \mathcal{C}$ such that $\forall\, \underline{X} = [x_1, \dots, x_n] \in \mathcal{O}^{\otimes}$, the $(\rho_i)_!$ exhibit $M(\underline{X}) \simeq \prod_{i=1}^{n} M(X_i)$

Thm $Mon_{\mathcal{O}}(\mathcal{C}) \simeq Alg_{\mathcal{O}}(\mathcal{C})$ where $\mathcal{C}$ is equipped with its Cartesian monoidal str. $\mathcal{C}^{\times}$

In particular, for $\mathcal{C} = Cat_\infty$, we define $\mathcal{O}$-monoidal ∞-categories

$$Alg_{\mathcal{O}}(Cat_\infty) \simeq Mon_{\mathcal{O}}(Cat_\infty) \simeq \left\{ \begin{array}{l} \text{opfibrations } \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \text{ exhibiting} \\ \mathcal{C}^{\otimes}_{\underline{O}} \simeq \prod_{i=1}^{n} \mathcal{C}^{\otimes}_{O_i} \end{array} \right\}$$

Ex/ → Symm. mon. ∞-categories $\mathcal{C}^{\otimes} \to Fin_*$ are commutative algebras in $Cat_\infty^{\times}$.

→ Monoidal ∞-categories should be $Alg_{\mathbb{E}_1^{\otimes}}(Cat_\infty) \simeq \left\{ \begin{array}{c} \mathcal{C}^{\otimes} \\ \downarrow \\ \mathbb{E}_1^{\otimes} \end{array} \text{with } \text{''} \right\}$

This agrees with our definition since $\Delta^{op} \to \mathbb{E}_1^{\otimes}$ is an "approximation"

(alternatively, can build a theory of non-symmetric ∞-operads as functors $\mathcal{O}^{\otimes} \to \Delta^{op}$ satisfying similar axioms, see [Gepner-Haugseng])

→ An $LM^{\otimes}$-monoid $LM^{\otimes} \xrightarrow{L} Cat_\infty$ consists of:

• A monoidal ∞-category $L|_{\mathbb{E}_1^{\otimes}}: \mathbb{E}_1^{\otimes} \to Cat_\infty$

$\to$ An $LM^{\otimes}$-monoid $LM^{\otimes} \to Cat_\infty$ consists of:

- A monoidal ∞-category $L|_{E_1^{\otimes}}: E_1^{\otimes} \to Cat_\infty$
- An object $L(\ell) \in Cat_\infty$
- A left-tensoring of $L(\ell)$ over $L|_{E_1^{\otimes}}$

$\to$ A $LM^{\otimes}_{pt}$-monoidal ∞-cat. is a $LM^{\otimes}$-monoidal one with a specified pointing $L_0 \in L(\ell)$.

Thm (Dunn additivity) For $\mathcal{C}$ s.m., equip $Alg_{\mathbb{E}_k}(\mathcal{C})$ with the pointwise (ie. absolute) symm. mon. structure. Then, $Alg_{\mathbb{E}_{n+k}}(\mathcal{C}) = Alg_{\mathbb{E}_n}(Alg_{\mathbb{E}_k}(\mathcal{C}))$

Ex/ What are $Alg_{\mathbb{E}_2}(Cat_1)$? $\to$ Alg(monoidal categories), ie. an 1-category $\mathcal{C}$ equipped with monoidal structures $\otimes$, $\boxtimes$ such that

- $1_{\boxtimes}: (1, \otimes) \to (\mathcal{C}, \otimes)$ is monoidal, ie. $1_{\boxtimes} = 1_{\otimes}$
- $\boxtimes: (\mathcal{C}, \otimes) \times (\mathcal{C}, \otimes) \to (\mathcal{C}, \otimes)$ is monoidal, so (Eckmann-Hilton)

$$X \boxtimes Y \cong (X \otimes 1) \boxtimes (1 \otimes Y) \underset{\text{ptwise}}{\cong} (X \boxtimes 1) \otimes (1 \boxtimes Y) \cong X \otimes Y$$

But: $(1 \otimes X) \boxtimes (Y \otimes 1) \cong (1 \boxtimes Y) \otimes (X \boxtimes 1) \cong Y \otimes X \Rightarrow$ Braiding!

Conversely, a natural trafo $\beta_{XY}: X \otimes Y \xrightarrow{\cong} Y \otimes X$ induces

$$(W \otimes X) \otimes (Y \otimes Z) \cong (W \otimes Y) \otimes (X \otimes Z)$$

exhibiting $\otimes: (\mathcal{C} \times \mathcal{C}, \otimes_{\text{ptwise}}) \to (\mathcal{C}, \otimes)$ as monoidal iff hexagon identities.

Rem: Recall $Mul_{\mathbb{E}_2}(X_1, \dots, X_n; Y) = Conf_n(\mathbb{R}^2)$, we have chosen one of these operations to decompose $\mathbb{E}_2 \cong \mathbb{E}_1 \otimes \mathbb{E}_1$ via Dunn additivity. Hence, the space of choices for $X_1 \otimes \dots \otimes X_n$ is acted on by the Artin braid group $\pi_1 Conf_n(\mathbb{R}^2)$

Def $Cat_{(n,1)} \subseteq Cat_{(\infty,1)}$ on those $\mathcal{C}$ where $Map_{\mathcal{C}}(X,Y)$ is $(n-1)$-truncated $\forall X, Y \in \mathcal{C}$

For $n=0$, empty or contractible $\Rightarrow$ posets

	sets	1-cats	(2,1)-cats	...
$\mathbb{E}_0$	pointed	pointed	pointed	...
$\mathbb{E}_1$	monoid	monoidal	monoidal	...
$\mathbb{E}_2$	commut. monoid	braided	braided	
$\mathbb{E}_3$	"	symmetric	sylleptic	
$\mathbb{E}_4$	"	"	symmetric	

Aside: Can also introduce braided or $\mathbb{E}_k$-operads as certain functors $\mathcal{O}^{\otimes} \to (\Delta^{op})^{\times k}$. See [Haugseng, "The higher Morita category of $\mathbb{E}_n$-algebras"]

$\leadsto$ also works for (n,m)-categories "Baez-Dolan stabilization" starting at $Alg_{\mathbb{E}_{n+2}}(Cat_{(n,m)})$

Stable ∞-categories

Recall: A category $\mathcal{C}$ is called abelian if

This is a property!

- It admits a zero object, products & coproducts

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- automatic (• The canonical map $X \sqcup Y \xrightarrow[\begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}]{} X \times Y$ is always an iso) $\Big\}$ additive
- It is enriched over abelian groups $\}$ induces the addition
- $\mathcal{C}$ admits kernels and cokernels
- $$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[dr, phantom, "\ulcorner"] & \mathrm{coker}(f) \\ 0 \arrow[r] & \mathrm{coim}(f) \arrow[r, "\cong"] & \mathrm{im}(f) \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

Def An ∞-category $\mathcal{C}$ is called stable $:\Longleftrightarrow$

$$\left(\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd} , \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(f) \end{tikzcd} \right)$$

- It admits a zero object 0
- It admits fibers and cofibers (analogs of kernels & cokernels)
- A sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ exhibits X' as fiber of g iff it exhibits X'' as the cofiber of f

Def For $X \in \mathcal{C}$, let
$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

then $\Omega : \mathcal{C} \overset{\simeq}{\longleftrightarrow} \mathcal{C} : \Sigma$ are inverse equivalences if $\mathcal{C}$ stable.

$\rightarrow$ Explains the name. Also write $X[1] = \Sigma X$.

Reason: $\Omega X \to 0 \to X$, $X \to 0 \to \Sigma X$ (co)fiber sequences

Note: $\pi_0 \mathrm{Map}(X,Y) = \pi_0 \mathrm{Map}(X, \Omega^2 \Sigma^2 Y) = \pi_0 \Omega^2 \mathrm{Map}(X, \Sigma^2 Y) = \pi_2 \mathrm{Map}(X,Y)$

$\Rightarrow$ We can add & subtract morphisms, abelian group. pointed by $X \to 0 \to Y$

In fact, get a spectrum / infinite loop space.

Note:
$$\begin{tikzcd} \mathrm{fib}(X \xrightarrow{0} \Sigma Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ \Omega\Sigma Y = Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd}$$
shows $X \times Y = \mathrm{fib}(X \xrightarrow{0} \Sigma Y) \in \mathcal{C}$

similarly $X \sqcup Y = \mathrm{cofib}(X[-1] \xrightarrow{0} Y)$

$\rightarrow X \sqcup Y \cong X \times Y$ since $h\mathcal{C}$ additive (Ab enriched ✓, has $\times, \sqcup, 0$ ✓)

Proposition: Any stable ∞-category $\mathcal{C}$ has biproducts, ie. $X \sqcup Y \xrightarrow{\cong} X \times Y$.
In fact, $h\mathcal{C}$ is additive.

Fact:
$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$
in $\mathcal{C}$ stable is a pullback iff it is a pushout square.
Thus, we can apply the pasting lemma in both directions.

Rem: The mirroring $\Delta^1 \times \Delta^1 \to \Delta^1 \times \Delta^1$ swapping the components acts on $X \xrightarrow{f} Y$ classified by
$$\begin{tikzcd} X[1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$
by sending it to $X \xrightarrow{-f} Y$, as it inverts the loops in $\mathrm{Map}(X,Y) = \Omega \mathrm{Map}(X[1], Y)$.

$\boxed{\textbf{Theorem}}$ If $\mathcal{C}$ is a stable ∞-category, then $h\mathcal{C}$ is triangulated, with shift functor $X[1] := \Sigma X$ and

dist. triangles
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

dist. triangles
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\perp"] & Y \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\perp"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "h"'] & X[1] \end{tikzcd}$$

Proof: • $X \xrightarrow{f} Y$ can always be completed to dist. tr. $X \xrightarrow{f} Y \to \mathrm{cofib}(f)$

• If $f = \mathrm{id}_X$, then $\mathrm{cofib}(f) = 0$

• Isomorphic to dist. tr. $\Rightarrow$ still one

• Shift
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & X[1] \arrow[d, "-f[1]"] \\ & 0 \arrow[r] & Y[1] \end{tikzcd}$$
— since mirrored
→ "turns loop around"

$X \to Y$ / $\begin{matrix} X \\ \downarrow \\ Y \end{matrix}$

• $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$ induce $\begin{matrix} \mathrm{cofib}(f) \\ \downarrow \\ \mathrm{cofib}(f') \end{matrix}$ fitting into dist. tr. $\Rightarrow$ functorial cone

• Octaeder axiom
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y/X \arrow[r] \arrow[d] & Z/X \arrow[r] \arrow[d] & X[1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ & 0 \arrow[r] & Z/Y \arrow[r] & Y[1] \arrow[r] & Y/X[1] \end{tikzcd}$$

(Could go on to find "higher octaeder".) □

Not every triangulated category arises this way, but "all the interesting ones", e.g.

- $\mathrm{Ch}(\mathcal{A}) = \{$chain cplxs, chain maps, chain homotopies, ...$\}$ is stable
- Any (pretriangulated) dg-category has an associated (stable) ∞-category
  A_∞
- For $\mathcal{A}$ Grothendieck abelian (for simplicity),

  $D(\mathcal{A}) = \{$injective chain cplxes, chain maps, chain homot, ...$\}$ is stable

  & has all limits and colimits (presentable stable)
- $D(\mathrm{Sh}(X;\mathcal{A})) = \mathrm{Sh}^{\mathrm{hyp}}_\infty(X; D(\mathcal{A}))$ is stable $\to$ Factorization algebras...
- $\mathrm{Sp} = \lim(S_* \xleftarrow{\Omega} S_* \xleftarrow{\Omega} \cdots)$ is stable, in fact universal among them

Rem: The cat. of abelian groups Ab is universal among additive categories as every add. cat. is Ab-enriched, and conversely an Ab-enriched category is "Cauchy-complete" if it stems from an idempotent complete add. cat.

Similarly: A Sp-enriched ∞-category is Cauchy-complete iff it stems from the natural Sp-enrichment of an idemp. compl. stable ∞-category.