Monoidal Structures, Operads and Stable Categories I

Symmetric Monoidal ∞-categories

Reminder: The Segal category Fin_* consists of finite pointed sets $\langle n\rangle = \{*, 1, \dots, n\}$, $n \geq 0$, and pointed maps (think: partially defined).

- $\alpha : \langle n\rangle \to \langle m\rangle$ is inert $:\Leftrightarrow$ Every $\forall k \in \langle m\rangle : \#\alpha^{-1}(\{k\}) = 1$ ($\rightsquigarrow$ injective inverse $\langle m\rangle \setminus \{*\} \subseteq \langle n\rangle \setminus \{*\}$)
- $\rho_i : \langle n\rangle \to \langle 1\rangle$, $1 \leq i \leq n$ standard inert maps $j \mapsto \begin{cases} 1, & j = i \\ *, & j \neq i \end{cases}$

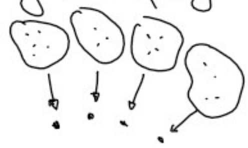
Idea: Let U, V, W be VS, then the reason why $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ is because both classify trilinear maps $\mathrm{Trilin}(U, V, W; X)$

$\Rightarrow$ New way to encode symm. monoidal structure $(\mathcal{C}, \otimes)$ via a new category $\mathcal{C}^{\otimes}$ with objects tuples $[c_1, \dots, c_n]$ in $\mathcal{C}$; morphisms $[c_1, \dots, c_n] \to [c'_1, \dots, c'_m]$ consisting of $\langle n\rangle \to \langle m\rangle$ in Fin_* and $\{ f_j : \bigotimes_{i \in \alpha^{-1}(\{j\})} c_i \to c_j \}_{1 \leq j \leq m}$

The canonical projection $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ is an opfibration:

Opfibrations: • Idea is that a map of sets $E \to B$ is specified by the family of inverse images $(f^{-1}(t))_{t \in B}$, in other words

$\mathrm{Set}_{/B} = \{\text{sets with a map to } T\} \simeq \mathrm{Fun}(B, \mathrm{Set})$ ← discrete category



• Replace B by a space $\hat{=}$ ∞-groupoid, then $\mathrm{Fun}(B, \mathrm{Set})$ should be the covering spaces over B, i.e. $\mathrm{Cov}(B) \simeq \mathrm{Fun}(B, \mathrm{Set})$ $\rightsquigarrow$ since Set is a 1-cat, $= \mathrm{Fun}(\pi_1(B), \mathrm{Set}) = \pi_1(B)\text{-Set}$

Most general version: An opfibration is a functor of ∞-categories $\mathcal{E} \xrightarrow{p} \mathcal{B}$ such that $\forall e \in \mathcal{E}\ \forall \alpha : p(e) \to b'$ there is a p-coCartesian morph. $\bar{\alpha} : e \to e'$ with $p(\bar{\alpha}) = \alpha$.

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(e', x) & \xrightarrow{-\circ\bar{\alpha}} & \mathrm{Map}_{\mathcal{E}}(e, x) \\ \downarrow p & \lrcorner & \downarrow p \\ \mathrm{Map}_{\mathcal{B}}(p(e'), p(x)) & \xrightarrow{-\circ\alpha} & \mathrm{Map}_{\mathcal{B}}(p(e), p(x)) \end{array}$$

This "coCartesian lift" is essentially unique, in fact we obtain a coCartesian transport functor $\alpha_! : \mathcal{E}_b \to \mathcal{E}_{b'}$

Thm [Lurie] $\forall \mathcal{B}$ ∞-category, $\left\{ \begin{array}{c} \text{opfibrations} \\ \text{over } \mathcal{B} \end{array} \right\} \simeq \mathrm{Fun}(\mathcal{B}, \mathrm{Cat}_\infty)$

$$(\mathcal{E} \to \mathcal{B}) \xmapsto{\text{straightening}} \left(\begin{array}{l} b \mapsto \mathcal{E}_b \\ \alpha \mapsto (\alpha_! : \mathcal{E}_b \to \mathcal{E}_{b'}) \end{array} \right)$$

$$\{\text{pairs } (b \in \mathcal{B}, x \in F(b))\} = \mathcal{B} \times_{\mathrm{Cat}_\infty} (\mathrm{Cat}_\infty)_{*/} \xleftarrow{\text{unstraightening}} F : \mathcal{B} \to \mathrm{Cat}_\infty$$

($\Rightarrow$ One of the main achievements of HTT)

Def A symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ is an opfibration $p : \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$, such that the transport maps $(\rho_i)_! : \mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle} =: \mathcal{C}$ induce an equivalence $\mathcal{C}^{\otimes}_{\langle n\rangle} \simeq \prod^n \mathcal{C}^{\otimes}_{\langle 1\rangle} =: \mathcal{C}^{\times n}$ $\forall n$

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induce an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \prod_{i=1}^{n} \mathcal{C}^{\otimes}_{\langle 1 \rangle} =: \mathcal{C}^{\times n} \quad \forall n$

Equivalently, it is a functor $\mathrm{Fin}_* \xrightarrow{C} \mathrm{Cat}_\infty$ such that $C(\rho_i) : C(\langle n \rangle) \to C(\langle 1 \rangle)$ exhibit $C(\langle n \rangle) \simeq \mathcal{C}^{\times n}$. In other words, a monoid object in Cat_∞.

Ex/ • If $\mathcal{C}$ has (co-)products, there exist $\mathcal{C}^{\sqcup} / \mathcal{C}^{\times} \to \mathrm{Fin}_*$

• $(\mathrm{Ch}(R), \otimes)$, $\quad (D(R), \otimes^L)$ recall: obj: injective chain cplxes, mor: chain maps, 2-mor: homotopies ...

We may read off the unit as $C(\langle 0 \rangle \xrightarrow{u} \langle 1 \rangle) = u_1 : \mathcal{C}^{\times 0} = \Delta^0 \to \mathcal{C}^{\times 1} = \mathcal{C}$

tensor product as $C(\langle 2 \rangle \xrightarrow[1,2 \mapsto 1]{\mu} \langle 1 \rangle) = \mu_1 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

braiding as $C(\langle 2 \rangle \xrightarrow[\substack{1 \mapsto 2 \\ 2 \mapsto 1}]{b} \langle 2 \rangle) = b_1 : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$

+ higher coherences from all the non-inert maps (e.g. ρ_i are just projections!)

Monoidal ∞-categories

Recall: Δ^{op} is the category of finite totally ordered sets $[n] = \{0 < \dots < n\}$, $n \geq 0$ and order-preserving maps $\alpha : [n] \leftarrow [m]$

α is inert $:\Leftrightarrow$ embeds $[m]$ as an interval of $[n]$, i.e. $\alpha(k) = \alpha(0) + k$

$\rho_i : [n] \leftarrow [1]$, $1 \leq i \leq n$ standard inert $:\Leftrightarrow \rho_i(0) = i-1$, $\rho_i(1) = i$

Def A monoidal ∞-category is an opfibration $p : \mathcal{C}^{\otimes} \to \Delta^{op}$, or equivalently a functor $\Delta^{op} \xrightarrow{C} \mathrm{Cat}_\infty$, such that ρ_i induce $\mathcal{C}^{\otimes}_{[n]} \cong (\mathcal{C}^{\otimes}_{[1]})^{\times n}$

Read off: Underlying is $\underbrace{C([1])}_{=:\mathcal{C}}$, and $[2] \leftarrow [1]$, $\substack{0 \leftarrow 0 \\ 2 \leftarrow 1}$ gives $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$[0] \leftarrow [1]$ gives unit

∞-Operads

Motivation: For some categories, e.g. kinds of topol. VS, the functor $\mathrm{Bilin}(V \times V', -)$ is not representable. Still, it is a weak kind of $\otimes$-structure.

Same idea: Given a colored operad $\mathcal{O}$ (symm.), define new category $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ with objects tuples of colors $[X_1, \dots, X_n]$

morphisms $[X_1, \dots, X_n] \to [Y_1, \dots, Y_m]$ consisting of $\alpha : \langle n \rangle \to \langle m \rangle$ and

$$(\phi_j \in \mathrm{Mul}(\{X_i\}_{i \in \alpha^{-1}(j)}, Y_j))_{1 \leq j \leq m}$$

composition induced by composition in $\mathcal{O}$, similarly identities.

Via this construction, $\mathcal{O}^{\otimes}_{\langle n \rangle} \cong (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^{\times n}$, but no opfibration, since e.g. there shouldn't be a tensor product functor to read off. However, there are projections

$\{n\text{-tuples in } \mathcal{O}\} \xrightarrow{\text{proj}} \{m\text{-tuples in } \mathcal{O}\}$ for $n \geq m$ inert maps!

Def An ∞-operad is a functor of ∞-categories $p: \mathcal{O}^\otimes \to \mathrm{Fin}_*$, such that

(i) $\forall \underline{X} \in \mathcal{O}^\otimes_{\langle n\rangle}$, $\alpha: \langle n\rangle \xrightarrow{\text{inert}} \langle m\rangle$ there is a p-coCartesian morphism $\bar\alpha: \underline{X} \to \underline{Y}$ in $\mathcal{O}^\otimes$ lifting α $\rightsquigarrow$ Transport functor $\alpha_!: \mathcal{O}^\otimes_{\langle n\rangle} \to \mathcal{O}^\otimes_{\langle m\rangle}$

(ii) The p-coCartesian lifts $(\rho_i)_!: \mathcal{O}^\otimes_{\langle n\rangle} \to \mathcal{O}^\otimes_{\langle 1\rangle}$ exhibit $\mathcal{O}^\otimes_{\langle n\rangle} \simeq (\mathcal{O}^\otimes_{\langle 1\rangle})^{\times n}$

Notation: Choose $\underline{X} \to X_i$ p-coCart lifts of ρ_i, then write $\underline{X} = [X_1, \dots, X_n]$

Notation: For $\alpha: \langle n\rangle \to \langle m\rangle$, $\underline{X} \in \mathcal{O}^\otimes_{\langle n\rangle}$, $\underline{Y} \in \mathcal{O}^\otimes_{\langle m\rangle}$ let $\mathrm{Map}^\alpha_{\mathcal{O}^\otimes}(\underline{X}, \underline{Y})$ be the fiber $\mathrm{Map}_{\mathcal{O}^\otimes}(\underline{X}, \underline{Y}) \times_{\mathrm{Map}_{\mathrm{Fin}_*}(\langle n\rangle, \langle m\rangle)} \{\alpha\}$, i.e. the morphisms over α.

(iii) $\mathrm{Map}^\alpha_{\mathcal{O}^\otimes}(\underline{X}, \underline{Y}) \xrightarrow[(\rho_i)_! \circ -]{\simeq} \prod_{1 \le i \le m} \mathrm{Map}^{\rho_i \circ \alpha}_{\mathcal{O}^\otimes}((X_i)_{i \in \alpha^{-1}(j)}, Y_j)$ $\rightsquigarrow$ We should allow multiple sources, but not multiple targets.

Denote $\mathrm{Mul}_{\mathcal{O}}(X_1, \dots, X_n; Y) := \mathrm{Map}^\alpha_{\mathcal{O}^\otimes}([X_1, \dots, X_n], Y)$ for $X_i, Y \in \mathcal{O} := \mathcal{O}^\otimes_{\langle 1\rangle}$, where $\alpha(1) = \dots = \alpha(n) = 1$

Ex/
- $\mathrm{Fin}_* \to \mathrm{Fin}_*$ is the commutative operad $\mathbb{E}^\otimes_\infty$
- Every (symm.) colored operad defines an ∞-operad $N(\mathcal{O}^\otimes \to \mathrm{Fin}_*)$ (nerve)
- —— " —— enriched in Kan complexes defines an ∞-operad. In fact, this is part of a Quillen-equivalence.
- Let $\mathbb{E}^\otimes_1$ be the category with objects $\langle n\rangle$, but Morph. are $\alpha: \langle n\rangle \to \langle m\rangle$ together with a specified total ordering on $\alpha^{-1}(\{k\})$ for all $1 \le k \le m$. Composition is defined by gluing together total orders "lexicographically"
  $\Rightarrow$ One multimorphism for every ordering
- $\mathrm{LM}^\otimes$ comes from the colored operad LM with two colors a, ℓ and
  $\mathrm{Mul}(\overbrace{a, \dots, a}^{n}; a) = \{\text{total orderings of } \{1, \dots, n\}\}$ as for $\mathbb{E}^\otimes_1$
  $\mathrm{Mul}(a, \dots, a, \ell; \ell) = \mathrm{Mul}(a, \dots, a; \ell) = \{ - " - \}$, otherwise $\emptyset$
  Composition again using lexicographical ordering.

Def A map of ∞-operads is a functor $F: \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ over Fin_*, sending coCart. lifts of inerts ($\Leftrightarrow$ inerts) to "/.
Call this an $\mathcal{O}$-algebra in $\mathcal{P}$ and define

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \subseteq \mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{P}^\otimes) \times_{\mathrm{Fun}(\mathcal{O}^\otimes, \mathrm{Fin}_*)} \{p\} = \mathrm{Fun}_{/\mathrm{Fin}_*}(\mathcal{O}^\otimes, \mathcal{P}^\otimes)$$

Def Let $\mathcal{D}$ be an ∞-cat. with products. A functor $M: \mathcal{O}^\otimes \to \mathcal{D}$ is an $\mathcal{O}$-monoid in $\mathcal{D}$ $:\Leftrightarrow$ The inert lifts $(O_1, \dots, O_n) \to (O_i)$ exhibit $M((O_1, \dots, O_n)) \simeq \prod_{i=1}^n M((O_i))$