Factorization algebras

References:

Costello, Gwilliam: Factorization algebras

Groth: A short course on ∞-categories

Lurie: Higher topos theory

Lurie: Higher algebra

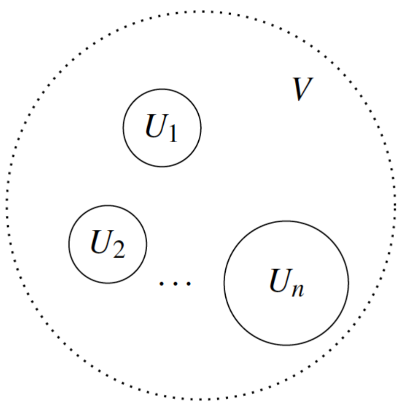
Content

1. A first definition

Def

M a topological space. A prefactorization algebra $\mathcal{F}$ on M taking values in vector spaces is a rule that assigns

- to each open set $U \subset M$ a vector space $\mathcal{F}(U)$
- For every finite collection of open sets $U_i \subset V \subset M$ with U_i pairwise disjoint a linear map

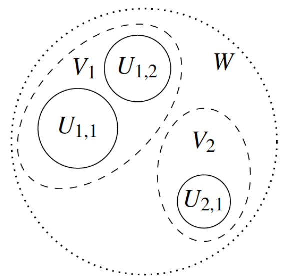


$$\rightsquigarrow m_V^{U_1,\dots,U_n} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$$

Compatibility:

- The maps are compatible in the obvious way, so that if $U_{i,1} \sqcup \cdots \sqcup U_{i,n_i} \subseteq V_i$ and $V_1 \sqcup \cdots \sqcup V_k \subseteq W$, the following diagram commutes.

$$\begin{tikzcd}
\bigotimes_{i=1}^{k} \bigotimes_{j=1}^{n_i} \mathcal{F}(U_j) \arrow[rr] \arrow[dr] & & \bigotimes_{i=1}^{k} \mathcal{F}(V_i) \arrow[dl] \\
& \mathcal{F}(W) &
\end{tikzcd}$$



$\rightsquigarrow$

$$\begin{tikzcd}
\mathcal{F}(U_{1,1}) \otimes \mathcal{F}(U_{1,2}) \otimes \mathcal{F}(U_{2,1}) \arrow[d] \arrow[dr] & \\
\mathcal{F}(V_1) \otimes \mathcal{F}(V_2) \arrow[r] & \mathcal{F}(W)
\end{tikzcd}$$

The case of $k = n_1 = 2, n_2 = 1$

- $\mathcal{F}(\emptyset)$ is commutative algebra.

Unital prefactorization algebra $\overset{}{\underset{\text{def}}{\iff}}$ $\mathcal{F}(\emptyset)$ is unital

$$\mathcal{F}(\emptyset) \hookrightarrow \mathcal{F}(M) \qquad \text{pointed vector space}$$

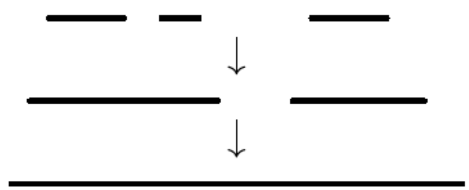
$$1 \longmapsto *$$

Example:

A associative alg $\Rightarrow$ locally constant prefact. algebra A^{fact} on $\mathbb{R}$

$$A^{fact}((a,b)) = A$$

$$A^{fact}\left(\coprod_j I_j\right) = \bigotimes_j A_j \qquad \text{locally constant}$$



$$\rightsquigarrow \quad \begin{array}{ccc} a \otimes b \otimes c & \in & A \otimes A \otimes A \\ \downarrow & & \downarrow \\ ab \otimes c & \in & A \otimes A \\ \downarrow & & \downarrow \\ abc & \in & A \end{array}$$

Figure 3.1. The prefactorization algebra A^{fact} of an associative algebra A.

multiplication of A

$\rightsquigarrow$

structure maps

2. Multicategories = coloured operads

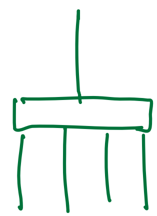
$$\begin{tikzcd}
\text{Operad} \arrow[dr] & \text{monoidal category} \arrow[d] & \text{category} \arrow[dl] \\
& \text{multicategory} &
\end{tikzcd}$$

Preview: prefactorization algebra w/ values in multicategory $\mathcal{C}$ is a functor of multicategories $\mathrm{Disj}_M \longrightarrow \mathcal{C}$

2.1. Operads

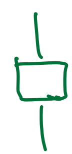
<u>Def</u>: An operad $\mathcal{O}$ in $\left\{\begin{matrix}\text{sets}\\ \text{vector spaces}\end{matrix}\right\}$ consists of

(i) A sequence $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$ $\left\{\begin{matrix}\text{of vector spaces}\\ \text{sets}\end{matrix}\right\}$, called vector space of operations



$\mathcal{O}(4)$

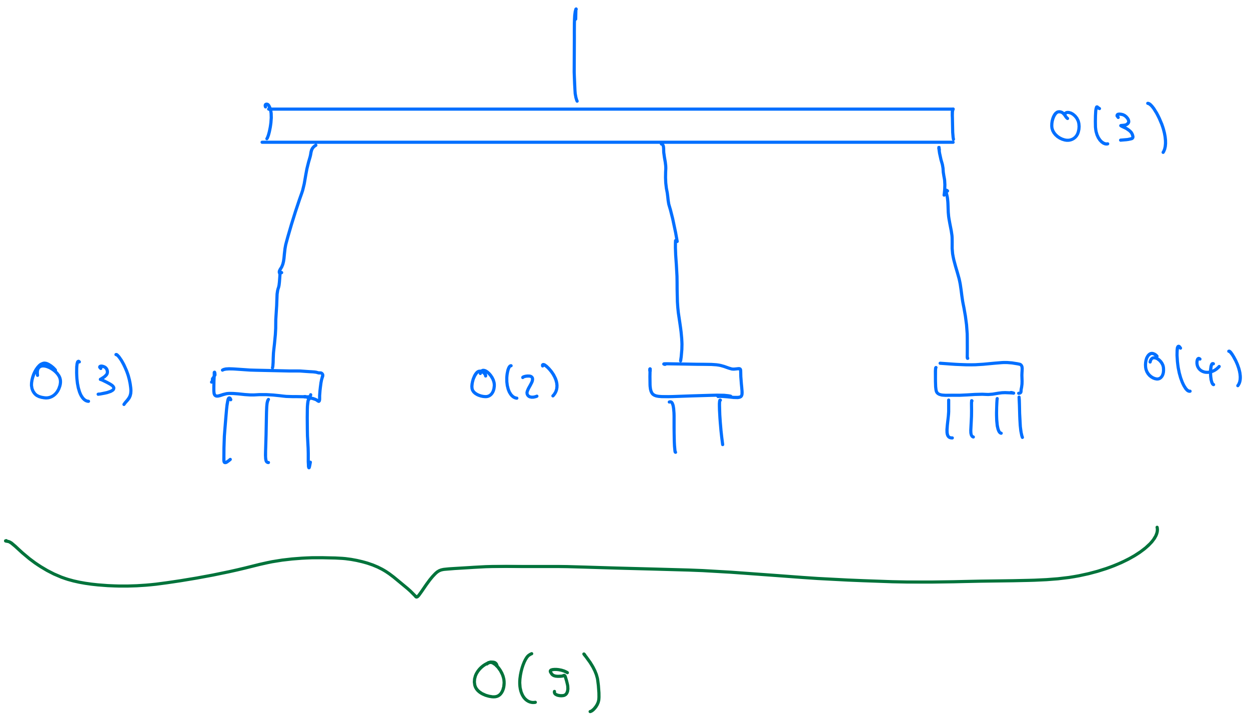
(ii) A unit element $\eta : K \longrightarrow \mathcal{O}(1)$



$1 \in \mathcal{O}(1)$

(iii) A collection of multilinear maps

$\circ_{n; m_1 \dots m_n} \quad \mathcal{O}(n) \otimes \left[\mathcal{O}(m_1) \otimes \dots \otimes \mathcal{O}(m_n) \right] \longrightarrow \mathcal{O}\left(\sum_{j=1}^{n} m_j \right)$



S_n - equivariance

These data are equivariant, associative, and unital in the following way.

(1) The n-ary operations $\mathcal{O}(n)$ have a right action of S_n.

(2) The composition maps are equivariant in the sense that the following diagram commutes,

$$\begin{tikzcd}
\mathcal{O}(n) \otimes (\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)) \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\circ"'] & \mathcal{O}(n) \otimes (\mathcal{O}(m_{\sigma(1)}) \otimes \cdots \otimes \mathcal{O}(m_{\sigma(n)})) \arrow[d, "\circ"'] \\
\mathcal{O}\left(\sum_{j=1}^n m_j\right) \arrow[r, "\sigma(m_{\sigma(1)},\ldots,m_{\sigma(n)})"] & \mathcal{O}\left(\sum_{j=1}^n m_j\right)
\end{tikzcd}$$

where $\sigma \in S_n$ acts as a block permutation on the $\sum_{j=1}^n m_j$ inputs, and the following diagram also commutes,

$$\begin{tikzcd}
\mathcal{O}(n) \otimes (\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)) \arrow[r, bend left, "\mathrm{id}\otimes(\tau_1\otimes\cdots\otimes\tau_n)"] \arrow[d, "\circ"'] & \mathcal{O}(n) \otimes (\mathcal{O}(m_{\sigma(1)}) \otimes \cdots \otimes \mathcal{O}(m_{\sigma(n)})) \arrow[d, "\circ"'] \\
\mathcal{O}\left(\sum_{j=1}^n m_j\right) \arrow[r, "\tau_1\oplus\cdots\oplus\tau_n"] & \mathcal{O}\left(\sum_{j=1}^n m_j\right)
\end{tikzcd}$$

where each τ_j is in S_{m_j} and $\tau_1 \oplus \cdots \oplus \tau_n$ denotes the blockwise permutation in $S_{\sum_{j=1}^n m_j}$.

Associativity

(3) The composition maps are associative in the following sense. Let n, m_1, $\ldots$, m_n, $\ell_{1,1}, \ldots, \ell_{1,m_1}, \ell_{2,1}, \ldots, \ell_{n,m_n}$ be positive integers, and set $M = \sum_{j=1}^n m_j$, $L_j = \sum_{i=1}^{m_j} \ell_{j,i}$, and

$$N = \sum_{i=1}^n L_j = \sum_{(j,k)\in M} \ell_{j,k}.$$

Then the diagram

$$\begin{tikzcd}
\mathcal{O}(n) \otimes \bigotimes_{j=1}^n \left(\mathcal{O}(m_j) \otimes \bigotimes_{k=1}^{m_j} \mathcal{O}(\ell_{j,k})\right) \arrow[r, "\text{shuffle}"] \arrow[d, "\mathrm{id}\otimes(\otimes_j\circ)"'] & \mathcal{O}(n) \otimes \bigotimes_{j=1}^n \mathcal{O}(m_j) \otimes \bigotimes_{j=1}^n \bigotimes_{k=1}^{m_j} \mathcal{O}(\ell_{j,k}) \arrow[d, "\circ\otimes\mathrm{id}"] \\
\mathcal{O}(n) \otimes \bigotimes_{j=1}^n \mathcal{O}(L_j) \arrow[d, "\circ"'] & \mathcal{O}(M) \otimes \bigotimes_{j=1}^n \bigotimes_{k=1}^{m_j} \mathcal{O}(\ell_{j,k}) \arrow[dl, "\circ"] \\
\mathcal{O}(N) &
\end{tikzcd}$$

commutes.

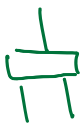
A map of operads $f : \mathcal{O} \longrightarrow \mathcal{P}$ is a sequence of maps

$$f(n) : \mathcal{O}(n) \longrightarrow \mathcal{P}(n)$$

compatible with composition and S_n- action.

Examples

1) Ass $\qquad Ass(n) = K[S_n]$ free S_n- module

Generator :  μ in $Ass(2)$

Relation :  in $Ass(3)$

2) Com with $Com(n) = K$ with trivial S_n action.

3) Given v.s. V: endomorphism operad End_V

$$\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$$ or multilinear maps

Composition is composition of multilinear maps

<u>Def</u>: $\mathcal{O}$ an operad. An algebra over $\mathcal{O}$ is a vector space V and a map of operads

$$\rho : \mathcal{O} \longrightarrow \mathrm{End}_V$$

Because of

$$\mathrm{Hom}(\mathcal{O}(n), \mathrm{Hom}(V^{\otimes n}, V)) \cong \mathrm{Hom}(\mathcal{O}(n) \otimes V^{\otimes n}, V)$$

$$\rho(n) : \mathcal{O}(n) \otimes V^{\otimes n} \longrightarrow V$$

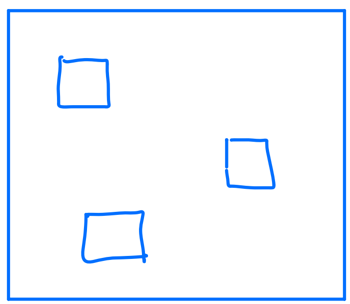
Think about this as spaces of operations.

Example

The ***little k-disk operad*** or ***little k-cubes operad*** (to distinguish from the framed little n-disk operad) is the topological operad/(∞,1)-operad E_k whose n-ary operations are parameterized by rectilinear disjoint embeddings of n k-dimensional cubes into another k-dimensional cube.

When regarded as a topological operad, the topology on the space of all such embedding is such that a continuous path is given by continuously moving the images of these little cubes in the big cube around.

Therefore the algebras over the E_k operad are "k-fold monoidal" objects. For instance k-tuply monoidal (n,r)-categories.



1) More than one vector space in vect_k $\longrightarrow$ multicategories = coloured operads

2) Need topological categories.

2.2. Multicategories

We work in an enriched setting: $(\mathcal{C}, \otimes)$ symmetric monoidal category

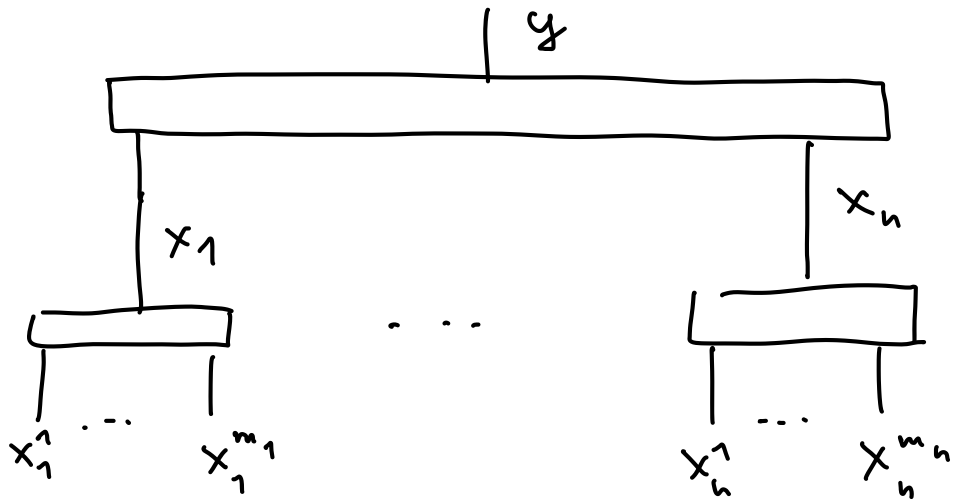
(e.g. $(\mathrm{Sets}, \times)$, $(\mathrm{vect}_k, \otimes_k)$)

with all necessary colimits

Def: A multicategory = coloured operad consists of

- collection of objects $\mathrm{Ob}\,\mathcal{M}$
- For each $(n+1)$ tuple of objects $(x_1, \dots, x_n \mid y)$ an object
$$\mathcal{M}(x_1, \dots, x_n \mid y) \in \mathcal{C}$$
- For each object x a unit $\eta_x : \mathbb{1}_{\mathcal{C}} \to \mathcal{M}(x \mid x)$
- Composition maps
$$\mathcal{M}(x_1 \dots x_n \mid y) \boxtimes \left(\mathcal{M}(x_1^1 \dots x_1^{m_1} \mid x_1) \boxtimes \dots \boxtimes \mathcal{M}(x_n^1 \dots x_n^{m_n}) \right)$$
$$\longrightarrow \mathcal{M}(x_1^1 \dots x_n^{m_n} \mid y)$$

$$\mathcal{M}(x_1 \ldots x_n \mid y) \boxtimes \left(\mathcal{M}(x_1^1 \ldots x_1^{m_1} \mid x_1) \boxtimes \ldots \boxtimes \mathcal{M}(x_n^1 \ldots x_n^{m_n}) \right)$$
$$\longrightarrow \mathcal{M}(x_1^1 \ldots x_n^{m_n} \mid y)$$



- For every $(n+1)$ tuple $(x_1, .., x_n \mid y)$ and $\sigma \in S_n$

$$\sigma^* : \mathcal{M}(x_1 \ldots x_n \mid y) \longrightarrow \mathcal{M}(x_{\sigma 1} \ldots x_{\sigma n} \mid y)$$

a morphism in $\mathcal{C}$

Assoc., unit, equivariance as for operads. η_x one sided unit in $\mathcal{M}(x|x)$.

Def: Map of multicategories $F : \mathcal{M} \to \mathcal{N}$

(i) For $x \in \mathcal{M}$ an object $F(x) \in \mathcal{N}$.

(ii) A morphism

$$F(x_1 \dots x_n | y) : \mathcal{M}(x_1, \dots x_n | y) \longrightarrow \mathcal{N}(Fx_1 \dots Fx_n | Fy)$$

in $\mathcal{C}$ preserving units, composition and S_n action.

Examples:

1) $\mathcal{B}$ ordinary category $\quad \mathcal{B}(x|y) = \mathcal{B}(x,y)$, $\mathcal{B}(x_1 \dots x_n | y) = \phi$ for $n \geq 2$.

2) Operad $\mathcal{O}$: single object $*$

$$\mathcal{O}(\underbrace{* \dots *}_{n} | *) = \mathcal{O}(n)$$

3) $(\mathcal{C}, \otimes)$ symmetric monoidal cat $\rightsquigarrow$ multicat. $\underline{\mathcal{C}}$ with same objects

$$\underline{\mathcal{C}}(x_1, \dots, x_n | y) = \mathcal{C}(x_1 \otimes \dots \otimes x_n, y) \quad \text{(multilinear maps)}$$

Symmetric monoidal cat $\longrightarrow$ Multicategory

has a "left adjoint" :

$\mathcal{M}$ multicategory , $S\mathcal{M}$

– objects : finite sequence of colours $[x_1 \dots x_m]$ $\quad x_i \in \mathcal{M}$

– morphism $\quad f : [x_1 \dots x_m] \to [y_1 \dots y_n]$

- surjection $\quad \underline{m} \to \underline{n} \qquad \underline{m} = \{1, \dots, m\}$
- $\forall\, j = 1 \dots n$

$$f_j \in \mathcal{M}\left(\{x_i\}_{i \in \phi^{-1} j} \mid y_j \right)$$

– monoidal structure concatenation.

product in $S\mathcal{M}$ is simply concatenation of formal sequences.

Finally, an *algebra over a colored operad* $\mathcal{M}$ with values in $\mathcal{N}$ is simply a functor of multicategories $F : \mathcal{M} \to \mathcal{N}$. When we view $\mathcal{O}$ as a multicategory and use the underlying multicategory $\underline{Vect}_{\mathbb{K}}$, then $F : \mathcal{O} \to \underline{Vect}_{\mathbb{K}}$ reduces to an algebra over the operad $\mathcal{O}$ as in the preceding subsection.

2.3. Factorization algebras and multicategories

Definition 3.1.1 Let Disj_M denote the following multicategory associated to M.

- The objects consist of all connected open subsets of M.
- For every (possibly empty) finite collection of open sets $\{U_\alpha\}_{\alpha \in A}$ and open set V, there is a set of maps $\mathrm{Disj}_M(\{U_\alpha\}_{\alpha \in A} \mid V)$. If the U_α are pairwise disjoint and all are contained in V, then the set of maps is a single point. Otherwise, the set of maps is empty.
- The composition of maps is defined in the obvious way.

A prefactorization algebra just *is* an algebra over this colored operad Disj_M.

Definition 3.1.2 Let $\mathcal{C}$ be a multicategory. A *prefactorization algebra* on M taking values in $\mathcal{C}$ is a functor (of multicategories) from Disj_M to $\mathcal{C}$.

Note :

- $\mathcal{F}(\emptyset)$ is a commutative algebra object of $\mathcal{C}$. $\mathcal{F}$ is called unital if $\mathcal{F}(\emptyset)$ is unital.
- Factorization algebras on sites of manifolds
- $$m_W^{U_1 U_2} \simeq m_W^V \circ m_V^{U_1, U_2}$$ up to homotopy, e.g. when $\mathcal{C}$ is Ch_k.

A theorem I would like to understand :

discs.

Theorem 5.1.3 *Let $\mathcal{F}$ be a holomorphically translation invariant prefactorization algebra on $\mathbb{C}$. Let $\mathcal{F}$ be equivariant under the action of S^1 on $\mathbb{C}$ by rotation, and let $\mathcal{F}_r^k$ denote the weight k eigenspace of the S^1 action on the complex $\mathcal{F}_r$. Assume that for every $r < s$, the extension map $\mathcal{F}_r^k \to \mathcal{F}_s^k$ associated to the inclusion $D(0,r) \subset D(0,s)$ is a quasi-isomorphism. Finally, we need to*

- on curves
- concrete constructions
- blocks

assume that the S^1 action on each $\mathcal{F}_r$ satisfies a certain technical "tameness" condition.

Then the vector space

$$V_{\mathcal{F}} = \bigoplus_{k \in \mathbb{Z}} H^*(\mathcal{F}_r^k)$$

has the structure of a vertex algebra. The vertex algebra structure map

$$Y_{\mathcal{F}} : V_{\mathcal{F}} \otimes V_{\mathcal{F}} \to V_{\mathcal{F}}[[z, z^{-1}]]$$

is the Laurent expansion of operator product map

$$H^*\mu : H^*(\mathcal{F}_{r_1}^{k_1}) \otimes H^*(\mathcal{F}_{r_2}^{k_2}) \to \mathrm{Hol}(\mathrm{Discs}(r_1, r_2 \mid s), H^*(\mathcal{F}_s)).$$

On the right-hand side, Hol *denotes the space of holomorphic maps.*

<u>Tasks</u>

- framework for top. categories + Ch_k
- Operads, monoidal cats in this setting.

3. ∞-categories

<u>Def</u> : Δ category of finite ordinals

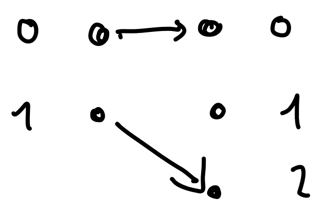
$$[n] = (0 < 1 < \dots n) \qquad n \geq 0$$

with order preserving maps.

<u>Generators for morphisms</u> :

- coface maps $\quad d^k : [n-1] \longrightarrow [n] \qquad 0 \leq k \leq n$

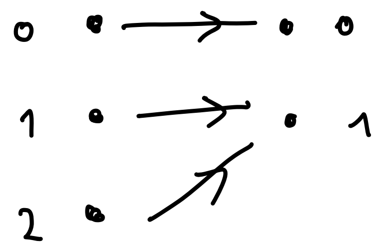
  injective, leaves out $k \in [n]$



d^1

- codegeneracy maps $\quad s^k : [n+1] \longrightarrow [n]$

  surjective, hits $\underset{[n]}{\underset{\cap}{k}}$ twice



S^1

<u>Def</u> : Category of simplicial sets $\quad [\Delta^{op}, Set]$

Def : Category of simplicial sets $\quad [\Delta^{op}, Set] =: sSet$

$$X_n = X([n]) \qquad X \in [\Delta^{op}, Set]$$

$$d_k = X(d^k) : X_n \to X_{n-1} \qquad s_k = X(s^k) : X_n \to X_{n+1}$$

$$X_0 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_2 \quad \cdots$$

Ex 1 : $X_n = *$ singleton set

Ex 2 : $\mathrm{Hom}_\Delta(-, [n])$ standard simplex

Example 1 : $\mathcal{C}$ a category, nerve $N(\mathcal{C}) \in sSet$

with

$$N(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$$

ordinal set, seen as a category

$N(\mathcal{C})_0$ = Objects of $\mathcal{C}$

$N(\mathcal{C})_1$ = Morphisms

$N(\mathcal{C})_2$ functors on

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr, "\text{composition}"'] & & 2 \end{tikzcd}$$

= pairs of composable morphisms

+ their composition

$d_1 : N(\mathcal{C})_2 \to N(\mathcal{C})_1$ maps to composition

Lemma: The nerve functor $N : \mathrm{Cat} \longrightarrow s\mathrm{Set}$ is fully faithful.

Characterize the essential image!

Example 2: X top. space $\quad |\Delta^n| \subset \mathbb{R}^n$ top simplex

$$|\Delta^n| = \left\{ (x_0, \dots, x_n) \;\middle|\; \sum_{i=0}^{n} x_i = 1 \right\}$$

$$\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$$

To learn about image of nerve functor: k-th n-horn $\Lambda^n_k \subseteq \partial\Delta^n$

subcomplex, i.e. $L_n \subset K_n$ s.t. restriction is a simplicial set

remove k-th face

$n = 2$

3 horns

$$\begin{tikzcd}
 & c_1 & \\
c_0 \arrow[ur] \arrow[rr] & & c_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & c_1 \arrow[dr] & \\
c_0 \arrow[ur] & & c_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & c_1 \arrow[dr] & \\
c_0 \arrow[rr] & & c_2
\end{tikzcd}$$

$k = 0 \qquad\qquad k = 1 \qquad\qquad k = 2$

Composition of $\mathcal{C}$ fills any horn Λ^2_1 of nerve $N(\mathcal{C})$ uniquely

$$\begin{tikzcd} & c_1 \arrow[dr] & \\ c_0 \arrow[ur] & & c_2 \end{tikzcd}$$

$k = 1$

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

inner horn

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outer horn

$$\begin{tikzcd} & c_1 \arrow[dr, "f"] & \\ c_0 \arrow[rr, "id"'] & & c_2 \end{tikzcd}$$

Λ^2_2

extends, if f has a left inverse.

Proposition 1.4. *Let X be a simplicial set.*

(i) There is an isomorphism $X \cong N(\mathcal{C}), \mathcal{C} \in \mathcal{C}at$, if and only if every inner horn $\Lambda^n_k \to X, 0 < k < n$, *can be* uniquely *extended to an n-simplex $\Delta^n \to X$.*

(ii) There is an isomorphism $X \cong N(\mathcal{G}), \mathcal{G} \in \mathcal{G}rpd$, if and only if every horn $\Lambda^n_k \to X, 0 \leq k \leq n$, *can be* uniquely *extended to an n-simplex $\Delta^n \to X$.*

Definition 1.5. A simplicial set X is a **Kan complex** if *every* horn $\Lambda^n_k \to X$ for $0 \leq k \leq n$ can be extended to an n-simplex $\Delta^n \to X$.

Recall : X top. space $\quad \mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$

$|\Lambda^n_k|$ retract of Δ^n

$\Rightarrow \quad \mathrm{Sing}(X)$ is a Kan complex

all horns extend uniquely

all horns extend

$$\begin{tikzcd} \mathrm{Grp} \arrow[r] \arrow[d] & \mathrm{Cat} \arrow[d] \\ \mathrm{Kan} \arrow[r] & \mathrm{sSet} \end{tikzcd}$$

inner horns extend uniquely

Definition 1.7. A simplicial set $\mathcal{C}$ is an ∞**-category** if every inner horn $\Lambda^n_k \to \mathcal{C}$, $0 < k < n$, can be extended to an n-simplex $\Delta^n \to \mathcal{C}$.

- space $\Rightarrow$ ∞ - category
- $\mathrm{Mfd}_n^{\sqcup}$ with embeddings $\Rightarrow$ ∞ - category
- ordinary category $\Rightarrow$ ∞ - category

Grothendieck hypothesis : "spaces and ∞ - groupoids are the same"

Terminology : ∞-category $\mathcal{C}$

objects = vertices $\quad x \in \mathcal{C}_0$

morphisms = 1-simplices

$$s = d_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$$

$$t = d_0 : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$$

$$id = s_0 : \mathcal{C}_0 \longrightarrow \mathcal{C}_1$$

Simplicial identities $\quad s^0 \circ d^0 :$
\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r, mapsto] & 0 \\ & 0 \arrow[ur, mapsto] & \end{tikzcd}
$= id_0$ and $s^1 \circ d^0 = id_0$

$$d_0 s_0 = d_1 s_0 = id_{\mathcal{C}_0} \quad \Rightarrow \quad id_x \in \mathrm{Hom}(x, x)$$

Composition ? $\quad x \xrightarrow{f} y$ and $y \xrightarrow{g} z$

$\Rightarrow$ inner horn

$$\lambda = (g, \bullet, f) : \Lambda^2_1 \to \mathcal{C}$$

\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] & & \cdot \end{tikzcd}

Non-unique extension $\quad \sigma : \Delta^2 \to \mathcal{C}$

$d_1 \sigma$ is a candidate for non-unique composition

We all know non-unique compositions:

- Algebra: $V, W \in \mathrm{vect}_k$. A tensor product is a v.s. + bilinear map
$$V \times W \xrightarrow{\otimes} V \otimes W$$
$\rightsquigarrow$ unique isomorphisms. Tensor product is a clique, a contractible diagram of vector spaces

- Fundamental group: paths do not compose uniquely

$s_1 : I \to X$

$s_2 : I \to X$

closed loops

$s_2 * s_1$ not unique

but up to homotopy

3.2. The homotopy category of an ∞-category

Def: $f, g : x \to y$ are (left) homotopic, $f \simeq g$, if there is a 2 simplex

$\sigma : \Delta^2 \to \mathcal{C}$

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "id"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

... right homotopic

Prop

1) right homotopic $\iff$ left homotopic

2) Homotopy is an equivalence relation. $[f]$ homotopy class of f.

Proposition 1.15. *Let $\mathcal{C}$ be an ∞-category. There is an ordinary category* $\mathrm{Ho}(\mathcal{C})$*, the* ***homotopy category*** *of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and morphisms the homotopy classes of morphisms in $\mathcal{C}$. Composition and identities are given by*

$$[g] \circ [f] := [g \circ f] \qquad \textit{and} \qquad \mathrm{id}_x := [\mathrm{id}_x] = [s_0 x],$$

where $g \circ f$ is an arbitrary *candidate composition of g and f.* ~~*Furthermore, there is a natural isomorphism of categories* $\mathrm{Ho}(\mathcal{C}) \cong \tau_1(\mathcal{C})$.~~

We have to show that any two choices of composition are homotopic.

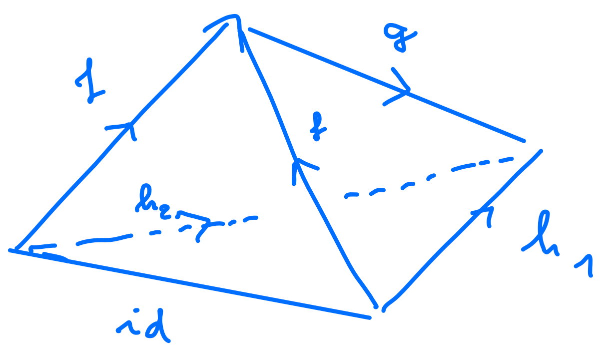
Take $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ and two compositions

$$\begin{tikzcd} & {} \arrow[dr, "g"] & \\ {} \arrow[ur, "f"] \arrow[rr, "h_1"'] & & {} \end{tikzcd} \qquad\qquad \begin{tikzcd} & {} \arrow[dr] & \\ {} \arrow[ur] \arrow[rr, "h_2"'] & & {} \end{tikzcd}$$

σ_1 $\qquad\qquad$ σ_2 $\qquad$ (2-simplices exhibit h_i as a composition)

Now consider

the 3-simplex



bottom face exhibits a homotopy that exhibits .

$h_1 \simeq h_2$

A slide of slogans

Classical math	∞-math
based on sets	based on spaces
vector space	chain complex
category of abelian groups	∞-category of spectra
category	∞-category
abelian categories	stable ∞-category

Set = "soln to an eqn"

Space: "redundant solutions"

many ways to identify them

homotopy types

Types

-1 - type no term $\emptyset$ or one term $*$

0 - type terms + for 2 terms a (-1) type of identifications

"equal or not equal"

$\rightsquigarrow$ set Question: is there a solution?

1 - type terms + for 2 terms a 0 - type.

A set of ways 2 elements are equal.

How are things equal?

2 - type type + 1 - type of identifications

How are things equal? Are 2 ways of seeing that they are equal equal?

...

Why should we care about the ways to see that "things are equal"? Isn't mathematical physics about finding an explicit set of solutions (explicitly).

Important point: local to global needs control on identifications

Ex Computing $\pi_1(X, x_0)$ is essentially understanding the coverings of X.

To glue coverings, we need morphisms of coverings.

An example for propositions in the theory:

Definition 1.20. An ∞-category is an **∞-groupoid** if the homotopy category is a groupoid.

following precise statement (see [Joy02, Corollary 1.4] or [Lur09c, p.35]).

Corollary 1.22. *An ∞-category is an ∞-groupoid if and only if it is a Kan complex.*

Algebraic categories $\xleftrightarrow[\text{math. physics}]{\text{TFT,}}$ geometric categories

$\downarrow$

Example!

$\mathcal{A}$ additive category. $Ch_{\mathcal{A}}$ chain complexes in $\mathcal{A}$ $\rightsquigarrow$ ∞-category

- objects chain complexes $\to M_1 \to M_0 \to M_{-1} \to \dots$
- chain morphisms
- chain homotopies

Thm: $\mathcal{A}$ abelian with enough projectives $\Rightarrow$ stable ∞-category $\bar{\mathcal{D}}(\mathcal{A})$.

Its homotopy category is the "old-fashioned" triangulated derived category.

3.3 Functors and natural transformations

Def: K simplicial set, $\mathcal{C}$ an ∞-category. A functor $F: K \to \mathcal{C}$ is a map of simplicial sets. (A simplicial set is a functor $\Delta^{op} \to \mathrm{Set}$, a map of simplicial sets is a nat. trafo)

Observation: ordinary categories $\mathcal{C}, \mathcal{D}$. A functor

$$F : \underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \quad \longrightarrow \quad \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

amounts to : two functors $F, G : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha : F \Rightarrow G$

$\Rightarrow$ A functor $\quad (\bullet \longrightarrow \bullet) \times \mathcal{C} \longrightarrow \mathcal{D}$

pair of functors + natural transformation

Def: A (pair of functors and α) natural transformation is a map $\Delta^1 \times K$ of simplicial sets.

Def: Space of functors

$$\mathrm{Fun}(K, \mathcal{C})_\bullet = \mathrm{Map}_{sSet}(K, \mathcal{C})_\bullet = \hom_{sSet}(\Delta^\bullet \times K, \mathcal{C})$$

is a simplicial set!

This extends the classical notion using the nerve functor $N : Cat \longrightarrow sSet$

Lemma 2.2. *For categories A, B there is a natural isomorphism of simplicial sets*
$$N(\mathrm{Fun}(A, B)) \cong \mathrm{Fun}(NA, NB).$$

general principles for extending categorical notions to ∞-categories:

[P1] The concepts are extensions of the ordinary concepts in that everything is compatible with the fully faithful nerve functor $N : \mathcal{C}at \to \mathrm{s}\mathcal{S}et$.

[P2] The notions are *coherent* variants of the classical notions, i.e., ∞-category theory realizes a homotopy coherent category theory.

[P3] The extensions of concepts are often defined for arbitrary simplicial sets, and when applied to ∞-categories we want these extensions to again give rise to ∞-categories.

[P4] All concepts are *invariant concepts*, i.e., an application of these constructions to equivalent input ∞-categories yields equivalent output ∞-categories.

4. Back to multicategories

4.1. Back to vector spaces

- Vector space $U \otimes_k V$ is defined only up to unique isomorphisms ("Clique")
- Hence, no reason to expect equality $(U \otimes V) \otimes W$ $U \otimes (V \otimes W)$
- Reason for existence of isomorphism: trilinear maps

$$\mathrm{Hom}_k\big((U \otimes V) \otimes W, X\big) \cong \mathrm{Trilin}(U, V, W \mid X) \cong \mathrm{Hom}_k\big(U \otimes (V \otimes W), X\big)$$

Idea: Monoidal cat $\mathcal{C}$ = category with (a lot of) structure

$\rightsquigarrow$ encode this via a new category $\mathcal{C}^{\otimes}$ + a nice functor to a "combinatorial" category

New category: given $(\mathcal{C}, \otimes)$ symmetric monoidal category $\mathcal{C}^{\otimes}$

- Objects: finite (possibly empty) sequences of objects
  $[c_1, \dots, c_n]$

- Morphism:
  $$[c_1, \dots, c_n] \longrightarrow [c'_1 \dots c'_m]$$

Two data:

- $S \subset \{1 \dots n\}$ + map $\alpha : S \to \{1 \dots m\}$

- family of morphisms
  $$\Big\{ f_j : \underbrace{\bigotimes_{\alpha(i)=j} c_i}_{\text{well-defined since } \mathcal{C} \text{ symmetric}} \longrightarrow c'_j \Big\}_{1 \leq j \leq m} \quad \text{in } \mathcal{C}$$

- <u>Composition</u> $[c_1, \dots c_n] \xrightarrow{f} [c'_1 \dots c'_m] \longrightarrow [c''_1 \dots c''_\ell]$

$S \subset \{1, \dots, n\}$ $\qquad$ $T \subset \{1, \dots, m\}$

$\alpha : S \longrightarrow \{1 \dots m\}$ $\qquad$ $\beta : T \longrightarrow \{1 \dots \ell\}$

$U := \alpha^{-1} T$ $\qquad$ $\beta \circ \alpha : U \longrightarrow \{1 \dots \ell\}$

$$\bigotimes_{(\beta\circ\alpha)(i)=k} C_i \cong \bigotimes_{\beta(j)=k} \bigotimes_{\alpha(i)=j} C_i \longrightarrow \bigotimes_{\beta(j)=k} C'_j \longrightarrow C''_k$$

<u>Def</u>: I a finite set $\rightsquigarrow$ $I_* := I \sqcup \{*\}$ finite pointed set

$\langle n \rangle^\circ := \{1, \dots, n\}$

$\langle n \rangle = \langle n \rangle^\circ_* = \{*, 1, \dots n\}$

Category Fin_* : objects are $\langle n \rangle$

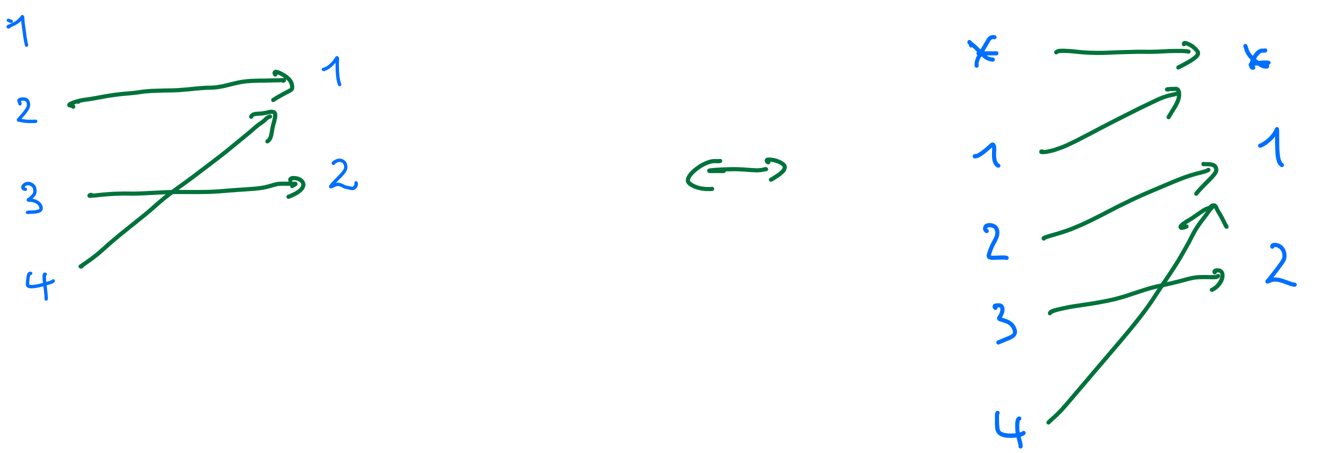
morphisms preserve marked element

Remarks on Fin_*

- equivalent to category of finite pointed sets
- Morphism $\langle n \rangle \longrightarrow \langle m \rangle$

$\longleftrightarrow$ partially defined map from $\langle n \rangle^\circ$ to $\langle m \rangle^\circ$

$$\begin{array}{ccc} \langle n \rangle^\circ & & \\ \cup & & \\ S & \longrightarrow & \langle m \rangle^\circ \end{array}$$



(" $*$ is the waste bin")

$\langle 4 \rangle^\circ$ $\quad$ $\langle 2 \rangle^\circ$ $\qquad\qquad$ $\langle 4 \rangle$ $\quad$ $\langle 2 \rangle$

- There is an obvious forgetful functor $\mathcal{C}^\otimes \longrightarrow Fin_*$

- There is an obvious forgetful functor $\mathcal{C}^{\otimes} \xrightarrow{p} Fin_*$

$$[c_1, \dots, c_n] \longmapsto \langle n \rangle$$

$$[c_1 \dots c_n] \longrightarrow [c'_1 \dots c'_m] \longmapsto \tilde{\alpha} : \langle n \rangle \longrightarrow \langle m \rangle$$

consists of
$$\begin{cases} \langle n \rangle^{\circ} \supset S \xrightarrow{\alpha} \{1, \dots, m\} = \langle m \rangle^{\circ} \\ \{f_j\} \end{cases}$$

The idea is that this functor $\mathcal{C}^{\otimes} \xrightarrow{p} Fin_*$ is so nice that it encodes the structure of a symmetric monoidal category.

- This then allows us to define symmetric monoidal categories in the ∞-world.

- If we replace finite sets by finite ordered sets, we get monoidal categories rather than symmetric monoidal categories

with order preserving maps as morphisms

Why is p nice? $\to$ op fibration

$\to$ Segal property

4.2 Opfibrations

Def $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{D}\end{array}$ fibre of p over $d \in \mathcal{D}$ is

$$\begin{tikzcd} \mathcal{E}_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\ {[0]} \arrow[r, "d"'] & \mathcal{D} \end{tikzcd}$$

$\mathcal{E}_d$ objects: $c \in \mathcal{E}$ s.t. $p(c) = d$

morphisms: $c \xrightarrow{f} c'$ s.t. $p(f) = id_d$

get an assignment $\mathcal{D} \rightsquigarrow \mathcal{C}at$

$d \longmapsto \mathcal{E}_d$

„Collection of categories parametrized by $\mathcal{D}$"

Q: Can we turn it into something like a functor?

Def : Situation :
$$\begin{tikzcd} \mathcal{E} \arrow[d, "p"] \\ \mathcal{D} \end{tikzcd}$$

Given $c_1 \xrightarrow{f} c_2$

with $d_1 \xrightarrow[p(f)=\alpha]{} d_2$ $\quad \downarrow p$

f is called a co-Cartesian lift of α, if

$\forall \quad c_1 \xrightarrow{h} c_3$

$d_1 \xrightarrow[\gamma = p(h)]{} d_3$

and all $\quad d_2 \xrightarrow{\beta} d_3$

$$\begin{tikzcd} c_1 \arrow[r, "f"] \arrow[rr, bend right, "h"'] & c_2 \arrow[r, dotted, "\exists !"] & c_3 \end{tikzcd}$$

$\downarrow p$

$$\begin{tikzcd} d_1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\gamma = p(h)"'] & d_2 \arrow[r, "\beta\ \forall"] & d_3 \end{tikzcd}$$

given h and β, get unique

$c_2 \to c_3$

NB :

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{E}}(c_2, c_3) \arrow[r, "f^*"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Hom}_{\mathcal{E}}(c_1, c_3) \arrow[d, "p"] \\ \mathrm{Hom}_{\mathcal{D}}(pc_2, pc_3) \arrow[r, "(pf)^* = \alpha^*"'] & \mathrm{Hom}_{\mathcal{D}}(pc_1, pc_3) \end{tikzcd}$$

is a pullback.

Lemma

Suppose $c \xrightarrow{f'} c'$, $c \xrightarrow{f''} c''$ are two co-Cartesian morphisms w/ same image under p in $\mathcal{D}$

$$\alpha = p f = p f'$$

Then

$$\begin{tikzcd}
c \arrow[r, "f'"] \arrow[dr, bend right, "f''"'] & c' \arrow[d, dashed, "\exists!"] \\
 & c''
\end{tikzcd}$$

$$\begin{tikzcd}
d \arrow[r, "\alpha"] \arrow[dr, bend right, "\alpha"'] & d' \arrow[d, dashed, "id"] \\
 & d'
\end{tikzcd}$$

„co-Cartesian lifts are essentially unique"

Def: $\begin{matrix}\mathcal{C} \\ \downarrow p \\ \mathcal{D}\end{matrix}$ is an <u>opfibration</u>, if for all $c_1 \in \mathcal{C}$ and all $p(c_1) \to d$ in $\mathcal{D}$, there is a co-Cartesian lift.

„enough co-Cartesian lifts"

Construction

Suppose $\begin{array}{c}\mathcal{C}\\ \downarrow\\ \mathcal{D}\end{array}$ is opfibration

Choose for each $c \in \mathcal{C}$ in $\mathcal{C}$ and each $p(c) \longrightarrow d$ in $\mathcal{D}$ a cc lift.

For $\alpha : d_1 \to d_2$ in $\mathcal{D}$ define a functor

$$\begin{aligned} \alpha_! \quad : \quad & \mathcal{C}_{d_1} \longrightarrow \mathcal{C}_{d_2} \\ & c_1 \longmapsto c_2 \end{aligned}$$

where c_2 is the codomain of the chosen lift $f : c_1 \to c_2$ of α.

$\rightsquigarrow$ functor

$\rightsquigarrow$ for $d_1 \xrightarrow{\alpha} d_2 \xrightarrow{\beta} d_3$

$$\begin{tikzcd} \mathcal{C}_{d_1} \arrow[r, "\alpha_!"] \arrow[rr, bend right, "(\beta\circ\alpha)_!"'] & \mathcal{C}_{d_2} \arrow[r, "\beta_!"] \arrow[d, phantom, "\Downarrow \exists !"] & \mathcal{C}_{d_3} \\ & {} & \end{tikzcd}$$

Pseudo functor $\quad \mathcal{D} \longrightarrow \mathcal{C}at$

("hide structure") $\quad d \longmapsto \mathcal{C}_d$

4.3 Back to symmetric monoidal categories

(M1) $\begin{array}{c}\mathcal{C}^{\otimes} \\ \downarrow p \\ \mathrm{Fin}_*\end{array}$ is an opfibration

Indeed, take $[c_1, \dots, c_n] \in \mathcal{C}^{\otimes}$ and $\langle n \rangle \xrightarrow{\alpha} \langle m \rangle$

We have to find a co-Cartesian lift for α.

Choose objects c_j' and isomorphisms for $j = 1 \dots m$

$$c_j' \xrightarrow{\sim} \bigotimes_{\alpha(i)=j} c_i$$

These isos combine into a morphism

$$[c_1, \dots, c_n] \longrightarrow [c_1', \dots c_m']$$

which is a co-Cartesian lift of α.

Every morphism $\langle m \rangle \xrightarrow{\alpha} \langle n \rangle$ in Fin_* induces a functor

$$\mathcal{C}^{\otimes}_{\langle m \rangle} \longrightarrow \mathcal{C}^{\otimes}_{\langle n \rangle}$$

well-defined up to canonical isomorphism.

Notation : for $1 \leq i \leq n$ denote $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$

$$\rho^i(j) = \begin{cases} 1 & \text{if } i = j \\ * & \text{else} \end{cases}$$

(M2) Denote by $\mathcal{C}^{\otimes}_{\langle n \rangle}$ the fiber of $\langle n \rangle$. [Segal condition]

Equivalence : $\mathcal{C}_{\langle 1 \rangle} \simeq \mathcal{C}$

$\rho^i : \langle n \rangle \to \langle 1 \rangle$ induces equivalence $\mathcal{C}_{\langle n \rangle} \simeq \underbrace{\mathcal{C} \times \ldots \times \mathcal{C}}_{n\text{-times}}$

We show a converse : suppose $\mathcal{D} \xrightarrow[p]{} Fin_*$ is an opfibration obeying the Segal condition.

Then $\mathcal{C} := \mathcal{D}_{\langle 1 \rangle}$ has the structure of a symmetric monoidal category.

(a) (M2) implies that $\mathcal{D}_0$ has, up to equ., one object.

Unique morphism $\langle 0 \rangle \to \langle 1 \rangle \rightsquigarrow$ functor $\mathcal{D}_{\langle 0 \rangle} \to \mathcal{D}_{\langle 1 \rangle} = \mathcal{C}$

$\rightsquigarrow$ object $\mathbb{1} \in \mathcal{C}$

(b) $\alpha : \langle 2 \rangle \to \langle 1 \rangle$

$$\begin{aligned} * &\mapsto * \\ 1 &\mapsto 1 \\ 2 &\nearrow \end{aligned}$$

$$\mathcal{C} \times \mathcal{C} = \mathcal{D}_{\langle 1 \rangle} \times \mathcal{D}_{\langle 1 \rangle} \xleftarrow{\rho^1 \times \rho^2} \mathcal{D}_{\langle 2 \rangle} \xrightarrow[\alpha]{} \mathcal{D}_{\langle 1 \rangle} = \mathcal{C}$$

equivalence by (M2) = Segal $\rightsquigarrow \otimes$

(c) $\sigma : \langle 2 \rangle \to \langle 2 \rangle$

$$\begin{aligned} * &\mapsto * \\ 1 &\searrow\swarrow 1 \\ 2 &\nearrow\nwarrow 2 \end{aligned}$$

$\Rightarrow$ isomorphism $\otimes^{op} \to \otimes$

Leads to definition:

Definition 2.0.0.7. A *symmetric monoidal ∞-category* is a coCartesian fibration of simplicial sets $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with the following property:

(∗) For each $n \geq 0$, the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$ induce functors $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ which determine an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$.

4.4 Back to multicategories

Construction 2.1.1.7. Let $\mathcal{O}$ be a colored operad. We define a category $\mathcal{O}^{\otimes}$ as follows:

(1) The objects of $\mathcal{O}^{\otimes}$ are finite sequences of colors $X_1, \ldots, X_n \in \mathcal{O}$.

(2) Given two sequences of objects

$$X_1, \ldots, X_m \in \mathcal{O} \qquad Y_1, \ldots, Y_n \in \mathcal{O},$$

a morphism from $\{X_i\}_{1 \le i \le m}$ to $\{Y_j\}_{1 \le j \le n}$ is given by a map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ together with a collection of morphisms

$$\{\phi_j \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in \alpha^{-1}\{j\}}, Y_j)\}_{1 \le j \le n}$$

in $\mathcal{O}$.

(3) Composition of morphisms in $\mathcal{O}^{\otimes}$ is determined by the composition laws on $\mathcal{F}\mathrm{in}_*$ and on the colored operad $\mathcal{O}$.

One can again reconstruct the operad from the forgetful functor

$$\mathcal{O}^{\otimes} \longrightarrow \mathrm{Fin}_*$$

$$\mathcal{O} := \mathcal{O}^{\otimes}_{\langle 1 \rangle} = \pi^{-1}\{\langle 1 \rangle\} \qquad\qquad \mathcal{O}^{\otimes}_{\langle n \rangle} \cong \mathcal{O}^{n}$$

$X_1 \dots X_n \in \mathcal{O} \quad \rightsquigarrow \quad \vec{X} \in \mathcal{O}^{\otimes}_{\langle n \rangle}$

$\mathrm{Mul}_{\mathcal{O}}(X_1 \dots X_n \mid Y) = \{ f : \vec{X} \longrightarrow Y$ in $\mathcal{O}^{\otimes}$ s.t.

$\pi(f) : \langle n \rangle \longrightarrow \langle 1 \rangle$ s.t.

$\pi(f)^{-1}\{*\} = \{*\}$

Definition 2.1.1.10. An ∞-*operad* is a functor $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ between ∞-categories which satisfies the following conditions: . . .