Refs: Costello-Gwilliam, volume 1
B. Williams: 1603.02349

1. Brief recap of Lie algebras

$K = \mathbb{C}$ (at least, $\mathfrak{g}$ should be projective K-module)

<u>Def</u>: Lie algebra $[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$
+ antisymmetry + Jacobi

<u>Ex</u>: 1. $M_n(K)$ $[A,B] = AB - BA$
2. $\mathrm{Vect}(M)$, M smooth mfd

<u>Def</u>: Module $(M, \rho : \mathfrak{g} \to \mathrm{End}(M))$
Category $\mathfrak{g}$-mod

<u>Ex</u> $C^\infty(M)$ is a $\mathrm{Vect}(M)$ module

<u>Two functors</u>:

<u>Invariants</u>: $(-)^{\mathfrak{g}} : \mathfrak{g}\text{-mod} \longrightarrow \mathrm{vect}_K$
$M \longmapsto M^{\mathfrak{g}}$

$$M^{\mathfrak{g}} = \{ m \mid x.m = 0 \ \forall x \in \mathfrak{g} \}$$

<u>Coinvariants</u> $(-)_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \longrightarrow \mathrm{vect}_K$
$M \longmapsto M_{\mathfrak{g}}$

$$M_{\mathfrak{g}} = M / \mathfrak{g}.M$$

Universal enveloping algebra

$$\begin{tikzcd} \mathrm{Ass\,Alg}_k \arrow[d, bend left=60, "\mathrm{Forget}"] \arrow[d, phantom, "\dashv"] \\ \mathrm{Lie\,Alg}_k \arrow[u, bend left=60, "U"] \end{tikzcd}$$

forget, take commutator

$$U\mathfrak{g} = \mathrm{Tens}(\mathfrak{g}) \Big/ x\otimes y - y\otimes x - [x,y]$$

- filtered associative algebra
- Hopf algebra, $\varepsilon : U(\mathfrak{g}) \to k$ augmentation
- $\mathrm{Hom}_{Lie}(\mathfrak{g}, \mathrm{Forget}\,\mathrm{End}(M)) \cong \mathrm{Hom}_{Alg}(U\mathfrak{g}, \mathrm{End}(M))$

$$\mathfrak{g}\text{-mod} \cong U\mathfrak{g}\text{-mod}$$

Hence $\mathfrak{g}$-mod has enough projectives.

- K trivial $\mathfrak{g}$-module $\quad x.k = 0 \quad \forall x \in \mathfrak{g},\ k \in K$
- K $U\mathfrak{g}$-bimodule $\quad K \cong U(\mathfrak{g})/\ker\varepsilon$

Monoidal: $U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$

<u>Ex</u>

$M_{\mathfrak{g}} = K \otimes_{U\mathfrak{g}} M$ coinvariants right exact

$M^{\mathfrak{g}} = \mathrm{Hom}_{U\mathfrak{g}}(K, M)$ left exact.

Def Lie algebra homology of $M \in \mathfrak{g}$-mod

$$H_*(\mathfrak{g}, M) = \mathrm{Tor}_*^{U\mathfrak{g}}(K, M)$$

Lie algebra cohomology of $M \in \mathfrak{g}$-mod

$$H^*(\mathfrak{g}, M) = \mathrm{Ext}^*_{U\mathfrak{g}}(K, M)$$

Fact: $H^1(\mathfrak{g}, M)$ classifies derivations mod inner derivations

Def An extension $\hat{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is a Lie algebra that fits in an exact sequence of Lie algebras

$$0 \to M \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

where M is an abelian Lie algebra.

Rmk:

Pick preimage $\hat{x} \in \hat{\mathfrak{g}}$ of $x \in \mathfrak{g}$. Then $[\hat{x}, m]$ is independent on choice of $\hat{x}$ and turns M into a $\mathfrak{g}$-module.

$H^2(\mathfrak{g}, M)$ classifies extensions of $\mathfrak{g}$ by M as Lie algebra

Def: A central extension of $\mathfrak{g}$ is an extension s.t. M is in the center of $\hat{\mathfrak{g}}$, i.e. $[\lambda, x] = 0 \quad \forall x \in \mathfrak{g}$.

Classified by $H^2(\mathfrak{g}) := H^2(\mathfrak{g}, M_{\mathrm{triv}})$

2. Standard cochain complexes

Key: Efficient free resolution of trivial $U(\mathfrak{g})$-module K.

$$C^n(\mathfrak{g}) := \Lambda^n \mathfrak{g} \otimes U(\mathfrak{g}) \qquad \text{free right } U(\mathfrak{g}) \text{ module.}$$

The situation becomes subtle concerning the tensor products, if $\mathfrak{g}$ is not finite-dimensional.

We assume that $\mathfrak{g}$ is a nuclear vector space w/ nuclear tensor product.

Cartan-Eilenberg differential:

$$y_1 \wedge \dots y_n \otimes (x_1 \dots x_m) \longmapsto$$

$$\sum_{k=1}^{n} (-1)^{n-k} (y_1 \wedge \dots \wedge \hat{y}_k \wedge \dots \wedge y_n) \otimes (y_k x_1 \dots x_m)$$

$$- \sum_{1 \le j < k \le n} (-1)^{j+k-1} ([y_j, y_k] \wedge y_1 \dots \hat{y}_j \wedge \dots \hat{y}_k \dots \wedge y_n$$

$$\otimes (x_1 \dots x_m)$$

Free resolution of K (cf. Weibel, chapter 7.7)

Thus :

$$K \otimes^{L}_{U(\mathfrak{g})} M \simeq \left(\to \cdots \to \Lambda^n \mathfrak{g} \otimes_K M \to \cdots \to \mathfrak{g} \otimes_K M \to M \right)$$

$$R\mathrm{Hom}_{U(\mathfrak{g})}(K, M) \simeq \left(M \to \mathfrak{g}^* \otimes_K M \to \cdots \to \Lambda^n \mathfrak{g}^* \otimes_K M \to \cdots \right)$$

Definition A.4.2 The *Chevalley–Eilenberg complex for Lie algebra homology* of the $\mathfrak{g}$-module M is

$$C_*(\mathfrak{g}, M) = (\mathrm{Sym}_{\mathbb{K}}(\mathfrak{g}[1]) \otimes_{\mathbb{K}} M, d)$$

where the differential d encodes the bracket of $\mathfrak{g}$ on itself and on M. Explicitly, we have

$$d(x_1 \wedge \cdots \wedge x_n \otimes m) = \sum_{1 \le j < k \le n} (-1)^{j+k} [x_j, x_k] \wedge x_1 \wedge \cdots \widehat{x_j} \cdots \widehat{x_k} \cdots \wedge x_n \otimes m$$
$$+ \sum_{j=1}^{n} (-1)^{n-j} x_1 \wedge \cdots \widehat{x_j} \cdots \wedge x_n \otimes [x_j, m].$$

Rmk: $\mathfrak{g}$ abelian, M trivial $\Rightarrow$ differential vanishes.

We often call this complex the *Chevalley–Eilenberg chains*.

The *Chevalley–Eilenberg complex for Lie algebra cohomology* of the $\mathfrak{g}$-module M is

$$C^*(\mathfrak{g}, M) = (\mathrm{Sym}_{\mathbb{K}}(\mathfrak{g}^\vee[-1]) \otimes_{\mathbb{K}} M, d)$$

where the differential d encodes the linear dual to the bracket of $\mathfrak{g}$ on itself and on M. Fixing a linear basis $\{e_k\}$ for $\mathfrak{g}$ and hence a dual basis $\{e^k\}$ for $\mathfrak{g}^\vee$, we have

$$d(e^k \otimes m) = -\sum_{i<j} e^k([e_i, e_j]) e^i \wedge e^j \otimes m + \sum_l e^k \wedge e^l \otimes [e_l, m]$$

and we extend d to the rest of the complex as a derivation of cohomological degree 1 (i.e., use the Leibniz rule repeatedly to reduce to the preceding text). We often call this complex the *Chevalley–Eilenberg cochains*.

Special case: trivial module K

$C^*(\mathfrak{g}) := C^*(\mathfrak{g}, K)$ commutative dg algebra

"functions on $B\mathfrak{g}$"

$d_{CE}(a \wedge b) = [a,b]$, continue as differential.

$C_*(\mathfrak{g}) := C_*(\mathfrak{g}, K)$ cocommutative dg coalgebra

"distributions on $B\mathfrak{g}$"

Generalization to

Def: A dg Lie algebra over a commutative ring R is a $\mathbb{Z}$-graded R-module $\mathfrak{g}$ with

(1) A differential

$$\xrightarrow{d} \mathfrak{g}^{-1} \xrightarrow{d} \mathfrak{g}^{0} \xrightarrow{d} \mathfrak{g}^{1}$$

$(\mathfrak{g}, d)$ is a dg R-module

(2) There is a bilinear bracket $[-,-] : \mathfrak{g} \otimes_R \mathfrak{g} \to \mathfrak{g}$ such that

- $[x,y] = -(-1)^{|x||y|}[y,x]$ (graded antisymmetry)
- $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$ (graded Leibniz rule)
- $[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$ (graded Jacobi rule),

where $|x|$ denotes the cohomological degree of $x \in \mathfrak{g}$.

Generalization: L_∞ algebras with higher brackets

Examples

(a) We construct the dg analog of $\mathfrak{gl}_n$. Let (V, d_V) be a cochain complex over $\mathbb{K}$. Let $\mathrm{End}(V) = \oplus_n \mathrm{Hom}^n(V,V)$ denote the graded vector space where Hom^n consists of the linear maps that shift degree by n, equipped with the differential

$$d_{\mathrm{End}\, V} = [d_V, -] : f \mapsto d_V \circ f - (-1)^{|f|} f \circ d_V.$$

The commutator bracket makes $\mathrm{End}(V)$ a dg Lie algebra over $\mathbb{K}$.

(b) For M a smooth manifold and $\mathfrak{g}$ an ordinary Lie algebra (such as $su(2)$), the tensor product $\Omega^*(M) \otimes_{\mathbb{R}} \mathfrak{g}$ is a dg Lie algebra where the differential is simply the exterior derivative and the bracket is

Differential forms

$$[\alpha \otimes x, \beta \otimes y] = \alpha \wedge \beta \otimes [x,y].$$

We can view this dg Lie algebra as living over $\mathbb{K}$ or over the commutative dg algebra $\Omega^*(M)$. This example appears naturally in the context of ~~gauge theory.~~ field theory.

CE complex for dg Lie algebras via double complexes.

$$d = d_{\mathfrak{g}} + d_{CE} \qquad\qquad d = d_{\mathfrak{g}}^{\vee} + d_{CE}^{\vee}$$

Theorem: Symmetric monoidal functor

$$CE : (\text{dg-Lie}, \oplus) \longrightarrow Ch_{\mathbb{K}}$$

Def

$\mathfrak{g}$ a dg-Lie algebra, $k \in \mathbb{Z}$. k-shifted central extension of $\mathfrak{g}$

$$0 \to \mathbb{C}[k] \longrightarrow \hat{\mathfrak{g}} \to \mathfrak{g} \longrightarrow 0$$

with $[\lambda, x] = 0$.

$H^{2+k}(\mathfrak{g})$ classifies extensions where $\hat{\mathfrak{g}}$ is an L_∞ algebra.

Example : Complex manifolds

X complex mfd, $T^{1,0}X \oplus T^{0,1}X$

$TX := T^{(1,0)}X$ holomorphic tangent bundle.

$d_{dR} : \mathcal{O}(X) \longrightarrow \Omega^1(X) = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X)$

splits $d_{dR} = \partial + \bar\partial$.

Dolbeault resolution of $\mathcal{O}^{hol}(TX)$ (holomorphic vector fields)

$$\Omega^{0,0}(X,TX) \xrightarrow[\bar\partial]{} \Omega^{0,1}(X,TX) \xrightarrow{\bar\partial} \Omega^{0,2}(X,TX) \longrightarrow \ldots$$

dg Lie algebra $\mathcal{L}^X = \Omega^{0,*}(X,TX)$

by extending Lie bracket of holomorphic vector fields.

3. $U\mathfrak{g}$ as a prefactorization algebra on $\mathbb{R}$

Definition A.5.3 A *precosheaf* of vector spaces on a space X is a functor $\mathcal{G}$: $\mathrm{Opens}_X \to \mathit{Vect}$. A *cosheaf* is a precosheaf such that for every open U and every cover $\mathfrak{U} = \{V_i\}_{i \in I}$ of U, we have

coproducts

$$\operatorname{colim}\left(\coprod_{i,j \in I} \mathcal{G}(V_i \cap V_j) \rightrightarrows \coprod_{i \in I} \mathcal{G}(V_i) \right) \xrightarrow{\cong} \mathcal{G}(U),$$

where the map into $\mathcal{G}(U)$ is the coproduct of the *extension* maps from the V_i to U and where, in the colimit diagram, the top arrow is extension from $V_i \cap V_j$ to V_i and the bottom arrow is extension from $V_i \cap V_j$ to V_j.

Ex for precosheaf:

say: real-valued

$\mathcal{E}_c(U)$ compactly supported functions on U

For $U \subset V$ extend $s \in \mathcal{E}_c(U)$ to

$ext_{U \subset V}(s)$ by zero on $V \setminus U$.

Def: $\left(\Omega^*_c(U) \otimes \mathfrak{g}, d_{dR} \right)_{U \in Op(\mathbb{R})} =: \mathfrak{g}^{\mathbb{R}}$

- Cosheaf of cochain complexes.
- Precosheaf of dg Lie algebras (coproduct of dg Lie algebras is not direct sum, cf. Agore 0905.2613.)

A *prefactorization algebra* $\mathcal{F}$ on M with values in $\mathcal{C}^{\otimes}$ is an assignment of an object $\mathcal{F}(U)$ of $\mathcal{C}$ for each open $U \subset M$ together with the following data:

- For $U \subset V$, a morphism $\mathcal{F}(U) \to \mathcal{F}(V)$.
- For any finite collection $\{U_i\}$ of pairwise disjoint opens in an open $V \subset M$ a morphism

$$\otimes_i \mathcal{F}(U_i) \to \mathcal{F}(V).$$

- Coherences between the above two sets of data.

Proposition [Costello - Gwilliam, Prop. 3.4.1]

Cohomology prefactorization algebra

$$\mathcal{H}(U) = H^*\left(C_*\left(\mathfrak{g}^{\mathbb{R}}(U)\right)\right)$$

CE complex

is locally constant. The corresponding associative algebra is isomorphic to universal enveloping algebra $U\mathfrak{g}$.

The construction generalizes from $\mathbb{R}$ to any smooth manifold M of any dimension.

Sketch of proof

- $I \subset J$ inclusion of intervals $\Rightarrow$ qis

$$\Omega^*_c(I) \otimes \mathfrak{g} \longrightarrow \Omega^*_c(J) \otimes \mathfrak{g}$$

- $$A_{\mathfrak{g}} := \mathcal{H}(\mathbb{R}) = H^*\left(C_*\left(\Omega^*_c(\mathbb{R}) \otimes \mathfrak{g}\right)\right)$$

  is associative algebra by general theorem.

- dg-Lie $\underbrace{\Omega^*_c(\mathbb{R}) \otimes \mathfrak{g}}_{\text{concentrated in degree } 0,1} \underset{\text{qis}}{\simeq} H^*_c(\mathbb{R}) \otimes \mathfrak{g} = \underbrace{\mathfrak{g}[-1]}_{\text{concentrated in degree 1.}}$

  cup product is zero $\Rightarrow$ abelian

- $$C_*\left(\Omega^*_c(\mathbb{R}) \otimes \mathfrak{g}\right) \underset{\text{qis}}{\simeq} CE\big(\underbrace{\mathfrak{g}[-1]}_{\text{abelian}}\big) = \mathrm{Sym}\, \mathfrak{g}$$

  $A_{\mathfrak{g}} \simeq \mathrm{Sym}\, \mathfrak{g}$ as a vector space.

- Fix bump function $\varepsilon \in H^1_c(I)$ $\quad \int_{\mathbb{R}} \varepsilon = 1$.

$$\Phi : \begin{array}{ccc} \mathfrak{g} & \longrightarrow & A_{\mathfrak{g}} \\ X & \longmapsto & X \otimes \varepsilon \end{array}$$

Claim: this is a morphism of Lie algebras

- Pick $f_0 \in C_c^\infty(-\delta, \delta)$ with $\int f_0 \, dx = 1$

$$f_t(x) := f_0(x-t) \in C_c^\infty(t-\delta, t+\delta)$$

Cochain repn. for $\Phi(X)$ with $X \in \mathfrak{g}$ is

$$f_t \, dx \otimes X \qquad \text{for any } t \in \mathbb{R}$$

- Take $\delta > 0$ so small that $(-\delta, \delta)$ and $(1-\delta, 1+\delta)$ are disjoint.

Product $\Phi(X) \cdot \Phi(Y)$ for $X, Y \in \mathfrak{g}$ is represented by

$$\begin{array}{c} (f_0 \, dx \otimes X)(f_1 \, dx \otimes Y) \in \mathrm{Sym}^2(\Omega_c^1(\mathbb{R}) \otimes \mathfrak{g}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \cap\! | \\ \qquad\qquad\qquad\qquad\qquad\qquad C_*(\Omega_c^*(\mathbb{R}) \otimes \mathfrak{g}) \end{array}$$

Commutator is represented by

$$(f_0 \, dx \otimes X)(f_1 \, dx \otimes Y) - (f_0 \, dx \otimes Y)(f_{-1} \, dx \otimes X)$$

$$\int (f_1 - f_{-1}) = 0 \implies \exists \; h \in C_c^\infty(\mathbb{R}) \text{ s.t.}$$

$$d_{dR} h = f_{-1} \, dx - f_1 \, dx$$

h takes value 1 on $(-\delta, \delta)$

Consider $\left(f_0 \, dx \otimes X \right) \left(h \otimes X \right) \in C_* \left(\Omega_c^*(\mathbb{R}) \otimes \mathfrak{g} \right)$

$$d\left(\left(f_0 \, dx \otimes X \right) \left(h \otimes X \right) \right)$$

$$= \left(f_0 \, dx \otimes X \right) \left(dh \otimes Y \right) + f_0 \, h \, dx \otimes [X, Y]$$

$$= \left(f_0 \, dx \otimes X \right) \left(\left(f_{-1} - f_1 \right) dx \otimes Y \right)$$

$$+ f_0 \, h \, dx \otimes [X, Y]$$

Since $h = 1$ on $(-\delta, \delta)$, we have $f_0 h = f_0$. $\square$

4. The factorization envelope

Goal: Kac-Moody vertex algebra and the Virasoro algebra as a twisted factorization envelope.

Motivations:

- in 2d CFT
- in 4d ?
- Morphism Fact envelope $\longrightarrow$ Fact. algebra for QFT expresses symmetries (anomalies ~> central extensions)

Def: A local dg Lie algebra on a mfd M are the data

- graded vector bundle L on M. $\mathcal{L}$ sheaf of smooth sections
- differential operator $d : \mathcal{L} \to \mathcal{L}$ of degree one and square zero.
- $[\ ,\ ] : \mathcal{L}^{\otimes 2} \to \mathcal{L}$ antisymmetric,

sheaf of L_∞ algebras

4.1 Idea

$\mathcal{L}$ (fine) sheaf of dg Lie algebras on M

$\mathcal{L}_c$ cosheaf of compactly supported sections of $\mathcal{L}$

of chain complexes, not of dg Lie algebras

$(\mathcal{L}_c, \oplus)$ prefactorization algebra

Recall:

A *prefactorization algebra* $\mathcal{F}$ on M with values in $\mathcal{C}^{\otimes}$ is an assignment of an object $\mathcal{F}(U)$ of $\mathcal{C}$ for each open $U \subset M$ together with the following data:

- For $U \subset V$, a morphism $\mathcal{F}(U) \to \mathcal{F}(V)$.
- For any finite collection $\{U_i\}$ of pairwise disjoint opens in an open $V \subset M$ a morphism

$$\otimes_i \mathcal{F}(U_i) \to \mathcal{F}(V).$$

- Coherences between the above two sets of data.

$$\bigoplus \mathcal{L}_c(U_i) = \mathcal{L}_c\left(\bigsqcup U_i\right) \longrightarrow \mathcal{L}_c(V)$$

for $\{U_i\}$ finite collection of disjoint opens in V.

Recall $CE : (\text{dg-Lie}, \oplus) \to Ch_k$ is monoidal

<u>Def</u>: Given $\mathcal{L}$ sheaf of dg Lie algebras on M, factorization envelope:

$$\mathbb{U}\mathcal{L}(V) := C_*\left(\mathcal{L}_c(V)\right)$$

$$\bigotimes_i C_* \mathcal{L}_c(U_i) \cong C_*\left(\bigoplus_i \mathcal{L}_c(U_i)\right)$$

$$\longrightarrow C_* \mathcal{L}_c(V)$$

(CG Thm 6.5.3 : is a factorization algebra, i.e. satisfies co-descent)

<u>Example</u>

$\Omega^*_{\mathbb{R}^n} \otimes \mathfrak{g} \rightsquigarrow$ locally constant factorization algebra E_n - enveloping algebra of $\mathfrak{g}$. (Knudsen HA)

Recall descent for sheaves:

$U \subset M$ open $\mathfrak{U} = \{U_i\}$ open cover of U, then

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram

Cosheaf : dualize : coequalizes

$$\coprod_{i,j} F(U_i \cap U_j) \rightrightarrows \coprod_i F(U_i) \longrightarrow F(U)$$

Impose this for

<u>Def</u> An open cover $\mathcal{U} = \{U_i\}$ of a top. space is a <u>Weiss cover</u>, if for any finite collection $\{x_1, \dots x_n\}$ of pts in M there is $U_i \in \mathcal{U}$ s.t. $\{x_1, \dots, x_n\} \subset U_i$.

<u>Examples</u> :

1. $M = [0,2]$ $\quad U_i = [0,2] \setminus \{\frac{1}{i}\}$ $\quad i \in \mathbb{N}$

2. M Riemannian mfd. Opens that are disjoint unions of balls of radius $\varepsilon > 0$.

<u>Remark</u>:

Weiss covers form a Grothendieck topology on the poset of open subsets of M.

Definition 2.1. A ***Grothendieck topology*** on a category C is a set T of families of maps $\{\phi_i : U_i \to U\}_{i \in I}$ (known as ***coverings***) such that

- for any isomorphism ϕ we have $\{\phi\} \in T$;
- if $\{U_i \to U\} \in T$ and $\{V_{i,j} \to U_i\} \in T$ for each i, then $\{V_{i,j} \to U\} \in T$; (covers refine)
- if $\{U_i \to U\} \in T$ and $V \to U$ is a morphism, then $U_i \times_U V$ exist and $\{U_i \times_U V \to V\} \in T$.

(covers pull back)

Example

Hol_1 objects are one-dimensional complex manifolds
morphisms holomorphic embeddings
Weiss covers define a Grothendieck topology on Hol_1.

Definition 5.1. A *universal* holomorphic prefactorization algebra (valued in the category $\mathrm{dgNuc}^{\otimes}$) is a symmetric monoidal functor

$$\mathrm{Hol}_1^{\sqcup} \to \mathrm{dgNuc}^{\otimes}.$$

A *universal* holomorphic factorization algebra is a universal holomorphic prefactorization algebra satisfying descent for Weiss covers.

My question: to what extent does this capture factorization?

5. Twisted factorization envelopes

Definition 3.6.3 Let $\mathcal{L}$ be a sheaf of dg Lie algebras on M. A *k-shifted central extension* of $\mathcal{L}_c$ is a precosheaf of dg Lie algebras $\widetilde{\mathcal{L}}_c$ fitting into an exact sequence

$$0 \to \underline{\mathbb{C}}[k] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0$$

of precosheaves, where $\underline{\mathbb{C}}[k]$ is the constant presheaf that assigns the one-dimensional vector space $\mathbb{C}[k]$ in degree $-k$ to every open. + Locality condition

Definition 3.6.4 In this situation, the *twisted factorization envelope* is the prefactorization algebra $\mathbb{U}\widetilde{\mathcal{L}}$ that sends an open set U to $C_*(\widetilde{\mathcal{L}}_c(U))$. (In the case that $\mathcal{L}$ is a local dg Lie algebra, we use the completed tensor product as above.)

The chain complex $C_*(\widetilde{\mathcal{L}}_c(U))$ is a module over chains on the Abelian Lie algebra $\mathbb{C}[k]$ for every k. Thus, we will view the twisted factorization envelope as a prefactorization algebra in modules for $\mathbb{C}[\mathbf{c}]$ where $\mathbf{c}$ has degree $-k-1$.

(1d)

Example: Let $\mathfrak{g}$ be a Lie algebra and consider the local Lie algebra $\Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$ on the real line $\mathbb{R}$, which we will denote $\mathfrak{g}^{\mathbb{R}}$. Given a skew-symmetric, invariant bilinear form ω on $\mathfrak{g}$, there is a natural shifted extension of $\mathfrak{g}^{\mathbb{R}}_c$ where

$$[\alpha \otimes X, \beta \otimes Y]_\omega = \alpha \wedge \beta \otimes [X,Y] + \int_{\mathbb{R}} \alpha \wedge \beta \, \omega(X,Y)\, \mathbf{c},$$

where we use $\mathbf{c}$ to denote the generator in degree 1 of the central extension. Let $\mathbb{U}_\omega \mathfrak{g}^{\mathbb{R}}$ denote the twisted factorization envelope for this central extension. By mimicking the proof of Proposition 3.4.1, one can see that the cohomology of this twisted factorization envelope recovers the enveloping algebra $U\widehat{\mathfrak{g}}$ of the central extension of $\mathfrak{g}$ given by ω. ◊

<u>Example</u> (2d) Kac-Moody factorization algebra

$\mathfrak{g}$ simple Lie algebra, $\langle\, , \rangle_{\mathfrak{g}}$ Killing form

Σ Riemann surface

$(\Omega^{0,*} \otimes \mathfrak{g}, \bar\partial)$ (Dolbeault analogue)

Cocycle: $\omega(\alpha,\beta) = \int_U \langle \alpha, \partial\beta \rangle_{\mathfrak{g}}$

for $\alpha, \beta \in \Omega_c^{0,*}(U) \otimes \mathfrak{g}$, $\partial : \Omega^{0,*} \to \Omega^{1,*}$

zero unless $\deg \alpha + \deg \beta = 1 \Rightarrow -1$ shifted cocycle.

$\mathbb{U}_\omega(\Omega^{0,*} \otimes \mathfrak{g})$ = Kac-Moody factorization algebra

$\longrightarrow$ Locally recover Kac-Moody vertex algebra.

<u>Example</u>:

X cpx mfd $\dim_{\mathbb{C}} X = n$ $\qquad \phi \in \Omega^{n-1,n-1}(X)$ closed

$$\omega(\alpha, \beta) = \int_X \langle \alpha, \partial \beta \rangle_{\mathfrak{g}} \wedge \phi$$

(see 2308.0441 for interesting ϕ
$\to$ generalized Cardy thm)

<u>Example</u>: Virasoro for Riemann surface Σ

$$\mathcal{L}^{\Sigma} := \left(\Omega^{0,*}(U, TU) \right)$$

sheaf of dg Lie algebras (in fact local dg Lie algebra)

Cocycle:

$$\omega : \mathcal{L}_c^{\mathbf{C}}(U) \otimes \mathcal{L}_c^{\mathbf{C}}(U) \to \mathbf{C}$$

given by

$$(\alpha \otimes \partial_z, \beta \otimes \partial_z) \mapsto \frac{1}{2\pi} \frac{1}{12} \int_U \left(\partial_z^3 \alpha_0 \beta_1 + \partial_z^3 \alpha_1 \beta_0 \right) \mathrm{d}^2 z$$

$\alpha = \alpha_0 + \alpha_1 dz \qquad \beta = \beta_0 + \beta_1 dz$

$$\mathcal{V}ir_0 := \mathbb{U}\, \mathcal{L}^c$$

$$\mathcal{V}ir := \mathbb{U}_\omega\, \mathcal{L}^c .$$

6. Equivariant prefactorization algebras and vertex algebras

M top. space, G discrete group ("no smoothness considered")

$M \circlearrowright G$

A prefactorization algebra with values in a multicategory $\mathcal{N}$ is a

$$\mathrm{Disj}_M \xrightarrow[\mathcal{A}]{} \mathcal{N} .$$

$g \in G$ provides <u>endofunctor</u> of Disj_M, act by precomposition

$$g\mathcal{A}(U) = \mathcal{A}(gU)$$

Def : Structure of G-Equivariant prefactorization algebra on A

isomorphisms $\sigma_g : A \to gA$

s.t.

$$\begin{tikzcd} A \arrow[r, "\sigma_h"] \arrow[dr, "\sigma_{gh}"'] & hA \arrow[d, "h\sigma_g"] \\ & ghA \end{tikzcd}$$

Define the complex manifold (disjoint holomorphic discs in $\mathbb{C}^k$)

$$\mathrm{Discs}(r_1, \dots, r_k) := \{z_1, \dots, z_k \in \mathbf{C} \mid D(z_1, r_1) \sqcup \cdots \sqcup D(z_k, r_k) \text{ disjoint}\} \subset \mathbf{C}^k.$$

The collection of these spaces form a $\mathbf{R}_{>0}$-colored operad in the category of complex manifolds, which we denote Discs. Applying the functor $\Omega^{0,*}$ we get a $\mathbf{R}_{>0}$-colored cooperad $\Omega^{0,*}(\mathrm{Discs})$ in the category of differentiable vector spaces. The main technical fact that we use to read off the structure of a vertex algebra is

Proposition 4.2 ([CG16a]). *Let $\mathcal{F}$ be a holomorphically translation-invariant factorization algebra on $\mathbf{C}$. Then, $\mathcal{F}$ defines an algebra over the $\mathbf{R}_{>0}$-colored cooperad $\Omega^{0,*}(\mathrm{Discs})$.*

~~This means~~ that at the level of cohomology as we let $p \in \mathrm{Discs}(r_1, \dots, r_k)$ vary the factorization maps

$$m[p] : \mathrm{H}^*\mathcal{F}(D(0, r_1)) \times \cdots \times \mathrm{H}^*\mathcal{F}(D(0, r_k)) \to \mathrm{H}^*\mathcal{F}(\mathbf{C})$$

lift to a map

$$\mu^{r_1, \dots, r_k}_{z_1, \dots, z_k} : \mathrm{H}^*\mathcal{F}(D(z_1, r_1)) \times \cdots \times \mathrm{H}^*\mathcal{F}(D(z_k, r_k)) \to \mathrm{Hol}\left(\mathrm{Discs}(r_1, \dots, r_k), \mathrm{H}^*\mathcal{F}(\mathbf{C})\right).$$

Translation invariance allows us to replace $\mathcal{F}(D(z_i, r_i)) \simeq \mathcal{F}(D(0, r_i))$ which we denote by $\mathcal{F}(r_i)$, so we can write this map as

$$\mu^{r_1, \dots, r_k}_{z_1, \dots, z_k} : \mathcal{F}(r_1) \times \cdots \times \mathcal{F}(r_k) \to \mathrm{Hol}\left(\mathrm{Discs}(r_1, \dots, r_k), \mathrm{H}^*\mathcal{F}(\mathbf{C})\right).$$

Taking a limit :

$$\mu_{z_1, \dots, z_k} : \left(\lim_{r \to 0} \mathrm{H}^*(\mathcal{F}(r))\right)^{\otimes k} \to \lim_{r \to 0} \mathrm{Hol}\left(\mathrm{Discs}_k(r), \mathrm{H}^*(\mathcal{F}(r))\right) \cong \mathrm{Hol}\left(\mathrm{Conf}_k(\mathbf{C}), \mathrm{H}^*\mathcal{F}(\mathbf{C})\right)$$

Theorem 4.3 (Theorem 5.2.2.1 [CG16a]). *Let $\mathcal{F}$ be a S^1-equivariant holomorphically translation invariant factorization algebra on $\mathbf{C}$. Suppose*

- *The action of S^1 on $\mathcal{F}(r)$ extends smoothly to an action of the algebra of distributions on S^1.*
- *For $r < r'$ the map*
$$\mathcal{F}^{(l)}(r) \to \mathcal{F}^{(l)}(r')$$
is a quasi-isomorphism.
- *The cohomology $\mathrm{H}^*(\mathcal{F}^{(l)}(r))$ vanishes for $l \gg 0$.*
- *For each l and $r > 0$ we require that $\mathrm{H}^*(\mathcal{F}^{(l)}(r))$ is isomorphic to a countable sequential colimit of finite dimensional vector spaces.*

Then $\mathrm{Vert}(\mathcal{F}) := \oplus_l \mathrm{H}^*(\mathcal{F}^{(l)}(r))$ (which is independent of r by assumption) has the structure of a vertex algebra.

Functor: $\mathrm{Vert} : \mathrm{PreFact}^{hol}_{\mathbb{C}} \longrightarrow \mathrm{VertexAlg}$

Thm

This gives Kac-Moody and Virasoro vertex algebras.

Thm

One can construct a universal Virasoro algebra on the site of complex curves.

Warning: the Virasoro cocycle is not independent on choice of coordinate. (Schwarzian derivative enters)

Fix: reduce to projective linear structure group or work with projective connections.

7. Factorization homology and conformal blocks

Factorization homology of $\mathcal{V}ir^{\Sigma^!}$ = global sections

$$\int_{\Sigma^!} \mathcal{V}ir^{\Sigma^!} := H^*\left(\mathcal{V}ir^{\Sigma^!}(\Sigma)\right)$$

dg nuclear vector space.

We compute this for the sphere $(g=0)$.

General observation

- The dg-Lie algebra $\left(\Omega^{0,*}(\Sigma^!, T_\Sigma), \bar\partial\right)$ is formal

$$\wr\wr \ \text{qis}$$

$$H^*_{\bar\partial}\left(\Omega^{0,*}(\Sigma^!, T_\Sigma)\right)$$

$$\Rightarrow H^*\left(\mathcal{V}ir^{\Sigma^!}(\Sigma^!)\right) = H^*\left(\mathrm{Sym}(H^*(\Sigma, T\Sigma) \oplus \mathbb{C}c, d_{CE}\right)$$

For genus $g=0$

$$H^*(\Sigma^!_0, T_{\Sigma_0}) \cong \mathfrak{sl}_2(\mathbb{C}) \ \ni \left\{ \begin{array}{l} \partial_z \\ z\,\partial_z \\ z^2\,\partial_z \end{array} \right.$$

Thus

$$\int_{\Sigma_0'} \mathrm{Vir}^{\Sigma_0} = H_*^{\mathrm{Lie}}(\mathfrak{sl}_2(\mathbb{C}))[c] \cong \mathbb{C}[y, c]$$

$$\deg y = 3 \qquad \deg c = 0.$$

Now towards Virasoro conformal blocks:

- Riemann surface Σ'
- $U_1 \ldots U_N$ opens, biholomorphic to $D_r(0) \subset \mathbb{C}$

Consider

$$(*) \qquad \Phi_{U_1 \ldots U_N} : \mathrm{Vir}^{\Sigma'}(U_1) \otimes \ldots \otimes \mathrm{Vir}^{\Sigma'}(U_N) \longrightarrow \mathrm{Vir}^{\Sigma'}(\Sigma')$$

$\mathrm{Vir}^{\Sigma'}$ is holomorphically, transl. invariant fact. algebra $/\mathbb{C}$

$\Rightarrow$ algebra over cooperad $\Omega^{0,*}(\mathrm{Disc})$

$\Rightarrow$ $(*)$ is a holomorphic function on

$$\mathrm{Disc}_n(r) \simeq \mathrm{Conf}_n(\mathbb{C})$$

Compute these functions.

- Fix complement to harmonic function & 1-forms and an inverse to $\bar\partial$ on this subspace

- Needs auxillary structure, e.g. metric $\to$ Green's function

$$\bar\partial G = \omega_{diag} \qquad (1,1) \text{ form on } \Sigma \times \Sigma$$

- Take $a_i \in \Omega^{0,*}(\Sigma, T\Sigma)$ $\bar\partial$ closed,

assume a_1 in orth. complement.

Compute

$$(\bar\partial + d_{Lie} + \omega)\left((\bar\partial^{-1}a_1)a_2\cdots a_n\right) = (\bar\partial\bar\partial^{-1}a_1)a_2\cdots a_n + \sum_{j=2}^{n}(-1)^{j+1}[\bar\partial^{-1}a_1, a_j]a_2\cdots\widehat{a_j}\cdots a_n$$

$$+ \sum_{j=2}^{n}(-1)^{j+1}\omega(\bar\partial^{-1}a_1, a_j)a_2\cdots\widehat{a_j}\cdots a_n.$$

a_i hol.

The first line follows from the fact that the only non trivial Lie bracket involving the elements $a_1, \ldots, a_n$ is between a_1 and a_j for $j \neq 1$. The second line follows from the fact that the cocycle ω is cohomologically degree one.

Since the term on the left hand side is exact in the cochain complex $\mathcal{V}ir^{\Sigma}(\Sigma)$ we have at the level of cohomology

$$\lfloor a_1\cdots a_n\rfloor = \sum_{j=2}^{n}(-1)^j \left\lfloor[\bar\partial^{-1}a_1, a_j]a_2\cdots\widehat{a_j}\cdots a_n\right\rfloor + \sum_{j=2}^{n}(-1)^j\omega(\bar\partial^{-1}a_1, a_j)\,\lfloor a_2\cdots\widehat{a_j}\cdots a_n\rfloor. \tag{5}$$

Recursion relation

In particular, we see that $\lfloor a\rfloor = 0$ for any a.

- We have now to make special choices for a_i

Take $(x_1 \ldots x_n) \in \mathrm{Conf}_n(\mathbb{C}P^1)$

$\varepsilon > 0$ s.t. $D(x_i, \varepsilon)$ pairwise disjoint

Idea: smear out

$f_{x_i}(r^2)$ bump functions on $D(x_i, \varepsilon)$

(0,1) form $\quad f_{x_i}(z,\bar z)\, d\bar z \in \Omega^{0,1}(D(x_i,\varepsilon))$

$\Rightarrow$ holomorphic vector field valued forms:

$$a_{x_i} := f_{x_i}(z,\bar z)\, d\bar z\, \partial_z \in \Omega^{0,1}(D(x_i,\varepsilon), T\mathbb{C}P^1)$$

$$\cap|$$

$$\mathcal{V}ir(D(x_i,\varepsilon))$$

- Compute

$$[a_{x_1} \cdots a_{x_n}] \in H^0\Big(\int_{\mathbb{C}P^1} \mathcal{V}ir\Big) \cong \mathbb{C}\, C$$

Using the explicit form of the operator $\overline{\partial}^{-1}$ on $\mathbf{C}P^1$ we find

$$\overline{\partial}^{-1}(a_{x_i}(z,\overline{z})) = \frac{1}{z-x_1}\partial_z.$$

For $a_i = a_{x_i}$, the recursive equation for the n-point function Equation (5) becomes

$$\begin{aligned} \lfloor a_{x_1} \cdots a_{x_n} \rfloor &= \sum_{j=2}^{n} (-1)^j \left\lfloor \left[\frac{1}{z-x_1}\partial_z, a_{x_j}(z,\overline{z}) \right] a_{x_2} \cdots \widehat{a}_j \cdots a_{x_n} \right\rfloor \\ &+ c \sum_{j=2}^{n} (-1)^j \omega \left(\frac{1}{z-x_1}\partial_z, f_j(z,\overline{z}) \mathrm{d}\overline{z}\partial_z \right) \left\lfloor a_{x_2} \cdots \widehat{a}_{x_j} \cdots a_{x_n} \right\rfloor . \end{aligned}$$

- One point blocks

$$[a_{x_1}] = 0$$

- 2 point blocks :

$$\lfloor a_{x_1} a_{x_2} \rfloor = c\omega\left(\bar{\partial}^{-1} a_{x_1}, a_{x_2}\right).$$

By definition of the cocycle ω, the right-hand side is equal to

$$c \cdot \frac{1}{12} \int_z \frac{1}{z - x_1} \partial_z^3 (f_{x_2}(z, \bar{z})) \mathrm{d}z \mathrm{d}\bar{z}.$$

Iterative application of integration by parts together with the fact that $\int \varphi(z) f_{x_2}(z, \bar{z}) \mathrm{d}z \mathrm{d}\bar{z} = \varphi(x_2)$ yields

$$\lfloor a_{x_1} a_{x_2} \rfloor = \frac{c}{2} \frac{1}{(x_1 - x_2)^4}.$$

- 3 point blocks

We can compute $\lfloor a_{x_1} a_{x_2} a_{x_3} \rfloor$ in a similar way. Since $\lfloor a_{x_i} \rfloor = 0$ the recursive formula implies

$$\lfloor a_{x_1} a_{x_2} a_{x_3} \rfloor = \left\lfloor \left[\frac{1}{z - x_1} \partial_z, a_{x_2}(z, \bar{z}) \right] \cdot a_{x_3} \right\rfloor - \left\lfloor \left[\frac{1}{z - x_1} \partial_z, a_{x_3}(z, \bar{z}) \right] \cdot a_{x_2} \right\rfloor.$$

Consider the first term above. We compute the Lie bracket

$$\left[\frac{1}{z - x_1} \partial_z, a_{x_2} \right] = \frac{1}{z - x_1} \partial_z (f_{x_2}(z, \bar{z})) \mathrm{d}\bar{z} \partial_z + \frac{1}{(z - x_1)^2} f_{x_2}(z, \bar{z}) \mathrm{d}\bar{z} \partial_z.$$

Applying $\bar{\partial}^{-1}$ to this expression yields the vector field

$$\left(-\frac{1}{(z - x_2)^2 (x_2 - x_1)} + \frac{2}{(z - x_2)(x_2 - x_1)^2} \right) \partial_z.$$

This calculation, combined with the fact that $\lfloor a \cdot b \rfloor = c\omega(\bar{\partial}^{-1} a, b)$ yields

$$\begin{aligned}
\left\lfloor \left[\frac{1}{z - x_1} \partial_z, a_{x_2}(z, \bar{z}) \right] \cdot a_{x_3} \right\rfloor &= -c\omega \left(\frac{1}{(z - x_2)^2 (x_2 - x_1)} \partial_z, a_{x_3}(z, \bar{z}) \right) + 2c\omega \left(\frac{1}{(z - x_2)(x_2 - x_1)^2} \partial_z, a_{x_3}(z, \bar{z}) \right) \\
&= -\frac{c}{12} \int_z \frac{1}{(z - x_2)^2 (x_2 - x_1)} \partial_z^3 (f_{x_3}(z, \bar{z})) \mathrm{d}z \mathrm{d}\bar{z} + \frac{c}{6} \int_z \frac{1}{(z - x_2)(x_2 - x_1)^2} \partial_z^3 (f_{x_3}(z, \bar{z})) \mathrm{d}z \mathrm{d}\bar{z} \\
&= \frac{c}{(x_3 - x_2)^4 (x_2 - x_1)} \left(-\frac{2}{x_3 - x_2} + \frac{1}{x_2 - x_1} \right)
\end{aligned}$$

The second term in (6) is obtained by sending $x_2 \leftrightarrow x_3$ in the above formula. In total, the sum is thus

$$\frac{c}{(x_3 - x_2)^4 (x_2 - x - 1)(x_3 - x_1)} \left(-2 \frac{x_3 - x_2}{x_3 - x_2} + \frac{x_3 - x_1}{x_2 - x_1} + 2 \frac{x_2 - x_1}{x_3 - x_2} + \frac{x_2 - x_1}{x_3 - x_1} \right).$$

This simplifies to the following expression for the 3-point correlator

$$\lfloor a_{x_1} a_{x_2} a_{x_3} \rfloor = \frac{c}{(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2}.$$

- General recursion relation, exhibiting Virasoro OPE

For general n, the recursive formula implies that can write the n-point function as

$$\begin{aligned}\lfloor a_{x_1}\cdots a_{x_n}\rfloor &= \sum_{j=2}^{n}\left(\frac{1}{x_j-x_1}\partial_{x_j}+\frac{1}{(x_j-x_1)^2}\right)\lfloor a_{x_2}\cdots a_{x_n}\rfloor \\ &+ \frac{c}{2}\sum_{j=2}^{n}(-1)^j\frac{1}{(x_j-x_1)^4}\left\lfloor a_{x_2}\cdots\widehat{a}_{x_j}\cdots a_{x_n}\right\rfloor.\end{aligned}$$