

Global Aspects of T-duality in Type II string theory

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The word “duality” means the equivalence of a pair of apparently different concepts.

In mathematics, one encounters “duality” for instance in projective geometry:

2 points determine a line;

2 lines determine a point,

giving a duality,

points \longleftrightarrow lines
--

In algebraic topology, “Poincaré duality” is one of many dualities: it establishes the equivalence for compact oriented manifolds,

homology \longleftrightarrow cohomology

In Pre-string theory physics, one encounters

Electro-Magnetic duality

Recall the vacuum Maxwell equations:

E = Electric field

B = magnetic field

$$\begin{aligned}\nabla \times B &= \frac{\partial E}{\partial t} \\ \nabla \times E &= -\frac{\partial B}{\partial t}\end{aligned}$$

these have a duality symmetry,

$$\begin{aligned}B &\rightarrow -E \\ E &\rightarrow B\end{aligned}$$

which continues to hold when one adds both electric and magnetic charges and currents.

In Quantum Electro Dynamics (QED), the coupling constant α in this case is the **fine structure constant** (which is precisely measured by the quantum Hall effect experiment),

$$\alpha = \frac{e^2}{4\pi\hbar c}$$

where e is the unit of electric charge,
 c is the speed of light
 \hbar is Planck's constant.

The **Montonen-Olive** symmetry in QED exchanging the electric and magnetic fields is

$$\begin{aligned} e &\longleftrightarrow \frac{4\pi\hbar c}{e} \\ \alpha &\longleftrightarrow \frac{1}{\alpha} \\ \text{(weak coupling)} &\longleftrightarrow \text{(strong coupling)} \end{aligned}$$

This duality also manifests itself in String Theory as **S-Duality**.

Supersymmetry is the conjectured duality:

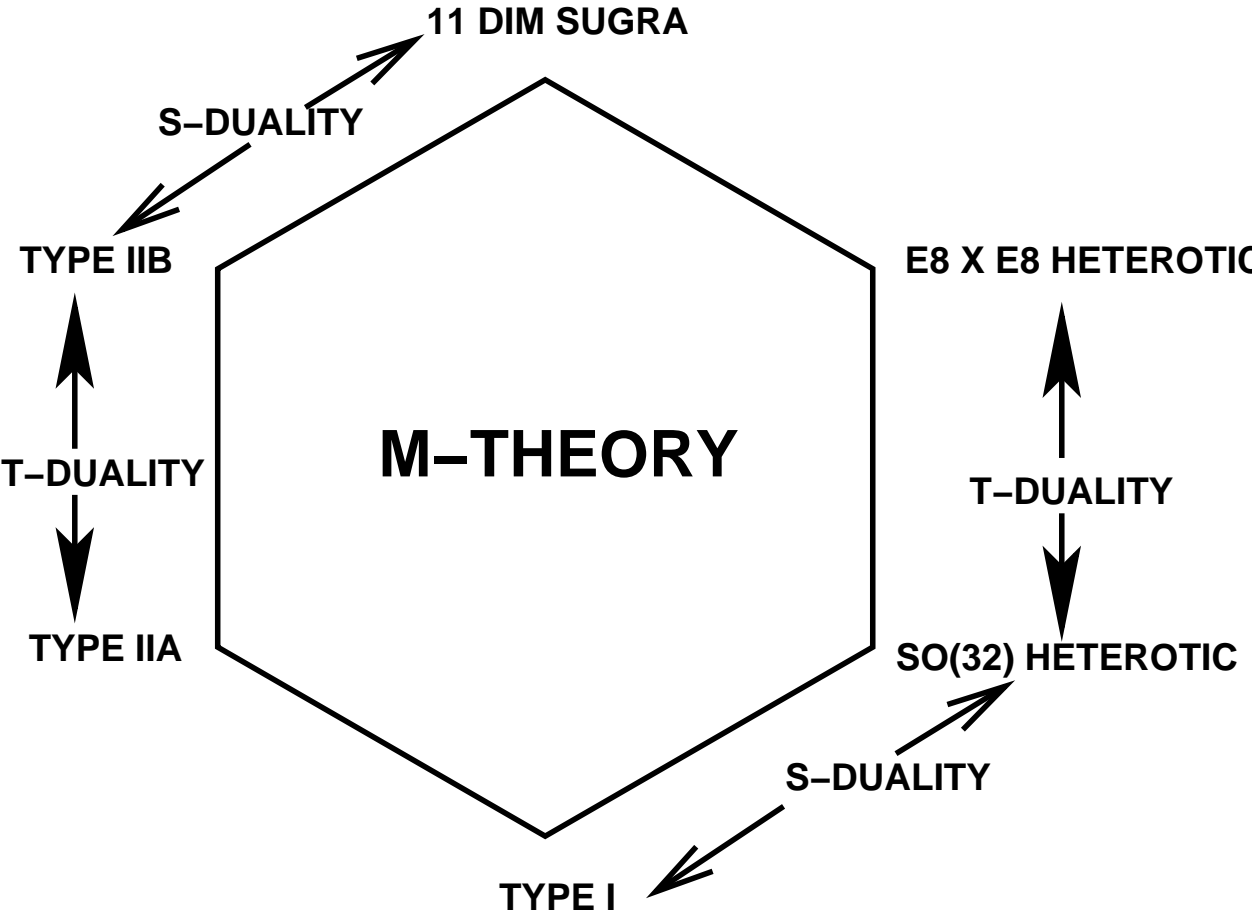
Fermions \longleftrightarrow Bosons

It is Quantum Mechanical, since Fermions are Quantum Mechanical.

Supersymmetry explains why Fermions have to exist, since Bosons are observed to exist.

Spacetime in relativity is enriched by both Bosonic and Fermionic coordinates, thanks to supersymmetry, leading eventually to 11D SUGRA.

GLOBAL DUALITIES IN STRING THEORY



References: (T-duality & K-theory)

*Survey of T-duality for **trivial torus bundles** in a topologically trivial background flux*

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*T-duality for **circle bundles** in a topologically **non-trivial** background flux*

- [BEM] P. Bouwknegt, J. Evslin and V. Mathai, *T-duality: Topology Change from H-flux*, Commun. Math. Phys. **249** no. 2 (2004) 383-415 [[hep-th/0306062](#)]
- [BEM2] P. Bouwknegt, J. Evslin and V. Mathai, *On the topology and H-flux of T-dual manifolds*, Physical Review Letters **92**, 181601 (2004) [[hep-th/0312052](#)].

*T-duality for **torus bundles** in a topologically **non-trivial** background flux, when the dual is **classical***

- [BHM] P. Bouwknegt, K. Hannabuss and V. Mathai, *T-duality for principal torus bundles*, JHEP **03** (2004) 018, 10 pages [[hep-th/0312284](#)].

*T-duality for **torus bundles** in a topologically **non-trivial** background flux, when the dual is a **field of noncommutative tori***

- [MR] V. Mathai, J. Rosenberg, *T-duality for torus bundles via noncommutative topology*, Commun. Math. Phys. (to appear) [**hep-th/0401168**]
- [MR2] V. Mathai, J. Rosenberg, *T-duality for torus bundles via noncommutative topology II: the high dimensional case and the T-duality group*, [**hep-th/0508084**]

*T-duality for **torus bundles** in a topologically **non-trivial** background flux, when the dual is a **field of nonassociative tori***

- [BHM2] P. Bouwknegt, K. Hannabuss and V. Mathai, *Nonassociative tori and their application to T-duality*, preprint [**hep-th/0412092**]
- [BHM3] P. Bouwknegt, K. Hannabuss and V. Mathai, *T-duality for principal torus bundles and dimensionally reduced Gysin sequences*, preprint [**hep-th/0412268**]

+ recent work of U. Bunke, T. Schick and collaborators.

The idea of T-duality

The simplest example is a free theory on a torus \mathbb{R}^n/Γ , where Γ is a lattice in \mathbb{R}^n . The partition function is a theta function,

$$Z_{\Gamma}(\beta) = \sum_{z \in \hat{\Gamma}} e^{-2\pi^2 \beta |z|^2}$$

where $\hat{\Gamma}$ is the dual lattice in $\hat{\mathbb{R}}^n$.

By the Poisson summation formula, this is equivalent to the partition function $Z_{\hat{\Gamma}}$ on the dual torus $\hat{\mathbb{R}}^n/\hat{\Gamma}$, where we notice that $r \mapsto 1/r$.

The situation however gets much more complicated when a background flux H is turned on, where $[H] \in H^3(\mathbb{R}^n/\Gamma, \mathbb{Z})$, and $[H] \neq 0$, and will be discussed later.

T-duality in the literature

Spacetime is $M \times S^1$, with trivial background flux - then the T-dual is topologically the same space $M \times \hat{S}^1$, and T-duality is realized by using the correspondence

$$\begin{array}{ccc}
 & M \times S^1 \times \hat{S}^1 & \\
 \swarrow p & & \searrow \hat{p} \\
 M \times S^1 & & M \times \hat{S}^1
 \end{array} \tag{1}$$

Poincaré line bundle \mathcal{P} : There is a canonical line bundle \mathcal{P} over the torus $S^1 \times \hat{S}^1$, defined as follows: Consider the free action of \mathbb{Z} on $\mathbb{R} \times \hat{S}^1 \times \mathbb{C}$ given by,

$$\begin{aligned}
 \mathbb{Z} \times (\mathbb{R} \times \hat{S}^1 \times \mathbb{C}) &\rightarrow \mathbb{R} \times \hat{S}^1 \times \mathbb{C} \\
 (n, (r, \rho, z)) &\rightarrow (r + n, \rho, \rho(n)z)
 \end{aligned}$$

The Poincaré line bundle is defined as $\mathcal{P} = (\mathbb{R} \times \hat{S}^1 \times \mathbb{C})/\mathbb{Z}$. its curvature is $\mathcal{F} = d\theta \wedge d\hat{\theta}$.

In this case, T-dualizing on S^1 , the Buscher rules for transforming the RR fields can be conveniently encoded in the formula

$$T_*G = \int_{\hat{S}^1} e^{\mathcal{F}} G, \quad (2)$$

$G \in \Omega^\bullet(M \times S^1)$ is the total RR fieldstrength,

$G = \sum_p G_{p+2}$, $p = 0, 2, 4, \dots, 8$ for **Type IIA**

$G = \sum_p G_{p+2}$, $p = -1, 1, \dots, 7$ for **Type IIB**.

$\mathcal{F} = d\theta \wedge d\hat{\theta}$ is the curvature of the Poincaré line bundle \mathcal{P} on $S^1 \times \hat{S}^1$, so that $e^{\mathcal{F}} = ch(\mathcal{P})$ is the Chern character of \mathcal{P} .

The RR field G is a closed form if and only if its T-dual \hat{G} is a closed form.

We can interpret (2) as an isomorphism

$$T_* : H^\bullet(M \times S^1) \xrightarrow{\cong} H^{\bullet+1}(M \times \hat{S}^1). \quad (3)$$

Recently, it was argued by Minasian-Moore and Witten that

Type IIA string theory

Fields are classified by $K^0(X)$

Charges are classified by $K^1(X)$

whereas,

Type IIB string theory

Fields are classified by $K^1(X)$

Charges are classified by $K^0(X)$

Note the parity shift!

The T-duality discussion given earlier can also be realized also in K-theory, and thus to the classification of D-branes on $M \times S^1$ and $M \times \hat{S}^1$, by using the correspondence

$$\begin{array}{ccc}
 & M \times S^1 \times \hat{S}^1 & \\
 p \swarrow & & \searrow \hat{p} \\
 M \times S^1 & & M \times \hat{S}^1
 \end{array} \quad (4)$$

Induces a T-duality isomorphism of K-theories

$$T_{\downarrow} : K^{\bullet}(M \times S^1) \xrightarrow{\cong} K^{\bullet+1}(M \times \hat{S}^1) \quad (5)$$

given by $T_{\downarrow} = \hat{p}_{\downarrow} (p^{\downarrow}(\cdot) \otimes \mathcal{P})$. That is, T-duality in the absence of a background field, gives an equivalence

Type IIA theory \longleftrightarrow Type IIB theory

N.B. no change in topology!

T-duality in the absence of a background flux can be summarized as the commutativity of the following diagram,

$$\begin{array}{ccc}
 K^\bullet(M \times S^1) & \xrightarrow[\cong]{T_!} & K^{\bullet+1}(M \times \widehat{S}^1) \\
 \text{\scriptsize } ch \downarrow & & \downarrow \text{\scriptsize } ch \\
 H^\bullet(M \times S^1) & \xrightarrow[\cong]{T_*} & H^{\bullet+1}(M \times \widehat{S}^1)
 \end{array}$$

where the horizontal arrows are isomorphisms, and ch is the Chern character. That is,

$$ch(T_!(Q)) = T_*ch(Q)$$

for all $Q \in K^\bullet(M \times S^1)$.

The aim: to generalize this to the case when there is a **non-trivial background flux**.

The case of circle bundles

In [BEM], we investigated the general case where E is an oriented S^1 -bundle over M

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \pi \downarrow \\ & & M \end{array} \quad (6)$$

classified by its first Chern class

$c_1(E) \in H^2(M, \mathbb{Z})$, with H -flux $H \in H^3(E, \mathbb{Z})$.

The **T-dual** of E is another oriented S^1 -bundle over M , denoted by \hat{E} ,

$$\begin{array}{ccc} \hat{S}^1 & \longrightarrow & \hat{E} \\ & & \hat{\pi} \downarrow \\ & & M \end{array} \quad (7)$$

which has first Chern class $c_1(\hat{E}) = \pi_* H$.

The Gysin sequence for E enables us to define a T-dual H -flux $\hat{H} \in H^3(\hat{E}, \mathbb{Z})$, satisfying

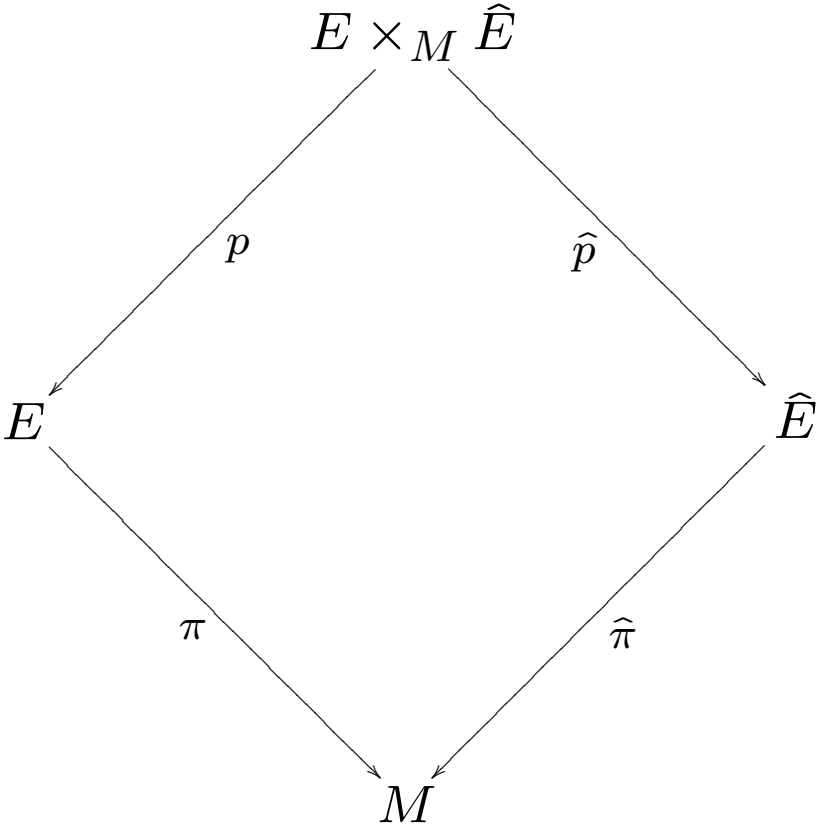
$$c_1(E) = \hat{\pi}_* \hat{H}, \quad (8)$$

where $\pi_* : H^k(E, \mathbb{Z}) \rightarrow H^{k-1}(M, \mathbb{Z})$, and similarly $\hat{\pi}_*$, denote the pushforward maps.

The surprising **new** phenomenon is that there is a **change in topology** when the H -flux is non-trivial. Note that this can also happen when spacetime is a product $E = M \times S^1$, eg $M = S^2$ and $H = a \cup b$, where $a = k.\text{vol} \in H^2(M, \mathbb{Z})$, b the generator of $H^1(S^1, \mathbb{Z})$. Then the T-dual circle bundle is the Lens space $L(1, k)$ and T-dual H-flux is zero. So T-duality in this case is the swapping

background flux \longleftrightarrow Chern class

T-duality & correspondence spaces



T-duality in a background flux

Choosing connection 1-forms A and \hat{A} , on the S^1 -bundles E and \hat{E} , respectively, the rules for transforming the RR fields can be encoded in the formula

$$T_*G = \int_{S^1} e^{A \wedge \hat{A}} G, \quad (9)$$

$G \in \Omega^\bullet(E)$ is the total RR fieldstrength,

$G = \sum_p G_{p+2}$, $p = 0, 2, 4, \dots, 8$ for **Type IIA**

$G = \sum_p G_{p+2}$, $p = -1, 1, \dots, 7$ for **Type IIB**,

where the right hand side is a form on $E \times_M \hat{E}$, and the integration is along the S^1 -fiber of E .

Recall that the twisted cohomology $H^\bullet(E, H)$ is defined as the cohomology of the complex

$$(\Omega^\bullet(E), d_H = d - H \wedge).$$

Let $F = dA$ and $\hat{F} = d\hat{A}$,

$$H = A \wedge \hat{F} - \Omega, \quad (10)$$

for some $\Omega \in \Omega^3(M)$, while the T-dual \hat{H} is given by

$$\hat{H} = F \wedge \hat{A} - \Omega. \quad (11)$$

We note that

$$d(A \wedge \hat{A}) = -H + \hat{H}, \quad (12)$$

so that T_* indeed maps d_H -closed forms G to $d_{\hat{H}}$ -closed forms T_*G . Therefore T-duality T_* induces a map

$$T_* : H^\bullet(E, H) \rightarrow H^{\bullet+1}(\hat{E}, \hat{H}).$$

The inverse is similarly defined, and using the fact that locally, we have $A = d\theta + \hat{\pi}_*\hat{B}$, $\hat{A} = d\hat{\theta} + \pi_*B$, one shows after a small computation that T-duality T_* is indeed an **isomorphism**.

Recently, it was argued by Witten, Bouwknegt-Mathai that in the presence of a background H -flux

Type IIA string theory

Fields are classified by $K^0(X, H)$

Charges are classified by $K^1(X, H)$

whereas,

Type IIB string theory

Fields are classified by $K^1(X, H)$

Charges are classified by $K^0(X, H)$

Note the parity shift!

The earlier discussion can also be realized in twisted K-theory and, in this more general setting, T-duality gives an isomorphism of the twisted K-theories of E and \widehat{E} ,

$$T_{\dagger} : K^{\bullet}(E, H) \rightarrow K^{\bullet+1}(\widehat{E}, \widehat{H})$$

defined by

$$T_{\dagger} = \widehat{p}_{\dagger} (p^{\dagger}(\cdot) \otimes \mathcal{L})$$

where \mathcal{L} is the substitute for the Poincaré line bundle. It is no longer necessarily a line bundle on the correspondence space $E \times_M \widehat{E}$, but rather a line bundle on the total space of a principal PU bundle Q over the correspondence space $E \times_M \widehat{E}$ with Dixmier-Douady invariant $\widehat{H} - H$. The first Chern class of \mathcal{L} is $c_1(\mathcal{L}) = A \wedge \widehat{A} - \widehat{B} + B$ determines \mathcal{L} , where we recall that $d(A \wedge \widehat{A}) = \widehat{H} - H = d(\widehat{B} - B)$.

Several of the constructions used in the definition of T-duality on twisted K-theory are adapted from joint work with Melrose and Singer.

T-duality in the presence of a background flux H can be summarized as the commutativity of the following diagram,

$$\begin{array}{ccc}
 K^\bullet(E, H) & \xrightarrow{T_!} & K^{\bullet+1}(\hat{E}, \hat{H}) \\
 ch_H \downarrow & & \downarrow ch_{\hat{H}} \\
 H^\bullet(E, H) & \xrightarrow{T_*} & H^{\bullet+1}(\hat{E}, \hat{H})
 \end{array} \tag{13}$$

That is,

$$ch_H(T_!(Q)) = T_*ch_{\hat{H}}(Q)$$

Sample calculations

Lens spaces $L(1, p) = S^3/\mathbb{Z}_p$, which is the total space of the circle bundle over the 2-sphere with Chern class equal to p times the generator of $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$.

$$K^i(L(1, j), H = k) \cong K^{i+1}(L(1, k), H = j).$$

In particular, since $L(1, 1) = S^3 = SU(2)$ we obtain an isomorphism

$$K^i(SU(2), H = k) \cong K^{i+1}(L(1, k), H = 1).$$

In particular, since $L(1, 0) = S^2 \times S^1$, we obtain an isomorphism

$$K^i(S^2 \times S^1, H = k) \cong K^{i+1}(L(1, k)) \quad (14)$$

which shows that it is a ring.

The case of principal \mathbb{T}^2 -bundles

In [MR], we investigated the general case where E is an oriented \mathbb{T}^2 -bundle over M .

$$\begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & E \\ & & \pi \downarrow \\ & & M \end{array} \quad (15)$$

classified by its first Chern class

$c_1(E) \in H^2(M, \mathbb{Z}^2)$, with H -flux $H \in H^3(E, \mathbb{Z})$.

The T-dual of E is **not** in general another oriented \mathbb{T}^2 -bundle over M !

A famous example of a principal torus bundle with non T-dualizable H-flux is provided by \mathbb{T}^3 , considered as the trivial \mathbb{T}^2 -bundle over \mathbb{T} with projection p , with H-flux given by k times the volume form $d\text{vol}$ on \mathbb{T}^3 , $k \neq 0$. Since $p_![H] = [\int_{\mathbb{T}} H] \neq 0$, H is non T-dualizable in the classical sense. Alternatively, there are no non-trivial principal \mathbb{T}^2 -bundles over \mathbb{T} , since $H^1(\mathbb{T}, \underline{\mathbb{T}}^2) \cong H^2(\mathbb{T}, \mathbb{Z}^2) = 0$, that is, there is no way to dualize the H-flux by a (principal) torus bundle over \mathbb{T} .

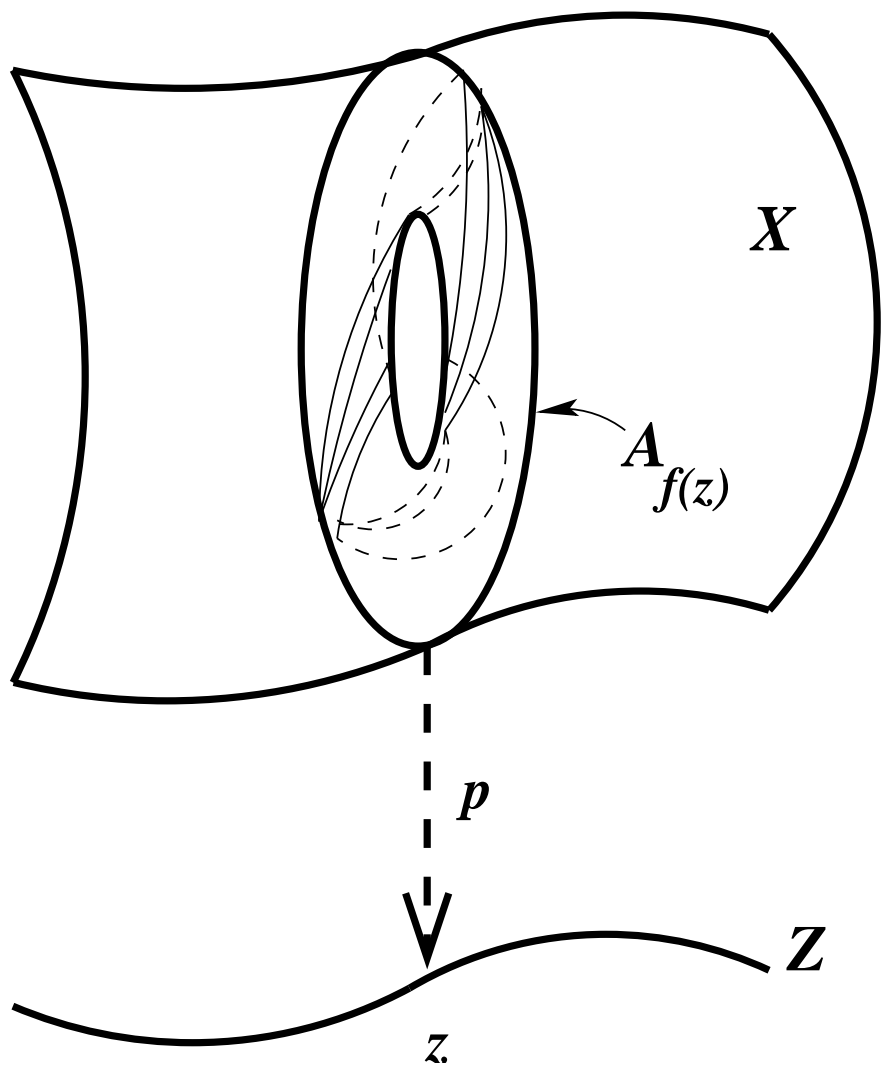
The T-dual is realized in this case by a continuous field of stabilized **noncommutative tori** fibered over \mathbb{T} . Let $\mathcal{H} = L^2(\mathbb{T})$ and consider the projective unitary representation $\rho_\theta : \mathbb{Z}^2 \rightarrow \text{PU}(\mathcal{H})$ given by the first \mathbb{Z} factor acting by multiplication by z^k and the second \mathbb{Z} factor acting by translation by $\theta \in \mathbb{T}$.

Let \mathbb{Z}^2 act on $C(\mathbb{T}, \mathcal{K}(\mathcal{H}))$ by α , which is given at the point θ by ρ_θ . Define the C^* -algebra

$$\begin{aligned} B &= \text{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} (C(\mathbb{T}, \mathcal{K}(\mathcal{H})), \alpha) \\ &= \left\{ f : \mathbb{R}^2 \rightarrow C(\mathbb{T}, \mathcal{K}(\mathcal{H})) : f(t+g) \right. \\ &\quad \left. = \alpha(g)(f(t)), t \in \mathbb{R}^2, g \in \mathbb{Z}^2 \right\} \end{aligned}$$

Then B is a continuous-trace C^* algebra having spectrum \mathbb{T}^3 , having an action of \mathbb{R}^2 whose induced action on the spectrum of B is the trivial bundle $\mathbb{T}^3 \rightarrow \mathbb{T}$. The crossed product algebra $B \rtimes \mathbb{R}^2 \cong C(\mathbb{T}, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^2$ has fiber over $\theta \in \mathbb{T}$ given by $\mathcal{K}(\mathcal{H}) \rtimes_{\rho_\theta} \mathbb{Z}^2 \cong A_\theta \otimes \mathcal{K}(\mathcal{H})$, where A_θ is the noncommutative 2-torus. Thus $B \rtimes \mathbb{R}^2$ is isomorphic to $C^*(H_{\mathbb{Z}}) \otimes \mathcal{K}$, where $H_{\mathbb{Z}}$ is the integer Heisenberg-type group,

$$H_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$



We can also consider the smooth subalgebra B^∞ of B defined by

$$\begin{aligned} B^\infty &= \text{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} (C^\infty(\mathbb{T}, \mathcal{K}^\infty(\mathcal{H})), \alpha) \\ &= \left\{ f : \mathbb{R}^2 \rightarrow C^\infty(\mathbb{T}, \mathcal{K}^\infty(\mathcal{H})) : f(t + g) \right. \\ &\quad \left. = \alpha(g)(f(t)), \quad t \in \mathbb{R}^2, g \in \mathbb{Z}^2 \right\} \end{aligned}$$

where $\mathcal{K}^\infty(\mathcal{H})$ denotes the algebra of smoothing operators on \mathbb{T} . Note that $B^\infty \rtimes \mathbb{R}^2 \cong C^\infty(\mathbb{T}, \mathcal{K}^\infty(\mathcal{H})) \rtimes \mathbb{Z}^2$ has fiber over $\theta \in \mathbb{T}$ given by $\mathcal{K}^\infty(\mathcal{H}) \rtimes_{\rho_\theta} \mathbb{Z}^2 \cong A_\theta^\infty \otimes \mathcal{K}^\infty(\mathcal{H})$, where A_θ^∞ is the smooth noncommutative torus and the tensor product is the projective tensor product. In this case, the crossed product $B^\infty \rtimes \mathbb{R}^2 \cong \mathcal{S}(H_{\mathbb{Z}}) \otimes \mathcal{K}^\infty(\mathcal{H})$, where $\mathcal{S}(H_{\mathbb{Z}})$ is the rapid decrease algebra.

Finally, T-duality can be expressed in this case by the following commutative diagram,

$$\begin{array}{ccc}
 K^\bullet(\mathbb{T}^3, k \, d\text{vol}) & \xrightarrow{T_!} & K_\bullet(C^*(H_{\mathbb{Z}})) \\
 \text{Ch}_H \downarrow & & \downarrow \text{Ch} \\
 H^\bullet(\mathbb{T}^3, k \, d\text{vol}) & \xrightarrow{T_*} & HP_\bullet(S(H_{\mathbb{Z}}))
 \end{array}$$

where $H = k \, d\text{vol}$,

Ch_H is the twisted Chern character

Ch is the Connes-Chern character.

Theorem 1. Let $\pi: E \rightarrow M$ be a principal \mathbb{T}^2 -bundle and $H \in H^3(E, \mathbb{Z})$ an H-flux on E .

1. [BHM] If $\pi_*H = 0 \in H^1(M, \mathbb{Z})$, then there is a uniquely determined classical T-dual to (π, H) , consisting of $\pi^\# : E^\# \rightarrow M$, which is a another principal \mathbb{T}^n -bundle over Z , and $H^\# \in H^3(E^\#, \mathbb{Z})$, the “T-dual H-flux” on $E^\#$. One obtains a picture exactly as obtained in the case of circle bundles.

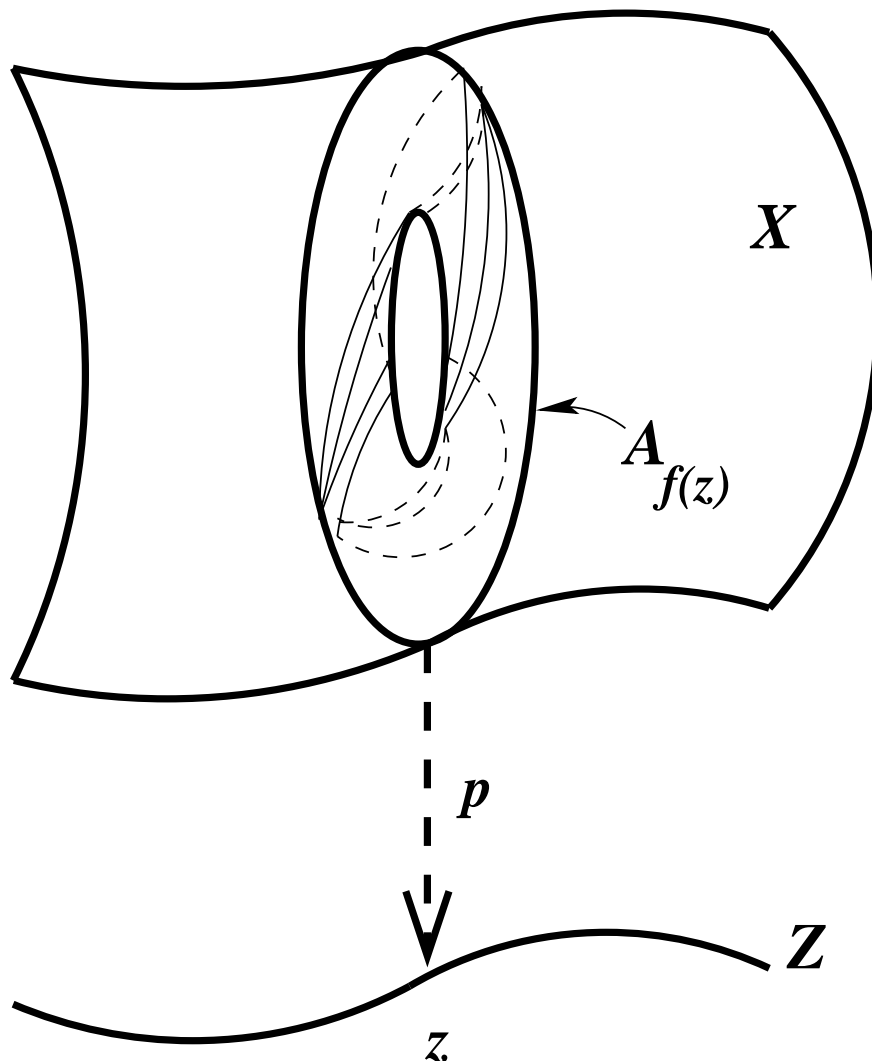
2. [MR] If $\pi_*H \neq 0 \in H^1(M, \mathbb{Z})$, then a classical T-dual as above does *not* exist. However, there is a “nonclassical” T-dual bundle of **noncommutative tori** over M . It is not unique, but the non-uniqueness does not affect its K -theory.

The mathematical theorem that we prove is that for \mathbb{T}^2 -torus bundles, the \mathbb{R}^2 action on E lifts to a \mathbb{R}^2 action on $CT(E, H)$, where $CT(E, H)$ is the algebra of sections of a bundle of compact operators with Dixmier-Douady invariant equal to H .

Then the T-dual of (E, H) is defined to be $CT(E, H) \rtimes \mathbb{R}^2$. It has an action of the dual group $\hat{\mathbb{R}}^2$ such that $CT(E, H) \rtimes \mathbb{R}^2 \rtimes \hat{\mathbb{R}}^2$ is Morita equivalent to $CT(E, H)$. We also have the commutative diagram,

$$\begin{array}{ccc}
 K^\bullet(E, H) & \xrightarrow[\cong]{T_!} & K_\bullet(CT(E, \delta) \rtimes \mathbb{R}^2) \\
 Ch_H \downarrow & & \downarrow Ch \\
 H^\bullet(E, H) & \xrightarrow[\cong]{T_*} & HP_\bullet(CT(E, \delta)^\infty \rtimes \mathbb{R}^2)
 \end{array}$$

where the horizontal arrows are isomorphisms, Ch_H is the twisted Chern character and Ch is the Connes-Chern character.



In the diagram, the fiber over $z \in Z$ is the noncommutative torus $A_{f(z)}$, which is represented by a foliated torus, with foliation angle equal to $f(z)$.

Theorem 2. [MR] If $E \xrightarrow{\pi} M$ is a torus bundle over M , H is an integral 3-form representing δ in de Rham cohomology, and where $CT(E, H)$ is the algebra of sections of a bundle of compact operators with Dixmier-Douady invariant equal to H .

Then the \mathbb{R}^n action on E lifts to a \mathbb{R}^n action on $CT(E, \delta)$ if and only if the restriction of H to each fibre E_x is exact, $\forall x \in M$. The T-dual in this case is the crossed product C^* -algebra $CT(E, \delta) \rtimes \mathbb{R}^n$, which is a continuous field of noncommutative tori. This field can be also described as follows: since $\pi_*(H) \in H^1(M, H^2(\mathbb{T}^n, \mathbb{Z})) = [M, \mathbb{T}^k]$, where $k = \binom{n}{2}$, there is a continuous representative $f : M \rightarrow \mathbb{T}^k$. Then the fiber of the T-dual at the point $x \in M$ is the noncommutative torus $A_{f(x)}$.

We also have a commutative diagram

$$\begin{array}{ccc}
 K^\bullet(E, H) & \xrightarrow[\cong]{T_!} & K_\bullet(CT(E, \delta) \rtimes \mathbb{R}^n) \\
 Ch_H \downarrow & & \downarrow Ch \\
 H^\bullet(E, H) & \xrightarrow[\cong]{T_*} & HP_\bullet(CT(E, \delta)^\infty \rtimes \mathbb{R}^n)
 \end{array}$$

where the horizontal arrows are isomorphisms, Ch_H is the twisted Chern character and Ch is the Connes-Chern character.

Nonassociative tori

Definition in 3D. Let U_1, U_2, U_3 be generators satisfying the relations

$$e^{2\pi i\phi} U_1(U_2 U_3) = (U_1 U_2) U_3,$$

$$U_i U_j = u_{ij} U_j U_i$$

where $\phi \in \mathbb{R}$ and u_{ij} are automorphisms, usually in $U(1)$. In general $e^{2\pi i\phi} \in H^3(\mathbb{Z}^n, U(1))$.

The algebra generated by U_1, U_2, U_3 is the non-associative torus - it is both noncommutative and nonassociative. Such tori have made their appearance earlier in physics, as Jackiw's non-associative anomaly in quantum field theory.

Theorem 3. [BHM2] If E is a torus bundle over M , H is an integral 3-form representing δ in de Rham cohomology, and we choose a smooth model for $CT(E, \delta)$ by taking a smooth bundle over X with fibers the smoothing operators.

Then the \mathbb{R}^n action on E lifts to a twisted \mathbb{R}^n action on $CT(E, \delta)$ and the T-dual of (E, δ) is $CT(E, \delta) \rtimes_{twist} \mathbb{R}^n$ which in general a bundle of nonassociative tori, if the restriction of H to the fibres E_x is not exact.

Summary of 4 important special cases:

1) The T-dual of the torus \mathbb{T}^3 with no background flux is the dual torus $\hat{\mathbb{T}}^3$. Similar if the background flux is topologically trivial.

2) $(\mathbb{T}^3, k\text{vol})$ considered as a trivial circle bundle over \mathbb{T}^2 . The T-dual of $(\mathbb{T}^3, k\text{vol})$ is the nilmanifold $(H_{\mathbb{R}}/H_{\mathbb{Z}}, 0)$, where $H_{\mathbb{R}}$ is the 3D Heisenberg group and $H_{\mathbb{Z}}$ a lattice in it.

3) $(\mathbb{T}^3, k\text{vol})$ considered as a trivial \mathbb{T}^2 -bundle over \mathbb{T} . The T-dual of $(\mathbb{T}^3, k\text{vol})$ is a continuous field of stabilized noncommutative tori, $C^*(H_{\mathbb{Z}}) \otimes \mathcal{K}$, since $\int_{\mathbb{T}^2} k\text{vol} \neq 0$.

4) $(\mathbb{T}^3, k\text{vol})$ considered as a trivial \mathbb{T}^3 -bundle over a point. The T-dual of $(\mathbb{T}^3, k\text{vol})$ is a nonassociative torus, A_{ϕ} , where ϕ is the tricharacter associated to $k\text{vol}$, where $\int_{\mathbb{T}^3} k\text{vol} \neq 0$.

Our results on the T-duality group: It is important to realize that a fixed space X can sometimes be given the structure of a principal torus bundle over Z in many different ways. For example, given a free action of a torus $T = \mathbb{T}^n$ on X , with quotient space $Z = X/T$, we can for every element $g \in \text{Aut}(\mathbb{T}^n) = GL(n, \mathbb{Z})$ define a new free action of T on X , twisted by g , by the formula $x \cdot_g t = x \cdot g(t)$. (Here $t \in T$, \cdot is the original free right action of T on X , and \cdot_g is the new twisted action.) If $c \in H^2(Z, \mathbb{Z}^n)$ was the Chern class of the original bundle, the Chern class of the g -twisted bundle is $g \cdot c$, with g acting via the action of $GL(n, \mathbb{Z})$ on \mathbb{Z}^n .

The group $GL(n, \mathbb{Z})$ embeds in $O(n, n; \mathbb{Z})$, the subgroup of $GL(2n, \mathbb{Z})$ preserving the quadratic form defined by $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, via $a \mapsto \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}$. This larger group is called the **T-duality group**.

In fact we will consider the still larger

generalized T-duality group

$$GO(n, n; \mathbb{Z}) = O(n, n; \mathbb{Z}) \rtimes (\mathbb{Z}/2)$$

of matrices in $GL(2n, \mathbb{Z})$ preserving the form $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ up to sign.

Suppose that we are in the basic setup as above, with Z simply connected, so that one is always guaranteed to have a classical T-dual. Then the generalized T-duality group $GO(n, n; \mathbb{Z})$ acts on the set of T-dual pairs (p, H) and $(p^\#, H^\#)$ to generate all related T-dual pairs. All of these pairs are physically equivalent. The restriction of the action to $GL(n, \mathbb{Z})$ (as embedded above) corresponds to twisting of the action on the same underlying space as above.

Consider the example of the 3 dimensional lens space $L(1, p) = S^3/\mathbb{Z}_p$, with H -flux $H = q$ times the volume form. Here $p, q \in \mathbb{Z}$, and initially we take $p, q > 0$. Then $L(1, p)$ is a circle bundle over the 2-dimensional sphere S^2 and has first Chern class equal to p times the volume form of S^2 . Then, as seen earlier $(L(1, p), H = q)$ and $(L(1, q), H = p)$ are T-dual to each other, and the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $O(1, 1; \mathbb{Z})$ interchanges them. The element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of the T-duality group $O(1, 1; \mathbb{Z})$ lies in the subgroup $GL(1, \mathbb{Z})$, embedded as above, and acts by twisting the S^1 action on $L(1, p)$. This twisted action makes $L(1, p)$ into a circle bundle over S^2 having first Chern class equal to $-p$ times the volume form of S^2 . This bundle is denoted $L(1, -p)$, and its total space is diffeomorphic to $L(1, p)$, though by an

orientation-reversing diffeomorphism. Therefore the action of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on the pair $(L(1, p), H = q)$ and $(L(1, q), H = p)$

gives rise to a new T-dual pair

$(L(1, -p), H = -q)$ and $(L(1, -q), H = -p)$.

The group $GO(1, 1; \mathbb{Z})$ is generated by the two elements of $O(1, 1; \mathbb{Z})$ just discussed and by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which replaces the original T-dual pair by the pair consisting of

$(L(1, p), H = -q)$ and $(L(1, -q), H = p)$.

Here we have tacitly assumed $p, q \geq 2$; we can extend things to other values of p and q by making the convention that $L(1, 1) = S^3$ and $L(1, 0) = S^2 \times S^1$. This refines earlier understood T-duality. Thus in general there are 8 different (bundle, H-flux) pairs with equivalent physics, corresponding to $(\pm p, \pm q)$ and $(\pm q, \pm p)$.