WZW Model with boundaries

Consider Riemann surface Σ with boundary, wolog $\partial\Sigma = S^1$.

1. "Naive Approach"

Ansatz for boundary conditions:

submanifold $\quad i: \; Q \to G$

and embeddings $\quad g: \; \Sigma \to G$

s.t. $\quad g(\partial\Sigma) \subseteq Q$.

Defn. of WZ term (1st attempt)

- require $\quad i^* H = d\omega \qquad \omega \in \Lambda^2(Q)$
- glue disc $D \subset Q$ s.t. $g(\Sigma) + D$ closed
- choose submfd M of G s.t.
$$\partial M = g(\Sigma) + D.$$
- WZ term
$$I_{WZ} = \int_M H - \int_D \omega$$

→ "Obstruction" $[g(\Sigma)] = 0 \in H_2(G, Q)$

→ Well-defined? choose $D_1, D_2 \in i(Q)$
$$\partial M_1 = g(\Sigma) + D_1$$
$$\partial M_2 = g(\Sigma) + D_2.$$

$$\left(\int_{M_1} H - \int_{D_1} \omega \right) - \left(\int_{M_2} H - \int_{D_2} \omega \right)$$

$$= \int_{M_1 - M_2} H - \int_{D_1 - D_2} \omega$$

$S := D_1 - D_2$ closed 2 mfd in $i(Q)$

$Z := M_1 - M_2$ $\quad \partial Z = D_1 - D_2 = S \subset i(Q)$

cycle (Z, S) with $[(Z,S)] \in H_3(G, Q)$

Pairing $\quad \int_Z H - \int_S \omega \in 2\pi \mathbb{Z}$

$\Rightarrow \quad [(H, \omega)] \in H^3(G, Q, 2\pi\mathbb{Z})$.

2. Problems

1. Problem: Are obstructions $H_3(G) = 0$ $\quad H_3(G, Q) = 0$ correct? ($\mathfrak{g}$ closed case)

Special case:

G simple compact connected Lie group.

- Algebraic results → scattering of bulk states → Q is a twisted conjugacy class:

$$\varphi : G \to G \quad \text{automorphism}$$

$$C_\omega(g_0) = \{ g \in G \mid g = h\, g_0\, \varphi(h)^{-1} \text{ with } h \in G \}$$

Confirmed by dynamics

- worldsheet action → conserved currents
- Born-Infeld

2. Problem

Take $SU(2)_k$ D-branes $\lambda = 0 \dots k$

BCFT $\longrightarrow$ ~~[illegible]~~ $SO(3)_k$, k even — D-branes are pairs $(0,k)$, $(1,k-1)$ … $(k/2)_\pm$

Two D-branes located at equator: same Q, same ω.

$\to$ Missing geometrical information.

Notice: $SO(3) = SU(2)/\mathbb{Z}_2$

$Q_{k/2}$ stabilized by $S \subseteq \mathbb{Z}$.

Alg. theory (S abelian): In general do not get $|S|$ different boundary conditions. Rather projective realization

$$\mathbb{C}S \rightsquigarrow \mathbb{C}_\varepsilon S \cong \bigoplus_{\psi \in U^*} \mathrm{Mat}_d(\mathbb{C})$$

with $U \subseteq S$ $\qquad d = \sqrt{\frac{|S|}{|U|}}$.

In string compactifications (FKLLSW 2000):

non-abelian $SU(d)$ gauge symmetry from <u>elementary</u> brane!

(3. Problem)

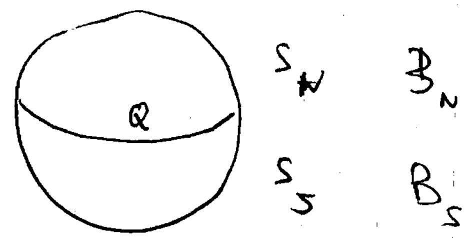
Thus: D-brane = Submfd + $U(1)$ gauge field is wrong.

Cf. also <u>4. Problem</u>: charges.

Locally on Q: $\quad F_\alpha = \omega - i^*(B_\alpha)$

$$dF_\alpha = d\omega - i^*(dB_\alpha) = 0 \to \text{line bundle!}$$

But, e.g. $SU(2)_k$



Compute $\int_Q F_N = \int_Q \omega - B_N$

$$\int_Q F_N - \int_Q F_S = \int_Q B_S - B_N = \int_{SU(2)} H = k$$

$\rightsquigarrow$ "Chern class mod k"

3. Gerbe modules

Defn: (i) Gerbe on G : $\mathcal{G} = (Y, B, L, \mu)$

$$\begin{array}{ccc} Y & L & \\ \downarrow \pi & \downarrow & B \in \Lambda^2(Y) \\ G & Y^{[2]} & \end{array}$$

$$\mu : p_{12}^* L \otimes p_{23}^* L \longrightarrow p_{13}^* L \quad \text{on } Y^{[3]}$$

s.t. associative on $Y^{[4]}$

Note $dd(\mathcal{G}) \in H^3(M, \mathbb{Z})$

(ii) Gerbe module $\mathcal{E} = (E, \rho)$

$$\begin{array}{c} E \\ \downarrow \\ Y \end{array} \qquad \rho : L \otimes p_2^* E \longrightarrow p_1^* E \quad \text{on } Y^{[2]}$$

s.t. on $Y^{[3]}$ $\rho \circ (\mu \otimes id) = \rho \circ (id \otimes \rho)$.

<u>Lemma</u> Gerbe modules exist $\Leftrightarrow$ $dd(\mathcal{G}) \in H^3(G, \mathbb{Z})$ is torsion.

<u>Defn</u> (still not final ...) : D-brane = gerbe module on (suitable) submanifold

$$i : Q \to M$$

$\rightsquigarrow$ $i^*(H) = d\omega$ in $\Lambda^3(Q)$

- suitable ?
- B-type branes

<u>Application to WZW</u>

Let $Y_Q = \pi^{-1}(Q)$, look for bundle E on Y_Q.

Conserved currents $\Rightarrow$ curvature on E is

$$F = \pi^* \omega - B|_{Y_Q} \qquad (*)$$

for a certain given $\omega \in \Lambda^2(Q)$.

<u>Puzzle 1</u> : Eqn. $(*)$ is an eqn. on Y_Q, descends to Q only for trivial gerbe.

<u>Puzzle 2</u> Gerbe modules with fixed curvature $(*)$ are torsor over flat Picard group.

Case of $SO(3)$:

$$\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2 \to \mathbb{Z}_2 \text{ torsor.}$$

<u>Puzzle 3</u> Gawedzki : irreducible gerbe modules of rank 2 occur in $Spin(4n) / \mathbb{Z}_2 \times \mathbb{Z}_2$.

<u>Lesson</u> : Can't avoid gerbes and twisted vector bundles !