

Generalized Geometry, Mirror Symmetry and T-duality

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References

Generalized Geometry

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A- and B-model

Closed strings on a Kähler manifold M are described by an $N = (2, 2)$ supersymmetric nonlinear σ -model. Twisting of this model gives a topological field theory

- **B-model:** uses complex structure J of M . The ring of observables is given by the Dolbeault cohomology $\bigoplus_{p,q} H_{\bar{\partial}}^p(\wedge^q T^{(1,0)} M)$. The category of D-branes is given by the bounded derived category of coherent sheaves $\mathcal{D}^b(M)$
- **A-model:** uses symplectic structure ω of M . The ring of observables is given by the "quantum cohomology ring" of M . The category of D-branes is (supposedly) the Fukaya category $\mathcal{F}(M)$

Mirror symmetry

Mirror symmetry interchanges the A and B -model, hence one should have an equivalence between categories $\mathcal{D}^b(M)$ and $\mathcal{F}(\widehat{M})$ (Kontsevich's homological mirror symmetry conjecture).

Problems

- From examples it has become clear that the Fukaya category (vector bundles on Lagrangian submanifolds equipped with unitary flat connections) needs to be enlarged by "coisotropic branes"
- Objects in $\mathcal{D}^b(M)$ and $\mathcal{F}(M)$ have a very different description. One would like to treat symplectic and complex structures on a more equal footing
- There exist more general $N = (2, 2)$ backgrounds (with NS flux), whose target spaces are non-Kähler, but otherwise have similar properties

SUSY nonlinear σ -models

$N = (1, 1)$ in 2D

$$S = \frac{1}{2} \int d^2\sigma d^2\theta (g_{ab}(\Phi) + B_{ab}(\Phi)) D_+ \Phi^a D_- \Phi^b$$

When does S have $N = (2, 2)$ SUSY?

$$\delta\Phi^a = \epsilon^+ J_+^a{}_b D_+ \Phi^b + \epsilon^- J_-^a{}_b D_- \Phi^b$$

Answer: iff (g, J_+, J_-, B) defines a **bi-hermitian geometry**,
i.e.

- pair of complex structures J_{\pm}
- g is hermitian with respect to both J_{\pm}
-

$$\nabla^{(\pm)} J_{\pm} = 0$$

where

$$\Gamma_{(\pm)}^a{}_{bc} = \Gamma^a{}_{bc} \pm \frac{1}{2} g^{ad} H_{dbc}$$

and $H = dB$

These data are equivalent to the existence of a twisted generalized Kähler structure $(\mathcal{G}, \mathcal{I}_1, \mathcal{I}_2)$ [Gualtieri]

\Rightarrow GENERALIZED GEOMETRY

Geometry of $T \oplus T^*$

Symmetric bilinear form on (sections of) $T \oplus T^*$

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\iota_X \eta + \iota_Y \xi)$$

Symmetries of $\langle \cdot, \cdot \rangle$ are given by orthogonal group

$$O(T \oplus T^*) \cong O(d, d)$$

Infinitesimal symmetries

Elements in the Lie algebra $\mathfrak{so}(T \oplus T^*)$ are given by

$$\begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix}$$

with $A \in \text{End } T$, $B : T \rightarrow T^*$, and $\beta : T^* \rightarrow T$ with B, β skew. Or, equivalently, $B \in \wedge^2 T^*$, $\beta \in \wedge^2 T$, through $\iota_X B = B(X)$

For a finite transformation

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : X + \xi \mapsto X + \xi + \iota_X B$$

Courant bracket on $T \oplus T^*$

The **Courant bracket** on (sections of) $T \oplus T^*$, is defined as

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}d(\iota_X \eta - \iota_Y \xi)$$

It is skew symmetric, but does not satisfy the Jacobi nor the Leibnitz equation

For $A, B, C \in T \oplus T^*$, and $f \in C^\infty(M)$, we have

$$\mathbf{Jac}(A, B, C) = [[A, B], C] + \mathbf{cycl} = d\mathbf{Nij}(A, B, C)$$

with

$$\mathbf{Nij}(A, B, C) = \frac{1}{3} (\langle [A, B], C \rangle + \mathbf{cycl})$$

While

$$[A, fB] = f[A, B] + (\rho(A)f)B - \langle A, B \rangle df$$

where $\rho : T \oplus T^* \rightarrow T$ is the projection

Twisted Courant bracket

For $B \in \Omega^2$, we have

$$[e^B(X + \xi), e^B(Y + \eta)] = e^B[X + \xi, Y + \eta] + \iota_Y \iota_X dB$$

So, for $dB = 0$ we find an automorphism of the Courant bracket.

More generally, for $H = dB \neq 0$, this suggests we should define a twisted Courant bracket by

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta] + \iota_Y \iota_X H$$

Relation to gerbes?

Courant bracket can be generalized to bracket on sections of $T \oplus \wedge^p T^*$, by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi)$$

For $B \in \Omega^{p+1} T^*$, we have

$$[e^B(X + \xi), e^B(Y + \eta)] = e^B[X, Y] + \iota_Y \iota_X dB$$

and, for a closed $H \in \Omega^{p+2}$, the twisted Courant bracket is defined by

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta] + \iota_Y \iota_X H$$

The case $p = 0$

On $T \oplus \wedge^0 T^* = T \oplus 1$

$$[X + f, Y + g] = [X, Y] + X(g) - Y(f)$$

we recognize the Lie bracket of invariant vector fields on $M \times S^1$, i.e. vector fields of the form

$$X + f \frac{\partial}{\partial \theta}$$

with X, f θ -independent

Maps $g : M \rightarrow S^1$ act on $M \times S^1$ as

$$(x, e^{i\theta}) \mapsto (x, g(x)e^{i\theta})$$

and map invariant vector fields to invariant vector fields.
They correspond to automorphisms of the bracket given by
1-form

$$A = -ig^{-1}dg \quad dA = 0$$

This can be generalized to principal circle bundles

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \pi \downarrow \\ & & M \end{array}$$

Atiyah sequence

We have a short exact sequence (**Atiyah sequence**)

$$0 \longrightarrow 1 \longrightarrow TE/S^1 \xrightarrow{\pi_*} TM \longrightarrow 0$$

which is split by a choice of connection $\nabla : TM \rightarrow TE/S^1$.
Thus $TE/S^1 \cong TM \oplus 1$, and we can translate the Lie bracket on TE/S^1 to $TM \oplus 1$ with the result

$$[X + f, Y + g]_F = [X + f, Y + g] + \iota_Y \iota_X F$$

where $\iota_Y \iota_X F = F(X, Y) = [\nabla(X), \nabla(Y)]$

Gerbe with connection

Given a gerbe with connection

$$H = dB_\alpha$$

$$B_\beta - B_\alpha = dA_{\alpha\beta}$$

$$A_{\beta\gamma} - A_{\alpha\gamma} + A_{\alpha\beta} = -ig_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma}$$

we construct a split short exact sequence

$$0 \longrightarrow TM^* \longrightarrow W \longrightarrow TM \longrightarrow 0$$

where W is a vector bundle obtained by ‘clutching’

On $U_{\alpha\beta}$

$$TM \oplus TM^* \Big|_{U_\alpha} \cong TM \oplus TM^* \Big|_{U_\beta}$$

through $X + \xi \sim X + \xi + \iota_X dA_{\alpha\beta}$. Well-defined on $U_{\alpha\beta\gamma}$.

The splitting $\nabla : TM \rightarrow W$ is defined by

$$\nabla(X) \Big|_{U_\alpha} = X + \iota_X B_\alpha.$$

This is well-defined because on $U_{\alpha\beta}$

$$X + \iota_X B_\alpha \sim X + \iota_X B_\alpha + \iota_X dA_{\alpha\beta} = X + \iota_X B_\beta$$

And the Courant bracket on W translated to the twisted Courant bracket on $TM \oplus TM^*$.

Cliff($T \oplus T^*$)

The **Clifford algebra** $\text{Cliff}(T \oplus T^*)$ is defined by

$$\{\gamma_{X+\xi}, \gamma_{Y+\eta}\} = 2\langle X + \xi, Y + \eta \rangle$$

and has a representation on $\wedge^\bullet T^*$, given by

$$\gamma_{X+\xi} \cdot \varphi = \iota_X \varphi + \xi \wedge \varphi$$

This gives rise to the **spinor** representation of $\text{Spin}(T \oplus T^*)$ such that, essentially, $\wedge^\bullet T^* = S^+ \oplus S^-$

$$S^+ = \wedge^{\text{even}} T^*$$

$$S^- = \wedge^{\text{odd}} T^*$$

$e^B \in \mathbf{Spin}(T \oplus T^*)$ acts as

$$e^B \varphi = \left(1 + B + \frac{1}{2} B \wedge B + \dots\right) \wedge \varphi$$

Pure spinors

The space

$$E_\varphi = \{X + \xi \in T \oplus T^* \mid \gamma_{X+\xi} \cdot \varphi = 0\}$$

is an isotropic subspace of $T \oplus T^*$. We say that φ is a **pure spinor** if E_φ is maximally isotropic

Note that, for $X + \xi, Y + \eta \in E_\varphi$,

$$\gamma_{[X+\xi, Y+\eta]_H} \cdot \varphi = \gamma_{X+\xi} \gamma_{Y+\eta} \cdot d_H \varphi$$

where $d_H \varphi = d\varphi + H \wedge \varphi$

Thus, a sufficient condition for E_φ to be closed under the (twisted) Courant bracket is for φ to be (twisted) closed

Generalized almost complex structure

A generalized almost complex structure (GACS) is a $\mathcal{J} \in O(T \oplus T^*)$, i.e. $\langle \mathcal{J}A, \mathcal{J}B \rangle = \langle A, B \rangle$, such that

$$\mathcal{J}^2 = -1$$

This implies that

$$E_{\pm} = \mathbf{Ker}(1 \mp i\mathcal{J}) = \{A \in T_{\mathbb{C}} \oplus T_{\mathbb{C}}^* \mid \mathcal{J}A = \pm iA\} \subset T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*$$

are maximally isotropic

Examples

- If J is an almost complex structure on T , then

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^T \end{pmatrix}$$

is a GACS, with $E_+ = T^{(0,1)} \oplus T^{*(1,0)}$

- If ω is a nondegenerate 2-form, then

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

defines a GACS

(twisted) Generalized complex structure

A (twisted) generalized complex structure (tGCS) is an AGCS \mathcal{J} , such that sections of E_{\pm} are closed under the (twisted) Courant bracket

- \mathcal{J}_J is a GCS iff J is a CS
- \mathcal{J}_ω is a GCS iff $d\omega = 0$

(twisted) Generalized CY structure

A (twisted) generalized Calabi-Yau (tGCY) structure is

1. a (twisted) closed form $\varphi \in \wedge^{\text{even}} \otimes \mathbb{C}$ (or $\wedge^{\text{odd}} \otimes \mathbb{C}$), which is a pure spinor for $\text{Cliff}(T \oplus T^*)$
2. $\langle \varphi, \bar{\varphi} \rangle \neq 0$ at each point

Recall,

$$\gamma_{[X+\xi, Y+\eta]_H} \cdot \varphi = \gamma_{X+\xi} \gamma_{Y+\eta} \cdot d_H \varphi$$

Thus, if (M, φ) is a tGCY structure, then E_φ defines a tGCS

Examples

- M a complex manifold (of complex dimension d) with nonvanishing holomorphic $\Omega \in \wedge^{d,0}$, then $\varphi = \Omega$ is closed pure spinor

$$\gamma_{X+\xi} \cdot \Omega = 0$$

for $X \in T^{(0,1)}$ and $\xi \in T^{*(1,0)}$

- (M, ω) a symplectic manifold, then $\varphi = e^{i\omega}$ is a closed pure spinor

(twisted) Generalized Kähler structure

A **(twisted) generalized Kähler structure** $(\mathcal{G}, \mathcal{J}_1, \mathcal{J}_2)$ is a pair of commuting (twisted) generalized complex structures, such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ defines a positive definite metric on $T \oplus T^*$

Example:

A Kähler manifold gives rise to a generalized Kähler structure by taking $\mathcal{J}_1 = \mathcal{J}_J$, $\mathcal{J}_2 = \mathcal{J}_\omega$. Then

$$\mathcal{G} = -\mathcal{J}_J\mathcal{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

Lie algebroids

A **Lie algebroid** is a vector bundle $E \rightarrow M$, equipped with a Lie bracket $[,]$ on $\Gamma(E)$, and a smooth map $\rho : E \rightarrow TM$ ('anchor'), which is a Lie algebra homomorphism, and satisfies the Leibnitz rule, i.e.

$$\rho([X, Y]) = [\rho(X), \rho(Y)]$$

$$[X, fY] = f[X, Y] + (\rho(X)f)Y$$

for $X, Y \in \Gamma(E)$, and $f \in C^\infty(M)$

Examples of Lie algebroids

- $E = T, \rho = id$
- $E = T^{(0,1)}, \rho : E \rightarrow T$ the inclusion map
- Isotropic, involutive subbundles E of $T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*$

Examples (cont'd)

Proof: For $A, B, C \in T \oplus T^*$, and $f \in C^\infty(M)$, we have

$$\mathbf{Jac}_H(A, B, C) = [[A, B], C]_H + \mathbf{cycl} = d\mathbf{Nij}_H(A, B, C)$$

with

$$\mathbf{Nij}_H(A, B, C) = \frac{1}{3} (\langle [A, B]_H, C \rangle + \mathbf{cycl})$$

and

$$[A, fB]_H = f[A, B]_H + (\rho(A)f)B - \langle A, B \rangle d_H f$$

E.g., when \mathcal{J} is a tGCS, $E_+ = \text{Ker}(1 - i\mathcal{J})$ is (maximally) isotropic and involutive, hence a Lie algebroid

Differential geometry on Lie algebroids

- a **differential** $d : \Omega^k \rightarrow \Omega^{k+1}$

$$d\omega(x_0, \dots, x_k) = \sum_{i=0}^k (-1)^i \rho(x_i) (\omega(x_0, \dots, \hat{x}_i, \dots, x_k)) \\ + \sum_{i < j} (-1)^{i+j} \omega([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k)$$

- a **contraction** $\iota_x : \Omega^k \rightarrow \Omega^{k-1}$

$$(\iota_x \omega)(x_1, \dots, x_{k-1}) = \omega(x, x_1, \dots, x_{k-1})$$

- a **Lie derivative** $\mathcal{L}_x : \Omega^k \rightarrow \Omega^k$

$$\mathcal{L}_x = d\iota_x + \iota_x d$$

Lie algebroid cohomology

Lie algebroid cohomology is the cohomology of the complex $(\Omega^\bullet(E), d)$.

Examples:

- For $E = T^{(0,1)}$, $\rho : E \rightarrow T$ the inclusion, we have $\Gamma(\wedge^p E^*) = \Omega^{0,p}$, and $d_E = \bar{\partial}$, hence

$$H_{d_E}^p = H_{\bar{\partial}}^{0,p}$$

• For $\mathcal{J} = \mathcal{J}_J$, we have $E = E_+ = T^{(0,1)} \oplus T^{*(1,0)}$, such that

$$\Gamma(\wedge^k E^*) = \bigoplus_{p+q=k} \left((\wedge^q T^{(0,1)}) \oplus (\wedge^p T^{*(1,0)}) \right)$$

while $d_E = \bar{\partial}$, hence

$$H_{d_E}^k = \bigoplus_{p+q=k} H_{\bar{\partial}}^p(\wedge^q T^{(0,1)})$$

Summary, results and questions

- If M has a twisted generalized Kähler structure $(\mathcal{G}, \mathcal{J}_1, \mathcal{J}_2)$, then there exists an $N = (2, 2)$ SUSY Nonlinear σ -model with target M . By twisting, we can construct topological A- and B-models, associated with the tGCS \mathcal{J}_1 and \mathcal{J}_2 , respectively
- The observables of the A- and B-models are given by the Lie algebroid cohomology of the Lie algebroids associated to \mathcal{J}_1 and \mathcal{J}_2 , respectively [Kapustin]
- What are the D-branes and can we associate in a natural way a (triangulated) category to a tGCS?

Application to T-duality

Given a principal circle bundle E with H-flux H

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \pi \downarrow \\ & & M \end{array} \quad H = \Omega + A \wedge \widehat{F}, \quad F = dA$$

there exists a T-dual principal circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & \widehat{E} \\ & & \widehat{\pi} \downarrow \\ & & M \end{array} \quad \widehat{H} = \Omega + \widehat{A} \wedge F, \quad \widehat{F} = d\widehat{A}$$

We have an isomorphism of differential complexes

$$\tau : (\Omega_{S^1}^\bullet(E), d_H) \rightarrow (\Omega_{S^1}^\bullet(E), d_{\widehat{H}})$$

$$\tau(\Omega_k + A \wedge \Omega_{k-1}) = -\Omega_{k-1} + \widehat{A} \wedge \Omega_k$$

Define a map $\phi : (TE \oplus TE^*)_{S^1} \rightarrow (T\widehat{E} \oplus T\widehat{E}^*)_{S^1}$

$$\phi(X + f\partial_A + \xi + gA) = -X - g\partial_{\widehat{A}} - \xi - f\widehat{A}$$

Theorem

- (a) The map ϕ is orthogonal wrt pairing on $T \oplus T^*$, hence it induces an isomorphism on the Clifford algebras
- (b) For $v \in (TE \oplus TE^*)_{S^1}$ we have

$$\tau(\gamma_v \cdot \Omega) = \gamma_{\phi(v)} \cdot \tau(\Omega)$$

hence τ induces an isomorphism of Clifford modules

$$\tau : \wedge^\bullet TE_{S^1}^* \rightarrow \wedge^\bullet T\hat{E}_{S^1}^*$$

- (c) ϕ preserves the twisted Courant bracket structure

END