The six-dimensional Hermitian Domain of Type IV

Diploma thesis

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Chapter 1

Introduction

By an automorphic function, one usually means a function f of a semisimple (or more general, a reductive) Lie group G with values in a complex vector space such that f is invariant under the operation of a certain discrete subgroup Γ of G. In order to study automorphic functions, one introduces the so-called automorphic forms which are functions that are only invariant under the group operation up to multiplication by a certain factor. This factor is either given by a representation of a maximal compact subgroup K of G or by a so-called factor of automorphy when defining the functions on the homogeneous space G/K. The equivalence of these two concepts is fairly hard to find in the literature and is carried out in detail in this thesis. However, this is only a minor aspect of this thesis.

The theory of automorphic forms on semisimple Lie groups is closely connected to the theory of symmetric spaces by the fact that the quotient G/K of a semisimple Lie group G by a maximal compact subgroup K is always a symmetric space. The other way round, every non-Euclidian symmetric space can be written as the quotient G/K of its semisimple group of isometries G by a maximal compact subgroup K of G. Furthermore, the quotient only depends on the Lie algebra $\mathfrak{g}$ of G. Both the non-Euclidian symmetric spaces and the semisimple Lie algebras were classified by Cartan in 1919.

A hermitian symmetric space is a symmetric space G/K that can be embedded into a complex vector space such that G operates on the image by holomorphic transformations. The realization of G/K in the complex vector space is called a hermitian domain in this case. When requiring automorphic forms on hermitian domains to be holomorphic, the theory becomes richer since methods of complex analysis can be applied. The classification of irreducible hermitian symmetric spaces reveals the existence of four infinite series labeled by capital Roman letters and two exceptional spaces. The symmetric spaces of the series III are those of the pair of the real symplectic group $Sp(n, \mathbb{R})$ and its maximal compact subgroup $U(n)$. The corresponding homogeneous space can be realized as the well-known Siegel space, the symplectic group operating by Moebius transformation, and the corresponding automorphic forms are the Siegel modular forms. The spaces IV_q of the series IV belong to the connected component $SO^+(2, q)$ of the simple Lie group $SO(2, q)$. The corresponding automorphic forms, sometimes called orthogonal modular forms, became acquainted with Borcherds who received the Fields medal for proving the Moonshine Conjecture. One important step was his discovery and application of a connection between the classical elliptic modular forms and orthogonal modular forms. The even members II_{2n} of the series II correspond to certain "hermitian symplectic" groups $Sp(n, \mathbb{H})$ over the quaternions with maximal compact subgroups $U(2n)$. The homogeneous space can be realized as a quaternary half-space in analogy to the Siegel space and the resulting automorphic forms are called modular forms of quaternions. In low dimensions, it turns out that some of the symmetric spaces of different series are isomorphic, thus allowing a comparison of different types of automorphic forms.

This thesis deals with a detailed study of an equivariant isomorphism

$$II_4 \cong IV_6$$

and some consequences concerning the corresponding automorphic forms. As the title suggests, both hermitian symmetric spaces can be realized as an open domain in a six-dimensional complex vector space. In the monograph [Sat], Satake introduces equivariant embeddings of different hermitian symmetric spaces into Siegel spaces in a very general context. This embedding is studied here in the given special case and many details applying only to this case are carried out explicitly.

In order to understand the above isomorphism, it is necessary to understand a certain amount of Lie theory, especially the notion of isogeny. By isogeneous Lie groups, one means two Lie groups having isomorphic Lie algebras. It turns out that this is equivalent to the fact that the two Lie groups have the same simply connected covering Lie group. All groups of an equivalence class of isogeneous groups are isomorphic to quotients of the universal covering group by discrete subgroups of its center. On the other hand, when taking quotients by maximal compact subgroups, isogeneous groups define isomorphic symmetric spaces. The connected component of the group of isometries of this symmetric space is necessarily isomorphic to the so-called adjoint form, which is defined as the quotient of the universal covering group by its center (assuming that the center is discrete). However, it has to be added that most of the semisimple Lie groups are also algebraic groups and the universal cover in the algebraic sense does not always agree with the topological universal cover.

The special case treated in this thesis concerns the isogeny induced by the isomorphism

$$\mathfrak{sp}(2, \mathbb{H}) \cong \mathfrak{so}(2, 6).$$

The corresponding (algebraically) simply connected group is well-known to be the Spin group of signature $(2, 6)$. The Spin group is usually defined as a certain subgroup of the Clifford algebra of the corresponding quadratic space. Its name is derived from physical applications concerning the sub-atomic spin. When the irreducible representations of the orthogonal groups were classified by Cartan, he found out that it was not possible to construct them all by applying linear algebra to the standard representation of $SO(n, \mathbb{C})$ on $\mathbb{C}^n$. However, the structure theory of real and complex Clifford algebras revealed further canonical representations. These representation of a Clifford algebra and the restrictions to the Spin group are called spin representations or spinors. All remaining irreducible representations of the orthogonal groups predicted by the general representation theory can be constructed with the help of these spin representations.

In the case of signature $(2, 6)$, there are two irreducible half-spin representations adding up to the spin representation. In the final chapter of this thesis, the spin representation is explicitly described on the standard generators of the Spin group, which are the so-called Eichler transformations. The image of the spin representation turns out to be contained in the direct product of two copies of $Sp(2, \mathbb{H})$. Furthermore, the two resulting projections correspond to the half-spin representations which are revealed to be two-sheeted coverings of $Sp(2, \mathbb{H})$. However, $SO^+(2, 6)$ and $Sp(2, \mathbb{H})$ are shown not to be isomorphic since the representations have different kernels. Moreover, an explicit isomorphism of the different realizations of the hermitian domains is given which is, of course, equivariant under the operation of the adjoint form. It should be remarked that nearly all proofs are carried out explicitly and without relying on the general Lie theory.

As a main result of this thesis, the image of the Spin group in the product $Sp(2, \mathbb{H}) \times Sp(2, \mathbb{H})$ is described with the help of an homomorphism $Sp(2, \mathbb{H}) \to Sp(2, \mathbb{H})/ \pm E$. As a consequence, it is possible to describe the operation of the Spin group on the Hermitian domain as Moebius transformation.

Another main result consists in the application of this description to determine the possible factors of automorphy. In order to achieve this, the maximal compact subgroup K of the Spin group is described from different points of view. In analogy to Siegel modular forms, a factor of automorphy with values in the complexification $K_{\mathbb{C}}$ of the maximal compact subgroup is given. The possible factors of automorphy are then obtained by composing representations of the complexification with this factor of automorphy.

Of course, the correspondence of quaternary and orthogonal modular forms is by no means exhausted by this thesis. Especially promising seems the comparison of Theta series which exist only in the quaternary world and Borcherds products which exist only in the orthogonal world.

I would like to express my gratitude to Prof. Freitag for proposing and supervising this thesis. Moreover, i thank my parents for their caring support.

Chapter 2

Matrix Groups over the Quaternion Algebra

2.1 The Quaternion Algebra

In this section, the basic properties of Hamilton's quaternions are listed without proofs. As a reference serves e.g. [Ebb].

Definition 2.1. The *quaternion algebra* $\mathbb{H}$ is defined as the four-dimensional real algebra with basis $\{1, \mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3\}$ where 1 denotes the identity element and the other basis vectors are multiplied according to the following rules:

$$\mathrm{i}_1^2 = \mathrm{i}_2^2 = \mathrm{i}_3^2 = \mathrm{i}_1\mathrm{i}_2\mathrm{i}_3 = -1$$
$$\mathrm{i}_1\mathrm{i}_2 = -\mathrm{i}_2\mathrm{i}_1 = \mathrm{i}_3, \quad \mathrm{i}_2\mathrm{i}_3 = -\mathrm{i}_3\mathrm{i}_2 = \mathrm{i}_1, \quad \mathrm{i}_3\mathrm{i}_1 = -\mathrm{i}_1\mathrm{i}_3 = \mathrm{i}_2$$

The subalgebra spanned by 1 will be identified with $\mathbb{R}$. So $1_{\mathbb{H}}$ can be regarded as $1_{\mathbb{R}}$ and will be omitted in the notation.

The most important properties of $\mathbb{H}$ are summarized in the following proposition. Recall that a division algebra is an associative algebra such that each element has a unique multiplicative inverse.

Proposition 2.2. *The quaternion algebra is an associative division algebra which is* not commutative. *The center of* $\mathbb{H}$ *is* $\mathbb{R}$.

Given two quaternions $x = x_0 + x_1\mathrm{i}_1 + x_2\mathrm{i}_2 + x_3\mathrm{i}_3$ and $y = y_0 + y_1\mathrm{i}_1 + y_2\mathrm{i}_2 + y_3\mathrm{i}_3$ the conjugate of a quaternion is defined as

$$\bar{x} := x_0 - x_1\mathrm{i}_1 - x_2\mathrm{i}_2 - x_3\mathrm{i}_3,$$

the real part as

$$Re(x) := x_0,$$

the canonical scalar product as

$$\langle x, y\rangle := 2(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3),$$

and the norm as

$$|x| := \sqrt{\frac{1}{2}\langle x, x\rangle} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

Lemma 2.3. *The most important rules for carrying out calculations in* $\mathbb{H}$ *are:*

(i) *The conjugation on* $\mathbb{H}$ *is an involution, i.e. it is* $\mathbb{R}$*-linear,* $\overline{xy} = \bar{y}\bar{x}$ *and* $\bar{\bar{x}} = x$*.*

(ii) $Re(x) = Re(\bar{x}) = \frac{1}{2}(x + \bar{x})$

(iii) $Re(xy) = Re(yx)$

(iv) $\langle x, y\rangle = 2Re(x\bar{y}) = 2Re(y\bar{x}) = x\bar{y} + y\bar{x} = \bar{x}y + \bar{y}x$ *is* $\mathbb{R}$*-bilinear, symmetric and positive definite.*

(v) $x\bar{x} = \bar{x}x = |x|^2$

(vi) $|x| = |\bar{x}|$

(vii) $|xy| = |x||y|$

(viii) $x^2 - 2Re(x)x + |x|^2 = 0$

2.2 Linear Algebra over the Quaternion Algebra

Due to the non-commutativity, one always has to be careful when trying to generalize linear algebra to matrices over quaternions. For example, the concept of a determinant no longer makes sense.

The set of $n \times n$-Matrices with entries in $\mathbb{H}$ will be denoted by

$$M_n(\mathbb{H})$$

and carries the structure of an associative unital $\mathbb{R}$-algebra. The identity element will always be denoted by E and the group of units by

$$GL(n, \mathbb{H}).$$

Lemma 2.4. *The equality*

$$\overline{AB}' = \bar{B}'\bar{A}' \tag{2.1}$$

holds for all matrices $A, B \in M_n(\mathbb{H})$*, thus entry-wise conjugation followed by transposition is an involution on* $M_n(\mathbb{H})$*.*

The proof is trivial, it might be remarked, though, that any similar equalities involving only transposition or only conjugation, which are familiar from commutative linear algebra, do not hold in general over non-commutative algebras.

The Gauss algorithm still works over a non-commutative skew field although calculations quickly become very laborious. Nevertheless, this characterizes $GL(n, \mathbb{H})$ as matrices with full rank and one can easily deduce a set of generators for $GL(n, \mathbb{H})$. In the case $n = 2$, the generators are especially simple.

Lemma 2.5.

$$GL(2, \mathbb{H}) = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{H} \rangle$$

Hermitian and Positive Definite Matrices

Hermitian and positive definite matrices with entries in $\mathbb{H}$ can often be treated like the well-known real and complex analogons, especially in the case $n = 2$.

Definition 2.6. A Matrix X $\in M_n(\mathbb{H})$ is called *Hermitian* if $X = \bar{X}'$.
The set of all hermitian matrices is called

$$Sym(n, \mathbb{H}) = \{X \in M_n(\mathbb{H}) \mid X = \bar{X}'\}.$$

The operation

$$X \mapsto X[A] := \bar{A}'XA$$

leaves $Sym(n, \mathbb{H})$ invariant for all $A \in M_{n\times m}(\mathbb{H})$, so especially $X[h]$ is real for all $h \in \mathbb{H}^n$.

A hermitian matrix Y is called *positive definite* if $Y[h] > 0$ for all $h \in \mathbb{H}^n$ which is abbreviated by $X > 0$. The set of all positive definite matrices is denoted by

$$Pos(n, \mathbb{H}) = \{Y \in Sym(n, \mathbb{H}) \mid Y > 0\}.$$

In [Kri], many properties of hermitian and positive definite matrices over the quaternions can be found. For example, one can define a real determinant for hermitian matrices generalizing the spectral theorem of linear algebra on hermitian matrices over $\mathbb{H}$. In the case $n = 2$, the determinant can simply be calculated for an $X \in Sym(2, \mathbb{H})$ by

$$det\, X = det \begin{pmatrix} x_0 & x_1 \\ \bar{x}_1 & x_2 \end{pmatrix} = x_0x_2 - \bar{x}_1x_1$$

which is always real because x_0 and x_2 obviously have to be real. This can be taken as a definition in this thesis which deals only with the case $n = 2$. As easily verified, the inverse matrix of a hermitian matrix X is

$$X^{-1} = \frac{1}{det\, X} \begin{pmatrix} x_2 & -x_1 \\ \bar{x}_1 & x_0 \end{pmatrix}.$$

As an example of the trouble induced by non-commutativity, the reader may calculate the inverse of an arbitrary 2 times 2 matrix with entries in $\mathbb{H}$ with the Gauss algorithm.

Another useful property in the case $n = 2$ is the fact that positive definite matrices can be characterized by positive principal minors in exactly the same way as if working over $\mathbb{R}$.

Lemma 2.7.

$$Pos(2, \mathbb{H}) = \{Y = \begin{pmatrix} y_0 & y_1 \\ \bar{y}_1 & y_2 \end{pmatrix} \mid y_0, y_2 \in \mathbb{R},\ y_1 \in \mathbb{H}\,;\ y_0 > 0\ ,\ det\,(Y) > 0\}$$

Symplectic Matrices

Both the real and complex symplectic groups $Sp_n(\mathbb{R})$ and $Sp_n(\mathbb{C})$ are defined as the invariance groups of standard alternating *bilinear* forms on $\mathbb{R}^{2n}$ respectively $\mathbb{C}^{2n}$. However, over the quaternions, the symplectic group $Sp(n, \mathbb{H})$ is defined as the invariance group of the standard alternating *sesquilinear* form on $\mathbb{H}^{2n}$ represented by the matrix $J := \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, that is $\langle x, y\rangle = \bar{x}'Jy$. In other words, one has

$$Sp(n, \mathbb{H}) := \{M \in M_{2n}(\mathbb{H}) \mid \bar{M}'JM = J\}.$$

The literature often uses the denotation $Sp(2n, \mathbb{H})$ or $SO^*(4n)$ (Helgason) or $SU^-(n, \mathbb{H})$ (Satake) for the same or an isomorphic group. An isomorphism to the most common form $SO^*(4n)$ is given via the matrix representations in the following section.

Lemma 2.8. *$Sp(n, \mathbb{H})$ is a subgroup of $GL(2n, \mathbb{H})$. Writing $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the following assertions are equivalent.*

(i) $M \in Sp(n, \mathbb{H})$

(ii) $\bar{M}' \in Sp(n, \mathbb{H})$

(iii) $\bar{A}'C - \bar{C}'A = \bar{B}'D - \bar{D}'B = 0 \ , \quad \bar{A}'D - \bar{C}'B = E$

(iv) $A\bar{B}' - B\bar{A}' = C\bar{D}' - D\bar{C}' = 0 \ , \quad A\bar{D}' - B\bar{C}' = E$

In this case the inverse matrix of M is

$$M^{-1} = \begin{pmatrix} \bar{D}' & -\bar{B}' \\ -\bar{C}' & \bar{A}' \end{pmatrix}$$

Relations (iii) and (iv) are called fundamental relations of symplectic matrices.

Important for many concrete calculations will be the following set of simple generators of the symplectic group.

Lemma 2.9.

$$Sp(n, \mathbb{H}) = \langle\, J \ , \begin{pmatrix} \bar{W}' & 0 \\ 0 & W^{-1} \end{pmatrix} \ , \begin{pmatrix} E & S \\ 0 & E \end{pmatrix} \mid W \in GL(n, \mathbb{H}) \ , \ S \in Sym(n, \mathbb{H}) \,\rangle$$

The center of the symplectic group is

$$\mathcal{Z}(Sp(n, \mathbb{H})) = \pm E.$$

The first two lemmata are shown in [Kri] whereas the last statement is proven by straightforward computation using the first lemma.

2.3 Matrix Representations

The matrix representations of $\mathbb{H}$ over $\mathbb{R}$ and $\mathbb{C}$ will be used pretty extensively in this thesis. Recall that a faithful representation of a unital K-algebra A is an injective homomorphism of unital K-algebras from A into the algebra of endomorphisms of some vector space V over K.

Define the mappings

$$\check{\ } : \mathbb{H} \to M_2(\mathbb{C}) \ , \quad x \mapsto \check{x} := \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$$

and

$$\hat{\ } : \mathbb{H} \to M_4(\mathbb{R}) \ , \quad x \mapsto \hat{x} := \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & -x_3 & x_2 \\ -x_2 & x_3 & x_0 & -x_1 \\ -x_3 & -x_2 & x_1 & x_0 \end{pmatrix} .$$

The latter matrix represents the linear mapping of $\mathbb{H}$ given by multiplication by $\bar{x}$ from the right with respect to the standard basis. The complex matrix representation can be defined similarly choosing an adequate basis of $\mathbb{H}$ as two-dimensional complex vector space.

Lemma 2.10. *The maps $\check{\ }$ and $\hat{\ }$ are faithful representations of the $\mathbb{R}$-algebra $\mathbb{H}$ having the following properties.*

(i) $det\,(\check{x}) = |x|^2 \ , \qquad det\,(\hat{x}) = |x|^4$

(ii) $(\check{\bar{x}}) = \bar{\check{x}}' \ , \qquad \hat{\bar{x}} = \hat{x}'$

Both representations can be extended to representations of the matrix algebras $M_n(\mathbb{H})$. Define

$$\check{}: M_n(\mathbb{H}) \to M_{2n}(\mathbb{C}) \ , \quad X = (x_{ij}) \mapsto \check{X} := (\check{x}_{ij})$$

and

$$\widehat{}: M_n(\mathbb{H}) \to M_{4n}(\mathbb{R}) \ , \quad X = (x_{ij}) \mapsto \widehat{X} := (\hat{x}_{ij}).$$

Lemma 2.11. *Both* $\check{}$ *and* $\widehat{}$ *are faithful representations of unital* $\mathbb{R}$*-algebras satisfying*

(i) $\overline{\check{A}}' = \overline{\check{A}}'$,

(ii) det $\check{X}$ *is always real,*

(iii) $(det\,\check{X})^2 = det\,\widehat{X}$,

(iv) If $X \in Sym(2,\mathbb{H})$ *additionally* $det\,(\check{X}) = (det\,X)^2$ *and* $det\,(\widehat{X}) = (det\,X)^4$,

(v) If $X \in Sp(n,\mathbb{H})$, $det\,(\check{X}) = 1$,

(vi) $X \in GL(n,\mathbb{H}) \ \Leftrightarrow \ \check{X} \in GL(2n,\mathbb{C}) \ \Leftrightarrow \ \widehat{X} \in GL(4n,\mathbb{R})$ *and*

(vii) $X \in Sp(n,\mathbb{H}) \ \Leftrightarrow \ \widehat{X} \in Sp(4n,\mathbb{R})$.

This lemma allows the definition of a substitute for the determinant of matrices over $\mathbb{H}$ via one of the embeddings.

A Complexification of $Sp(n,\mathbb{H})$

This section deals with the extension of scalars of the $\mathbb{R}$-algebra $\mathbb{H}$ by tensoring with $\mathbb{C}$. The isomorphisms

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{H} \oplus \mathrm{i}\mathbb{H}$$

and

$$M_n(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}) \cong M_n(\mathbb{H}) \oplus \mathrm{i}M_n(\mathbb{H})$$

are well-known. A $\mathbb{C}$-linear involution on $\mathbb{H} \oplus i\mathbb{H}$ is obviously given by

$$X + \mathrm{i}Y \mapsto \bar{X}' + \mathrm{i}\bar{Y}'.$$

Thus, the complexifications $Sp(n,\mathbb{H}\otimes\mathbb{C})$ and $Sym(n,\mathbb{H}\otimes\mathbb{C}) \cong Sym(n,\mathbb{H}) \oplus \mathrm{i}Sym(n,\mathbb{H})$ can be defined in the obvious way.

Lemma 2.12. *The following maps are isomorphisms of* $\mathbb{C}$*-algebras.*

(i) $\mathbb{H} \oplus \mathrm{i}\mathbb{H} \xrightarrow{\sim} M_2(\mathbb{C})$, $x + \mathrm{i}y \mapsto \check{x} + \mathrm{i}\check{y}$

(ii) $M_n(\mathbb{H}) \oplus \mathrm{i}M_n(\mathbb{H}) \xrightarrow{\sim} M_{2n}(\mathbb{C})$, $X + \mathrm{i}Y \mapsto \check{X} + \mathrm{i}\check{Y}$

The proof consists mainly in comparing dimensions.

However, the relation $\overline{\check{A}}' = \overline{\check{A}}'$ no longer holds over $\mathbb{H} \oplus \mathrm{i}\mathbb{H}$ since the complex conjugation on the right hand side is not $\mathbb{C}$-linear.

Lemma 2.13. *After applying the isomorphism* $\mathbb{H}\otimes\mathbb{C} \xrightarrow{\sim} M_2(\mathbb{C})$*, the* $\mathbb{C}$*-linear quarternary conjugation on* $\mathbb{H}\otimes\mathbb{C}$ *corresponds to the mapping*

$$M \mapsto A^{-1}M'A \qquad \textit{with} \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This implies on the level of $n \times n$-matrices with entries in $\mathbb{H} \otimes \mathbb{C}$ that quaternary conjugation composed with transposition corresponds to the mapping

$$M \mapsto \tilde{J}^{-1} M' \tilde{J} \qquad \text{with} \qquad \tilde{J} = \begin{pmatrix} A & & & \\ & A & & \\ & & \dots & \\ & & & A \end{pmatrix}$$

on $M_{2n}(\mathbb{C})$.

Proof. The quarternary conjugation on a matrix $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ obviously results in $\begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$. The given matrix multiplication, which is clearly $\mathbb{C}$-linear, yields the same result. On matrix level, the proof is still trivial. □

As a consequence of this lemma, the symplectic relation $\bar{M}'JM = J$ in $GL(2n, \mathbb{H} \otimes \mathbb{C})$ translates to

$$\tilde{J}^{-1} M' \tilde{J} J M = J \qquad \Leftrightarrow \qquad M' \tilde{I} M = \tilde{I}$$

in $GL(4n, \mathbb{C})$ with

$$\tilde{I} := \tilde{J} J = \begin{pmatrix} & & & A & & \\ & & & & \dots & \\ & & & & & A \\ -A & & & & & \\ & \dots & & & & \\ & & -A & & & \end{pmatrix}$$

when applying the embedding $\check{}$. The matrix $\tilde{I}$ is invertible and symmetric because of $-A = A'$, thus defines a symmetric non-degenerate bilinear form. As explained in the next chapter, the image of $Sp(n, \mathbb{H})$ must therefore be contained in an orthogonal group isomorphic to $SO(4n, \mathbb{C})$. A comparison of dimensions delivers that the complexification $Sp(n, \mathbb{H} \otimes \mathbb{C})$ is isomorphic to $SO(4n, \mathbb{C})$.

The matrix embedding $\check{}$ also induces an isomorphism of $Sp(n, \mathbb{H})$ with the group

$$SO^*(4n) := \{M \in SL(4n, \mathbb{C}) \mid \bar{M}'JM = J\,,\ M'M = E\}$$

which is defined for example in [Hel].

An easy calculation yields the image of $GL(n, \mathbb{H})$ in $GL(2n, \mathbb{C})$ under the embedding $\check{}$ which is

$$GL(n, \mathbb{H}) \cong \{M \in GL(2n, \mathbb{C}) \mid \tilde{J}^{-1} M \tilde{J} = \bar{M}\}.$$

Thus,

$$\begin{aligned} Sp(n, \mathbb{H}) &\cong \{M \in SL(4n, \mathbb{C}) \mid \tilde{J}^{-1} M \tilde{J} = \bar{M}\,,\ M' \tilde{I} M = \tilde{I}\} \\ &= \{M \in SL(4n, \mathbb{C}) \mid \bar{M}' \tilde{I} \tilde{J}^{-1} M = \tilde{I} \tilde{J}^{-1}\,,\ M' \tilde{I} M = \tilde{I}\} \\ &\cong SO^*(4n) \end{aligned}$$

where the last isomorphism is given by an adequate change of basis.

Chapter 3

Classical Orthogonal Groups

By a classical orthogonal group, one usually means the real and complex orthogonal groups and all groups having the same Lie algebra. The classical references for the whole chapter are [Che] and [Die]. The definitions and some details about the Clifford algebra and the Spin groups are taken from more recent books like [Ha-OM] or [Law].

3.1 Quadratic Spaces

The section summarizes the standard facts about quadratic spaces over fields of characteristic different from 2.

Bilinear forms and quadratic forms

Let V be a finite-dimensional vector space over a field K. If characteristic K is different from 2, there is a one-to-one correspondence between symmetric bilinear forms on V and quadratic forms on V. The correspondence is given by assigning to a symmetric bilinear form

$$\langle\ ,\ \rangle : V \times V \to K$$

the quadratic form

$$q(x) := \frac{1}{2}\langle x, x\rangle.$$

and the other way round by the polarization formula

$$\langle x, y\rangle = q(x+y) - q(x) - q(y).$$

After choosing a basis $e_1, ..., e_n$ of V, a bilinear form is represented by the so-called Gram matrix

$$G = (g_{ij}) := (\langle e_i, e_j\rangle).$$

When the basis is changed by a matrix $A = (a_{ij})$, $f_i = \sum_j a_{ij} e_j$ the Gram matrix transforms to $A'GA$.

A symmetric bilinear form is called non-degenerate if the linear map

$$V \to V^*\ ,\ x \mapsto \langle ., x\rangle$$

is injective for all $x \in V$. Equivalent is the property that the Gram matrix representing the bilinear form is invertible for one and hence for any basis.

In this thesis, a field is always considered to have a characteristic different from 2.

Quadratic Spaces

Definition 3.1. (i) A pair $(V, \langle\ ,\ \rangle)$ of a finite-dimensional vector space over a field of characteristic different from 2 and a non-degenerate symmetric bilinear form on V will be called a *quadratic space.* (V, q) shall denote the same quadratic space where q is the corresponding quadratic form.

(ii) An *isometry* of two quadratic spaces $(V_i, \langle\ ,\ \rangle_i)$ over the same field K is a linear map $\sigma : V_1 \to V_2$ satisfying $\langle \sigma(x), \sigma(y)\rangle_1 = \langle x, y\rangle_2$.

(iii) Two quadratic space are called isomorphic (as quadratic spaces), if there is a bijective isometry.

(iv) An *orthogonal transformation* of a quadratic space $(V, \langle\ ,\ \rangle)$ is an isometric automorphism of V.
The *orthogonal group* $O(V) = O(V, \langle, \rangle)$ is defined as the group of orthogonal transformations of a quadratic space. In other words, an orthogonal group is the invariance group of a non-degenerate symmetric bilinear form on a finite-dimensional vector space.

(v) The *special orthogonal group* $SO(V)$ denotes the subgroup of $O(V)$ of orthogonal transformations of determinant 1.

Orthogonal transformations have necessarily a determinant of 1 or -1, which is easily proved introducing a basis. Hence, $SO(V)$ has index 2 in $O(V)$.

Lemma 3.2. *For $dim\, V \geq 3$, the center of $O(V)$ is $\{\pm id_V\}$. Hence, the center of $SO(V)$ is $\{\pm id_V\}$ in even dimension and trivial in odd dimension in this case.*

The following facts about quadratic spaces over $\mathbb{C}$ and $\mathbb{R}$ are well known from linear algebra.

Proposition 3.3. *(i) Every quadratic space (V, q) over $\mathbb{C}$ of dimension n is isomorphic to $\mathbb{C}^n$ equipped with the quadratic form*

$$q(x) = x_1^2 + ... + x_n^2.$$

So an isomorphism class of a complex quadratic space is already determined by the dimension of V.

(ii) Let $\mathbb{R}^{p,q}$ denote the quadratic space $(\mathbb{R}^{p+q}, q)$ with the quadratic form

$$q(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2.$$

Every quadratic space (V, q) over $\mathbb{R}$ is isomorphic to an $\mathbb{R}^{p,q}$. So an isomorphism class of real quadratic spaces is determined by the pair (p, q) which is called the signature of (V, q).

The orthogonal groups of these standard quadratic spaces can be identified with some well-known matrix groups.

$$\begin{aligned} O(n,\mathbb{C}) &:= \{A \in GL(n,\mathbb{C}) \mid A'A = E\} \\ O(p,q) &:= \{A \in GL(n,\mathbb{R}) \mid A'E^{p,q}A = E^{p,q}\} \\ O(n) &:= O(n,\mathbb{R}) = O(n,0) = \{A \in GL(n,\mathbb{R}) \mid A'A = E\} \end{aligned}$$

$E^{p,q}$ denotes a diagonal matrix with diagonal entries p times 1 and q times -1. By the proposition, all orthogonal groups over $\mathbb{C}$ and $\mathbb{R}$ are isomorphic to one of these matrix groups.

The classification of non-degenerate hermitian forms on complex vector spaces looks completely analogous. This motivates the following definition of the standard unitary groups.

$$\begin{aligned}
U(p,q) &:= \{A \in GL(n,\mathbb{C}) \mid \bar{A}' E^{p,q} A = E^{p,q}\} \\
SU(p,q) &:= \{A \in SL(n,\mathbb{C}) \mid \bar{A}' E^{p,q} A = E^{p,q}\} \\
U(n) &:= \{A \in GL(n,\mathbb{C}) \mid \bar{A} A = E\} \\
SU(n) &:= \{A \in SL(n,\mathbb{C}) \mid \bar{A}' A = E\}.
\end{aligned}$$

Proposition 3.4. *Every quadratic space admits an orthogonal basis* $e_1, ...e_n$*, that is* $\langle e_i, e_j\rangle = a_i \delta_{ij}$.

Over $\mathbb{R}$, the a_i can obviously be normalized to ± 1 and the resulting basis is called orthonormal.

Definition 3.5. (i) The orthogonal direct sum $V_1 \perp V_2$ of two quadratic spaces $(V_i, \langle\ ,\ \rangle_i)$ is defined as the quadratic space $(V_1 \oplus V_2, \langle\ ,\ \rangle)$ with the bilinear form

$$\langle x_1 + x_2, y_1 + y_2\rangle = \langle x_1, y_1\rangle_1 + \langle x_2, y_2\rangle_2.$$

(ii) Two vectors x,y of a quadratic space are called *orthogonal* to each other, if $\langle x, y\rangle = 0$. A vector x with $q(x) = 0$ is called *isotropic*, otherwise anisotropic.

(iii) The orthogonal complement $W^\perp$ of a subspace W of a quadratic space V is defined as the subspace of all vectors orthogonal to all vectors of W.

The equality

$$dimW + dimW^\perp = dimV$$

holds for any subspace W, whereas

$$W \perp W^\perp = V$$

if and only if the restriction of the bilinear form to W is non-degenerate.

Reflections

The previous observation leads to the decomposition $V = K \cdot a \perp a^\perp$ for any anisotropic vector a. In this case, the reflection σ_a along a is the automorphism of V which maps a to $-a$ and operates trivially on the hyperplane $a^\perp$. This reflection σ_a can also be expressed by the formula

$$\sigma_a(x) = x - \frac{\langle x, a\rangle}{q(a)} a$$

As easily proven, a reflection is an orthogonal transformation of determinant -1. The following important result is still not hard to prove.

Proposition 3.6. *Let* $(V, \langle\ ,\ \rangle)$ *be a quadratic space. The orthogonal group O(V) is generated by the reflections along anisotropic vectors. An orthogonal transformation belongs to the subgroup SO(V) if and only if it can be written as a product of an even number of reflections.*

Hyperbolic Planes

A hyperbolic plane is a two dimensional quadratic space V which admits a basis f_1, f_2 with the Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the case $K = \mathbb{R}$, a hyperbolic plane has the signature (1,1), which is immediately verified when changing to the basis $\frac{1}{\sqrt{2}}(f_1 \pm f_2)$.

The other way round starting with the prototype of an indefinite real quadratic space $\mathbb{R}^{p,q}$, $p, q \geq 1$, as defined above with standard basis $e_1, ..., e_n$ one finds a hyperbolic plane as the subspace spanned by $\frac{1}{\sqrt{2}}(e_1 \pm e_{p+1})$. The orthogonal complement of this subspace is isomorphic to $\mathbb{R}^{p-1,q-1}$. Induction implies the following lemma.

Lemma 3.7. *Let $V = \mathbb{R}^{p,q}$ be a real quadratic space with $p \leq q$. Then V is isomorphic to a direct sum of p hyperbolic planes and a negative definite subspace V_0.*

$$V \cong H_1 \perp ... \perp H_p \perp V_0$$

Witt's Theorem

The most fundamental result about orthogonal groups is Witt's Theorem. The proof takes some time but can be found in any of the references.

Proposition 3.8. *(Witt's Theorem) Let $(V, \langle\ ,\ \rangle)$ be a quadratic space as defined above. Let W_1, W_2 be subspaces of V and let $\sigma : W_1 \to W_2$ be an isometry. Then σ can be extended to an orthogonal transformation of $O(V)$.*

The Vector Representation of an Orthogonal Group

To begin with, some definitions concerning representations are recalled.

A representation of a group G on a vector space V is a group homomorphism ρ of G in the group $GL(V)$ of automorphisms of V. Often the vector space V itself is called a representation of G (or a G-module) if G acts on V via a representation ρ. An equivariant or G-linear map of two representations V and W is a linear map $V \to W$ that commutes with the action of G. Hence, two representations are called equivalent if there is an equivariant isomorphism $V \to W$. A representation V is called irreducible (or simple) if there is no proper nonzero subspace W that is invariant under the action of G (that is G-invariant) and completely reducible if there is a direct sum decomposition of V into G-invariant subspaces.

Given a quadratic space (V, q), the standard action of the orthogonal group $O(V)$ on V is obviously a representation which is called the vector representation of $O(V)$. It is a well-known fact, that this representation is irreducible when the characteristic of K is different from 2 (cf. [Che], prop. I.6.2).

If x is an element with norm $q(x) \neq 0$, the subspace of V spanned by all vectors v of the same norm as x is obviously invariant under the operations of $O(V)$. The irreducibility of the vector representation immediately implies the following lemma.

Lemma 3.9. *Let (V, q) be a quadratic space with char $K \neq 2$ and q non-degenerate and let x be an element with norm $q(x) \neq 0$. Then V has a basis of vectors of the same norm $q(x)$.*

3.2 Clifford Algebras

Informally, the Clifford algebra of a quadratic space V is the free associative algebra generated by V modulo the relation $v^2 = q(v)$. In modern algebraic language, this can be expressed slightly more formally.

Definition 3.10. Let (V, q) be a quadratic space. An associative K-algebra $\mathscr{C} = \mathscr{C}(V) = \mathscr{C}(V, q)$ together with a linear map $\iota : V \to \mathscr{C}(V, q)$ is a Clifford algebra of V if and only if any linear mapping $\psi : V \to A$ into any associative K-algebra A satisfying $\psi(x)^2 = q(x) \cdot 1_A$ factors through $\mathscr{C}(V, q)$. More exactly, there is a unique homomorphism $\phi : \mathscr{C}(V, q) \to A$ of K-algebras making the following diagram commutative.

$$\begin{tikzcd} V \arrow[rr, "\iota"] \arrow[dr, "\psi"'] & & \mathscr{C}(V,q) \arrow[dl, "\exists!\phi"] \\ & A & \end{tikzcd}$$

By abstract nonsense, a Clifford algebra is uniquely determined up to a unique isomorphism. The definition implies that

$$\iota(a)^2 = q(a) \cdot 1_{\mathscr{C}} \qquad \forall a \in V$$

or equivalently by polarization

$$\iota(a)\iota(b) + \iota(b)\iota(a) = \langle a, b\rangle \cdot 1_{\mathscr{C}} \qquad \forall a, b \in V.$$

Proposition 3.11. *The Clifford algebra $\mathscr{C}(V, q)$ exists for any quadratic space (V, q). The map ι is injective and $\mathscr{C}(V, q)$ has dimension 2^n over K, if $n = dim(V)$.*

Proof. One defines the Clifford algebra as the quotient of the tensor algebra $\bigoplus V^{\otimes i}$ by the two-sided ideal generated by all elements $x \otimes x - q(x) \cdot 1_{\mathscr{C}}$ for $x \in V$. Then the embedding of V and the universal property follow from the corresponding properties of the tensor algebra. Details and the proof of the dimension can be found e.g. in [La]. □

For the sake of readability, the vector space V will be identified with $\iota(V)$ and ι will be omitted in the notation. After choosing a basis of V, the Clifford algebra $\mathscr{C}$ is spanned as a vector space over K by

$${e_1}^{\nu_1} ... {e_n}^{\nu_n} \qquad \text{with } \nu_i = 0 \text{ or } 1 \text{ for } i = 1, ..., n$$

Using the defining relations

$${e_i}^2 = q(e_i) \quad \text{and} \quad e_i e_j + e_j e_i = \langle e_i, e_j\rangle \quad \text{for } i \neq j,$$

basis vectors can be multiplied in a convenient way especially when an orthonormal basis is chosen.

Example 3.12. The Clifford algebras of $\mathbb{R}^{0,0}$, $\mathbb{R}^{0,1}$ and $\mathbb{R}^{0,2}$ are isomorphic to $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ respectively. From this point of view, the concept of a Clifford algebra may be regarded as a generalization of the quaternion algebra.

The even and odd part, $\mathscr{C}^+ = \mathscr{C}^+(V, q)$ and $\mathscr{C}^- = \mathscr{C}^-(V, q)$, of a Clifford algebra are defined as the span of products of an even respective odd number of basis elements. This definition provides the Clifford algebra with the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded algebra which means that

$$\mathscr{C} = \mathscr{C}^+ \oplus \mathscr{C}^- , \quad \mathscr{C}^+\mathscr{C}^- \subset \mathscr{C}^- , \quad \mathscr{C}^-\mathscr{C}^+ \subset \mathscr{C}^- , \quad \mathscr{C}^+\mathscr{C}^+ \subset \mathscr{C}^+ , \quad \mathscr{C}^-\mathscr{C}^- \subset \mathscr{C}^+.$$

Both parts have dimensions 2^{n-1}, as follows from counting basis vectors. Furthermore, $\mathscr{C}^+$ is obviously generated by the two-products of basis vectors of an arbitrary basis of V.

Remark 3.13. (i) The linear embedding $V \to \mathscr{C}(V)$, $v \mapsto -v$ lifts by the universal property of the Clifford algebra to the unique main automorphism $^* : \mathscr{C} \to \mathscr{C}$ of the Clifford algebra, that is

$$(a + b)^* = a^* + b^* , \quad (ab)^* = a^*b^* , \quad v^* = -v , \quad \lambda^* = \lambda^*$$

for $a, b \in \mathscr{C}$, $v \in V$, $\lambda \in K$.

(ii) There is a unique involution $'$: $\mathscr{C} \to \mathscr{C}$ of the Clifford algebra called the main involution with the properties

$$(a+b)' = a' + b' \,, \quad (ab)' = b'a' \,, \quad v' = v \,, \quad \lambda' = \lambda$$

for $a, b \in \mathscr{C}$, $v \in V$, $\lambda \in K$.

Both the main automorphism * and the main involution $'$ respect the $\mathbb{Z}/2\mathbb{Z}$-grading and commute with each other. In addition, the main automorphism acts trivially on the even part $\mathscr{C}^+$ of the Clifford algebra.

Chiral Elements and the Center of a Clifford Algebra

After choosing an orthogonal basis $e_1, ..., e_n$ of V, the element

$$\chi := e_1 ... e_n$$

can be defined which is called a chiral element of the Clifford algebra $\mathscr{C}(V)$ by some authors. The line $K \cdot \chi$ does not depend on the choice of the basis as a change of basis by an invertible matrix A yields the new chiral element $\tilde{\chi} = det\, A\chi$. Easy calculations result in the equation

$$\chi e_i = (-1)^{n-1} e_i \chi$$

and the center $\mathscr{Z}$ of a Clifford algebra which is (confer [Die])

Lemma 3.14.

$$\mathscr{Z}(\mathscr{C}) = \begin{cases} K & \textit{if dim } V \textit{ is even,} \\ K \oplus K \cdot \chi & \textit{if dim } V \textit{ is odd,} \end{cases}$$

and

$$\mathscr{Z}(\mathscr{C}^+) = \begin{cases} K \oplus K \cdot \chi & \textit{if dim } V \textit{ is even,} \\ K & \textit{if dim } V \textit{ is odd.} \end{cases}$$

The Structure of the Clifford Algebra

Recall that a quadratic space is defined here over a field of characteristic different from two and that its quadratic form is always assumed to be non-degenerate. Under this presumptions, some important results about the structure of the Clifford algebra are proved in [Che]. Actually, the following proposition is already a corollary using the results about the center calculated above. Recall that a central simple K-algebra is a K-algebra with center K having no non-trivial two-sided ideals.

Proposition 3.15. *(i) Assume (V,q) is a quadratic space of even dimension. Then the Clifford algebra $\mathscr{C}(V,q)$ is a central simple algebra.*
The even part $\mathscr{C}^+(V,q)$ is either simple or the direct sum of two simple ideals. If $\mathscr{C}^+$ is not simple, there is a chiral element χ of square 1 and the two simple ideals of $\mathscr{C}^+$ are spanned by $1+\chi$ and $1-\chi$.

(ii) If the dimension of V is odd, $\mathscr{C}^+$ is central simple. The whole Clifford algebra is isomorphic to $\mathscr{Z}(\mathscr{C}) \otimes \mathscr{C}^+$ and is either simple or the direct sum of two simple ideals.

A finite-dimensional simple algebra over a field K of characteristic 0 is always isomorphic to a matrix algebra over a skew field in which K is central. In the case of $\mathbb{R}$, the only skew fields with this property are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. The Clifford algebra of the quadratic space $\mathbb{R}^{p,q}$ is denoted by $\mathscr{C}(p,q)$. The complete list of all real Clifford algebras $\mathscr{C}(p,q)$ identified with matrix algebras can be found in the monograph [Law] including proofs.

The structure of nearly all even parts $\mathscr{C}^+(p,q)$ of real Clifford algebras follows from this classification by the following lemma.

Lemma 3.16. *Suppose $q \geq 1$ and let $e_1, ..., e_{p+q}$ denote the standard orthonormal basis of $\mathbb{R}^{p,q}$. Then the map defined on generators by*

$$\mathscr{C}^+(p,q) \to \mathscr{C}(p,q-1)\,,\ e_{p+q}e_i \mapsto e_i \quad 1 \leq i \leq p+q-1$$

is an isomorphism of algebras.

In this thesis, the isomorphism

$$\mathscr{C}^+(2,6) \cong \mathscr{C}(2,5) \cong M_4(\mathbb{H}) \oplus M_4(\mathbb{H})$$

is crucial and will be constructed explicitly.

3.3 Spin Groups

In order to adequately introduce the Pin groups together with the Spin groups, the definitions in this chapter are given in a slightly more general context than needed. However, this is useful for orthogonal modular forms and elucidates possible generalizations of this thesis. [Law] serves as a reference for this section.

Definition 3.17. Define the twisted Clifford group $G = G(V,q)$ of a quadratic space (V,q) as the group of invertible elements x of the Clifford algebra $\mathscr{C}$ such that $x^*vx^{-1} \in V$ for all $v \in V$. The resulting group homomorphism

$$\pi : G(V,q) \to GL(V)\,,\ x \mapsto (v \mapsto x^*vx^{-1})$$

is called the twisted adjoint representation of the Clifford group. Define further the Clifford norm N as the mapping

$$N : \mathscr{C} \to \mathscr{C}\,,\ x \mapsto xx^{*\prime}.$$

The most important properties of π and N are summarized in the following proposition.

Proposition 3.18. *The Clifford norm restricted to the twisted Clifford group defines a homomorphism*

$$N : G \to K^*.$$

The following sequence of groups is exact.

$$1 \longrightarrow K^* \longrightarrow G(V,q) \xrightarrow{\pi} O(V,q) \longrightarrow 1$$

An anisotropic vector $a \in V$ viewed as an element of the Clifford algebra is mapped by π to the reflection σ_a along a.

Proof. The first statement, the computation of the kernel of π and the orthogonality of the transformation $\pi(x)$ are Propositions 2.5, 2.4 and 2.6 in [Law]. The calculation

$$a^*va^{-1} = -av\frac{a}{q(a)} = -\frac{a}{q(a)}(\langle a,v\rangle - av) = -\frac{\langle a,v\rangle}{q(a)}a + v = \sigma_a(v)$$

proves the second statement which implies the surjectivity of π. $\square$

Definition 3.19. Define the Pin group and the Spin group of a quadratic space (V, q) as

$$\begin{aligned} Pin(V) &:= \{x \in G \mid N(x) = 1\} \\ Spin(V) &:= Pin(V) \cap \mathscr{C}^+. \end{aligned}$$

This definition of the Spin group agrees with the classical definition of [Che] since the main automorphism acts trivially on the even part $\mathscr{C}^+$ of the Clifford algebra.

The Clifford norm on $K^* \cdot 1_{\mathscr{C}}$ is obviously given by $N(\lambda) = \lambda^2$. Then a short chase in the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & K^* \arrow[r] \arrow[d, "N"'] & G(V,q) \arrow[r, "\pi"] \arrow[d, "N"'] & O(V,q) \arrow[r] \arrow[d, dotted, "SN"'] & 1 \\ 1 \arrow[r] & (K^*)^2 \arrow[r] & K^* \arrow[r] & K^*/(K^*)^2 \arrow[r] & 1 \end{tikzcd}$$

defines the so-called Spinor norm homomorphism $SN : O(V, q) \to K^*/(K^*)^2$.
A reflection σ_a lifts to a $\lambda a \in G(V, q)$ with $\lambda \in K^*$ and then $N(\lambda a) = \lambda^2 a a^{*\prime} = -q(a)\lambda^2$ implies

$$SN(\sigma_a) = -q(a) \cdot (K^*)^2. \tag{3.1}$$

This minus sign is again due to the twisting and vanishes on the even part. Hence SN corresponds to the classical Spinor norm when restricted to the Spin group.

Proposition 3.20. *The restriction of the twisted adjoint representation π to Pin(V) and Spin(V) yields the exact sequences*

$$1 \longrightarrow \{\pm 1\} \longrightarrow Pin(V) \xrightarrow{\pi} O(V) \xrightarrow{SN} K^*/K^{*2}$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow Spin(V) \xrightarrow{\pi} SO(V) \xrightarrow{SN} K^*/K^{*2}$$

Proof. Let $g \in O(V)$ be in the kernel of the Spinor norm, that is $g = \pi(x)$ with $x \in G$ and $N(x) = \lambda^2$ for an $\lambda \in K^*$. But then $N(\frac{x}{\lambda}) = 1$ and $\pi(\frac{x}{\lambda}) = \pi(x) = g$, hence g is the image of $\frac{x}{\lambda} \in Pin(V)$ as needed. The kernel of π restricted to $Pin(V)$ is $\{\pm 1\}$ because $\lambda \in K^*$ with $N(\lambda) = \lambda^2 = 1$ implies $\lambda = \pm 1$. An even product of reflections lifts to an element of the even part of the Clifford algebra and therefore the exact sequence involving $Spin(V)$ and $SO(V)$ is immediately implied by the first sequence. □

The kernel of the Spinor norm homomorphism restricted to $SO(V)$ respectively the image of $Spin(V)$ under the homomorphism π is called the Spinorial kernel and denoted by

$$SO^+(V).$$

In accordance, the whole kernel of the Spinor norm, that is the image of $Pin(V)$ in the orthogonal group $O(V)$, is denoted by

$$O^+(V)$$

such that $SO^+(V) = SO(V) \cap O^+(V)$. When the underlying field is $\mathbb{R}$, one obviously has

$$\mathbb{R}^*/(\mathbb{R}^*)^2 \cong \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Then (3.1) shows that $O^+(V)$ is generated by the reflections along vectors a of norm $q(a) = -1$ and two-products of reflections along vectors a and b of norm $q(a) = q(b) = 1$. In the positive definite case, this implies that $O^+ = SO^+ = SO$ and in the negative definite case $O^+ = O$ and $SO^+ = SO$.

The Case $\mathbb{R}^{2,n}$

The quadratic space $\mathbb{R}^{2,n}$ is the only one that matters in this thesis as orthogonal modular forms can be defined only for the groups $O(2,n)$. The denotations $Spin(2,n)$ and $Pin(2,n)$ do not need explanations.

The signature $(2,n)$ guarantees the existence of vectors of positive and negative norm. Hence, the Spinorial kernel $SO^+(2,n)$ is a subgroup of index 2 in $SO(2,n)$ and of index 4 in $O(2,n)$.

When $e_1, ..., e_{n+2}$ denotes the standard orthonormal basis of $\mathbb{R}^{2,n}$, consider the chiral element

$$\chi = e_1 ... e_{n+2}$$

corresponding to this basis. Obviously one has $N(\chi) = (-1)^{2+n}(-1)^n = 1$. Furthermore, one calculates

$$\chi^* e_i \chi^{-1} = (-1)^{n+2} e_1 ... e_{n+2} e_i (-1)^n e_{n+2} ... e_1 = \underbrace{(-1)^n}_{\text{signature (2,n)}} \underbrace{(-1)^{n+1}}_{\text{transpositions}} e_i = (-1)^{2n+1} e_i.$$

Therefore, $\chi \in Pin(2,n)$ and

$$\pi(\chi) = -id_V. \tag{3.2}$$

In other words, $-id = \sigma_{e_1} ... \sigma_{e_{n+2}}$ is the product of the reflections along all the basis vectors.

Lemma 3.21. *If n is even, the center of $Spin(2,n)$ is*

$$\mathcal{Z}(Spin(2,n)) = \{\pm 1, \pm\chi\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

and $\{\pm 1\}$ if n is odd.

Proof. First of all, observe that the Spin group generates $\mathscr{C}^+$ as an $\mathbb{R}$-algebra. To see this, choose a basis $e_1, ..., e_n$ according to Lemma 3.9 of vectors of the same norm $q(e_i) = a \neq 0$. Then the elements $a^{-1}e_ie_j$ have the norm $N(a^{-1}e_ie_j) = a^{-2}q(e_i)q(e_j) = 1$ and generate $\mathscr{C}^+$. The relation $(e_ie_j)^* v (e_ie_j)^{-1} \in V$ is obvious.
Therefore, the center of the Spin group is contained in $\mathcal{Z}(\mathscr{C}^+)$. The rest is easily seen when calculating the Clifford norm of the elements of $\mathcal{Z}(\mathscr{C}^+)$. □

Eichler Calculus

Eichler transformations are introduced to find a set of generators of most Spin groups, not including those of quadratic spaces without isotropic vectors. They are especially important as they preserve most of their properties when the field K is replaced by a ring.

Let V be a quadratic space having an isotropic vector. Let u be such a vector and let v be a vector in the orthogonal complement of u or in terms of the Clifford algebra

$$u^2 = 0\,, \quad uv = -vu.$$

These relations imply after a short calculation that $1 + uv$ lies in the Spin group. The same calculation yields the image of $1 + uv$ in the Spinorial kernel $SO^+(V)$ which is denoted by $E(u,v)$ and acts on V as

$$a \mapsto E(u,v)(a) = a - \langle a,u\rangle v + \langle a,v\rangle u - q(v)\langle a,u\rangle u$$

These orthogonal transformations are called Eichler transformations, as well as their inverse images

$$\pm(1 + uv)$$

in Spin(V).

Proposition 3.22. *Suppose the quadratic space V has an isotropic vector and $dim(V) \geq 3$. Then the Spinorial kernel $SO^+(V)$ is generated by Eichler transformations.*

This is Theorem 6.4.27 in [Ha-OM].

The prerequisites are always satisfied for the relevant quadratic spaces $\mathbb{R}^{2,n}$, n ≥ 1. If Lemma 3.7 is applied to $V = \mathbb{R}^{2,n}$, n ≥ 2, that is V decomposes into

$$V = H_1 \perp H_2 \perp V_0 = \mathbb{R}f_1 \oplus \mathbb{R}f_2 \perp \mathbb{R}f_3 \oplus \mathbb{R}f_4 \perp V_0\,,$$

the negative definite subspace V_0 has no isotropic vectors. Hence, the proposition provides a practical set of generators of Spin(2,n) which consists of $-1_{\mathscr{C}}$ and all

$$1 + f_i v\,, \qquad 1 \leq i \leq 4 \tag{3.3}$$

where f_i is one of the hyperbolic basis vectors and v is in V_0 or one of the two hyperbolic basis vectors orthogonal to the current f_i.

3.4 Spin Representations

The classification of all irreducible representations revealed the existence of representations of real and complex orthogonal groups that could not be obtained as quotients of tensor products of the standard vector representation. These remaining representations, the spin representations, can be constructed via the Clifford algebra as follows.

Recall that a representation of an associative algebra A on a vector space V is a homomorphism in the algebra of endomorphisms $End(V)$ of a vector space V. This definition is equivalent to the notion of an A-module. Equivalence and irreducible representations correspond to A-linear mappings and simple A-modules.

There is a well-known theorem that all irreducible representations of a simple algebra are equivalent (confer for example [La], XVII, §4).

Consider now the Clifford algebra $\mathscr{C}(V,q)$ of a quadratic space of *even* dimension. By Proposition 3.15 and this theorem, $\mathscr{C}$ has only one irreducible representation up to isomorphism. Any representative of this isomorphism class is called a spin representation of $\mathscr{C}$. By restriction, the spin representations of the even part $\mathscr{C}^+$ and the Spin group are defined.

If $\mathscr{C}^+$ decomposes in the sum of two simple ideals, this representation decomposes in the sum of two *inequivalent* irreducible representations which are called the half-spin representations. The inequivalence follows again from a theorem of the representation theory of K-algebras. However, in the special case $\mathbb{R}^{2,6}$ considered here it will be shown explicitly in the final chapter. The properties of the spin representation of the Spin group are summarized in the following proposition.

Proposition 3.23. *Let char $K \notin \{2,3\}$. If $\mathscr{C}^+$ is a simple algebra, the spin representation of the Spin group is irreducible. If $\mathscr{C}^+$ is not simple, the spin representation is the sum of the inequivalent half-spin representations of the Spin group.*

Proof. This is Proposition II.4.3 in [Che]. The proof relies on the fact that the Spin group generates $\mathscr{C}^+$ as a K- algebra (as shown for $\mathbb{R}$ in the proof of Lemma 3.21) and therefore the restricted spin representation has the same properties as the spin representation of $\mathscr{C}^+$. □

In the odd case, the spin representation is defined as the only irreducible representation of the even part $\mathscr{C}^+$ which lifts to two spin representations of the whole Clifford algebra as anticipated by Proposition 3.15.

Chapter 4

Lie Theory

Lie theory is a vast subject. In this chapter, the main definitions and facts are outlined as short as possible. Only the notion of isogeny and its connection to the theory of covering spaces is described in detail. Furthermore, a lot of facts concerning the matrix groups involved in this thesis are summarized. A general reference would be [Hel]. Many details were taken from the monographs [Bump], [War] and [Ful].

4.1 Lie Groups and Lie Algebras

A *Lie algebra* over a field K is a vector space over K endowed with an alternating bilinear operation $(x, y) \mapsto [x, y]$ (called bracket) satisfying the Jacobi identity, that is

$$[x, x] = 0 \text{ and } [x, [y, z] + [z, [x, y]] + [y, [z, x]] = 0.$$

A homomorphism of Lie algebras is a K-linear mapping respecting the bracket operation.

A (real) *Lie group* is a group with the structure of a smooth manifold such that multiplication and inversion are smooth. A homomorphism of Lie groups is a group homomorphism that is also a smooth map. Complex Lie groups and homomorphisms of complex Lie groups are defined by simply replacing "smooth" by "complex analytic" in the previous definitions.

The Lie algebra $\mathfrak{g}$ of a Lie group G can be defined either as the tangent space at the identity element or as the vector space of left-invariant vector fields both with the commutator as bracket operation. The differential of any Lie group homomorphism at the identity turns out to be a homomorphism of Lie algebras.

An ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$ and an ideal is Abelian if $[\mathfrak{h}, \mathfrak{h}] = 0$. A Lie algebra is called *simple* if it has no non-trivial ideals and *semisimple* if it has no Abelian ideals. Equivalently a Lie algebra is semisimple if and only if it decomposes uniquely in the direct sum of simple ideals. A Lie group is called simple respectively semisimple if its Lie algebra has these properties.

There is a basic result that all closed subgroups of the complex general linear group are real Lie groups. Obviously, all matrix groups introduced in the preceding chapters are closed in some general linear group and hence real Lie groups. Among those, only the groups $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$ turn out to be complex Lie groups. The last three are simple and $GL(n, \mathbb{C})$ is not even semisimple. Similarly, $U(n)$ and $GL(n, \mathbb{R})$ are not semisimple, whereas $SL(n, \mathbb{R})$, $Sp(n, \mathbb{R})$, $SO(p, q)$ and $Sp(n, \mathbb{H})$ are simple.

To calculate the Lie algebra of a Lie group, the *exponential map* has to be introduced, one of the most important means in Lie theory. For a matrix group it can simply be defined as the exponential series of matrices, that is

$$exp : M_n(\mathbb{C}) \to GL(n,\mathbb{C})\,,\; X \mapsto \sum_{i=0}^{\infty} \frac{X^i}{i!}.$$

The series converges for all matrices, but the functional equation of the one-dimensional exponential map holds only for commuting matrices. However, this property suffices to make the map

$$t \mapsto exp(tX)$$

a one-parameter subgroup of $GL(n,\mathbb{C})$, that is a continuous homomorphism $\mathbb{R} \to GL(n,\mathbb{C})$. In the case of a general Lie group, the exponential map of a tangential vector X at e is constructed by solving a certain ordinary differential equation. Since this thesis works only with matrix groups, the general exponential map is omitted.

Proposition 4.1. *Let G be a closed subgroup of $GL(n,\mathbb{C})$.*

(i) The set $\mathfrak{g}$ of all matrices $X \in M_n(\mathbb{C})$, such that $exp(tX) \in G$ for all t, is a Lie algebra of the same dimension as G as a manifold. This is the Lie algebra of G in the general sense defined above.

(ii) The map $X \mapsto exp(X)$ gives a diffeomorphism of a neighborhood of the origin in $\mathfrak{g}$ onto a neighborhood of the identity in G.

(iii) The exponentials of all elements of the Lie algebra generate the connected component of the identity of G.

The second part of the proposition allows to write the multiplication of the Lie group sufficiently close to the identity in terms of the Lie algebra via the exponential map and its inverse. The resulting formula is called Campbell-Hausdorff formula. This is meant in the first place when asserting that the group structure of the connected component of G is encoded in the Lie algebra when also considering the third part.

The first part of the proposition gives a handy way to calculate the Lie algebras of matrix Lie groups. They will always be denoted by the small German version of the letter of the Lie group. The calculation for some of the matrix groups mentioned above yields the following Lie algebras (confer [Hel]).

$$\mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C})$$
$$\mathfrak{sl}(n,\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \text{ tr X} = 0\}$$
$$\mathfrak{so}(n,\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid X' = -X \text{ (i.e. X skew symmetric)}\}$$
$$\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) \mid \bar{X}' = -X \text{ (i.e. X skew hermitian) ; tr } X = 0\}$$
$$\mathfrak{so}(p,q) = \{\begin{pmatrix} X_1 & X_2 \\ X_2{}' & X_3 \end{pmatrix} \mid X_1 \in M_p(\mathbb{R}),\ X_2 \in M_{p,q}(\mathbb{R}),\ X_3 \in M_q(\mathbb{R}),\ X_1' = -X_1,\ X_3' = -X_3\}$$
$$\mathfrak{sp}(n,\mathbb{H}) \cong \{\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \mid Z_1, Z_2 \in M_n(\mathbb{C}), Z_1' = -Z_1,\ \bar{Z}_2' = Z_2\}$$

Two Lie groups are called *isogenous* if their Lie algebras are isomorphic. The crucial exceptional isogeny in this thesis is

$$\mathfrak{so}(2,6) \cong \mathfrak{sp}(2,\mathbb{H}) \quad (\cong \mathfrak{so}^*(8)\ ).$$

Another exceptional isogeny that will be of importance is

$$\mathfrak{su}(4) \cong \mathfrak{so}(6).$$

Connectivity of Matrix Groups

The concepts of connected and path-connected are equivalent in the case of Lie groups since they are manifolds.

Lemma 4.2. *(i) The groups $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $SL(n,\mathbb{R})$, $U(n)$, $SU(p,q)$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{R})$, $Sp(n,\mathbb{C})$ and $Sp(n,\mathbb{H})$ are connected.*

(ii) $SO(p,q)$ is connected if and only if p or q equals 0. Otherwise, it has two connected components.

(iii) $Spin(p,q)$ is connected, if p or $q \geq 2$.

Proof. The first two items can be found literally in [Hel], Ch. X, § 2, Lemmata 2.2 and 2.4. The connectedness of $Spin(p,q)$, p or q ≥ 2, can be seen as follows.
Consider any Eichler transformation $1+uv$, that is $u^2 = 0$ and $uv + vu = 0$. Then $1 + tuv$ is still an Eichler transformation for all $t \in [0,1]$. Hence any product of Eichler transformations can be connected to 1_{Spin} by a path in the Spin group. In order to take care of the last missing generator -1_{Spin}, one has to find a pair of vectors x and y with $\langle x, y\rangle = 0$ and $q(x)q(y) = 1$. The existence is obvious when choosing orthonormal basis vectors of the same norm and considering the restriction that p or $q \geq 2$. For arbitrary real numbers s and t the equivalence

$$z = s + txy \in Spin(p,q) \Leftrightarrow (s + txy)(s + txy)^{*\prime} = s^2 + t^2 = 1$$

holds when using $xy+yx = 0$ and $x^2y^2 = 1$. The relation $zvz^{-1} \in V$ is an easy calculation. Hence, -1_{Spin} corresponding to $(s,t) = (-1,0)$ can be continuously transformed to 1_{Spin} corresponding to $(1,0)$ without leaving the Spin group. □

Corollary 4.3. The Spinorial kernel $SO^+(p,q)$ is connected if p or q ≥ 2 as continuous image of the Spin group. Since the identity is contained in $SO^+(p,q)$, it is the connected component of the identity of $SO(p,q)$.

4.2 Covering Spaces and Fundamental Group

To understand the full consequences of two Lie groups being isogenous, one needs the theory of covering spaces. The theory will be used again as the critical means in determining the maximally compact subgroup of the Spin group.

The whole section belongs to the standard content of any lecture or book on algebraic topology and can be found e.g. in [Hat].

Fundamental Group

A loop in a topological space X is a path with identical start and ending point, that is a continuous map $f : I \to X$ of the unit interval $I = [0,1]$ into X with $f(0) = f(1)$. The idea of deforming loops continuously is formalized by defining a homotopy of loops as a family $\{f_t\}$ of loops starting at the same base point x_0 such that the associated map $F : I \times I \to X$, $(s,t) \mapsto f_t(s)$ is continuous. Homotopy of loops defines an equivalence relation. Two homotopy classes can be multiplied by simply concatenating two representing loops. The set of classes of this equivalence relation corresponding to a fixed base point x_0 forms a group that is called the *fundamental group*

$$\pi_1(X, x_0).$$

A continuous map $\phi : X \to Y$ of topological spaces taking a fixed base point $x_0 \in X$ to another fixed base point $y_0 \in Y$ induces a well-defined group homomorphism

$$\phi_* : \pi_1(X, x_o) \to \pi_1(Y, y_0)$$

of fundamental groups by defining $\phi_*([f]) := [\phi \circ f]$. This makes the fundamental group a covariant functor. A standard implication is that homeomorphic topological spaces have isomorphic fundamental groups.

If X is path-connected, the fundamental group is, up to isomorphism, independent of the choice of the base point and can be abbreviated by $\pi_1(X)$. A topological space is called *simply-connected* if it is path-connected and has trivial fundamental group.

Example 4.4. Theorems 13.5 and 13.6 in [Bump] calculate the fundamental groups of the relevant matrix groups.

$$\pi_1(SU(n)) \cong \pi_1(SL(n,\mathbb{C})) \text{ is trivial and}$$
$$\pi_1(SO(n)) \cong \pi_1(SL(n,\mathbb{R})) \cong \begin{cases} \mathbb{Z} & if\ n = 2 \\ \mathbb{Z}/2\mathbb{Z} & if\ n > 2 \end{cases}$$

The proofs use some non-trivial algebraic topology as the long exact homotopy sequence of a fibration. In a very similar way it can be shown that

$$\pi_1(U(n)) \cong \pi_1(GL(n,\mathbb{C})) \cong \mathbb{Z}.$$

The following useful facts about the fundamental group are standard.

Lemma 4.5. *(i) Let X and Y be path-connected topological spaces. Then*

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y).$$

(ii) The fundamental group of any topological group and especially of any Lie group is abelian.

Covering Spaces

A *covering space* of a topological space X is a topological space $\tilde{X}$ together with a continuous map $p : \tilde{X} \to X$ satisfying the following condition: There exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α.

If X is path-connected, the cardinality of $p^{-1}(x)$ is constant over X and called the number of *sheets* of the covering.

The standard example of a covering space is $p : \mathbb{R} \to S^1$, $x \mapsto exp(ix)$ having a countable number of sheets, which might be visualized as the projection of a helix onto a circle.

In the following, the main interrelations of covering spaces and the fundamental group are listed. The most important properties of covering spaces needed to prove these propositions are certain lifting properties which are omitted here.

Proposition 4.6. *The number of sheets of a covering space $p : \tilde{X} \to X$ with X and $\tilde{X}$ path-connected equals the index of $p_*(\pi_1(\tilde{X}))$ in $\pi_1(X)$.*

The main theorem classifies all covering spaces of a given topological space. Recall that a topological space is called locally path-connected if every neighborhood of every point contains a path-connected neighborhood and semilocally simply-connected if each point has a neighborhood U such that the inclusion-induced map $\pi_1(U,x) \to \pi_1(X,x)$ is trivial.

Theorem 4.7. *(Classification of covering spaces) Let X be a path-connected, locally path-connected and semilocally simply-connected topological space.*

(i) For every subgroup H of $\pi_1(X,x_0)$, there is a path connected covering space $p: X_H \to X$ such that $p_*(\pi_1(X_H,\tilde{x}_0)) = H$ for a suitably chosen base point $\tilde{x}_0 \in X_H$.

(ii) X_H is unique up to an isomorphism of covering spaces, that is a homeomorphism preserving the inverse images of all points of the underlying space X.

(iii) Ignoring base points there is a bijection between isomorphism classes of path-connected covering spaces $p: \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X)$.

This theorems implies the existence of a simply-connected covering space $p: \tilde{X} \to X$ which is called the *universal cover* of X.

4.3 More Lie Theory

Covering Spaces of Lie Groups and Isogenies

The application of the theory of covering spaces to Lie groups is rather straightforward.

As a first step, the local topological properties demanded in the main theorem are trivially fulfilled by any manifold. Furthermore, a covering space $p: \tilde{G} \to G$ of any smooth manifold G can be endowed with a differentiable structure in a unique way if the covering map p is required to be smooth. Finally, if $p: \tilde{G} \to G$ is a covering of a Lie group and an element $\tilde{e}$ is chosen in the kernel of p, there is a unique Lie group structure on $\tilde{G}$ such that $\tilde{e}$ is the identity and p is a homomorphism of Lie groups. Then the kernel of p lies in the center of $\tilde{G}$. The proof of all these statements is straightforward using the lifting properties of covering spaces (confer [War]).

When applying the main theorem of covering spaces, it follows that any Lie group G has a simply-connected covering Lie group which is unique up to isomorphism.

Example 4.8. The homomorphism

$$Spin(V) \to SO^+(V)$$

of section 3.3 is a two-sheeted covering map. This provides $Spin(V)$ with the structure of a Lie group. For $n \geq 2$, $\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ implies that $Spin(n)$ is the simply connected form of $SO(n) = SO^+(n)$ because of Proposition 4.6.
It might be a common mistake to assume all Spin groups being simply connected. There is a general theorem that the fundamental group of a connected Lie group is the same as the fundamental group of a maximal compact subgroup. A maximal compact subgroup of $SO^+(2,n)$ is $SO(2) \times SO(6)$ and hence $\pi_1(SO^+(2,n)) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This implies that $\pi_1(Spin(2,n))$ is far from trivial because it is isomorphic to a subgroup in $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of index 2.
Nevertheless, viewed as algebraic groups, Spin groups are always simply connected in the algebraic sense.

The construction that is inverse in a sense to finding a covering space of a Lie group is the following. Let Γ be any discrete subgroup of the center of a Lie group G. Then there is a unique Lie group structure on the quotient group G/Γ such that the canonical quotient map is a homomorphism of

Lie groups. Obviously, this is a covering map by be discreteness of Γ. If the center $\mathcal{Z}(G)$ of G is discrete, it is easy to show, that the quotient $G/\mathcal{Z}(G)$ has trivial center.

Recall that two Lie groups are called *isogenous* if their Lie algebras are isomorphic. This defines obviously an equivalence relation.

Proposition 4.9. *(i) A homomorphism of connected Lie groups is a covering map if and only if its differential at the identity is an isomorphism of Lie algebras.*

(ii) Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively and with G simply connected. Given a homomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$, there exists a unique homomorphism $\varphi : G \to H$ with differential ψ.

(iii) If $\mathfrak{g}$ is a Lie algebra, then there is a Lie group G with Lie algebra $\mathfrak{g}$.

The first two statements are proved in [War] (Prop. 3.26 and 3.27). The third statement is a deep result due to Ado.

The proposition characterizes two Lie groups as being isogenous if and only if there is a third Lie group covering both of them. By the third statement, any Lie algebra defines an equivalence class of isogenous Lie groups. There is a topmost element in each class, unique up to isomorphism, which is a covering space of all other elements of this class. This is called the *simply connected form* of a Lie algebra. Therefore all elements of an isogeny equivalence class can be realized up to isomorphism by finding the simply connected form and building quotients by all discrete subgroups of its center. The quotient of the simply connected form by its whole center (assuming that this is discrete) is called the *adjoint form* of a Lie algebra. It is an undermost element of the equivalence class in the sense that all other members are covering the adjoint form.

The following example will be used in the final chapter.

Example 4.10. Since the Lie algebras of $SU(4)$ and $SO(6)$ are isomorphic, their simply connected coverings must be the same. However, the fundamental groups given in Proposition 4.4 imply that $SU(4)$ is the simply connected form and is therefore isomorphic to $Spin(6)$. The adjoint form can be computed when the center of an arbitrary member of the equivalence class of these groups is known, hence it is isomorphic to $SO(6)/\pm id$.
The two-sheeted covering map from $SU(4)$ to $SO(6)$ can be constructed explicitly as follows. Consider the six-dimensional real vector space V of real anti-symmetric 4×4-matrices. Provided with the the quadratic form $\frac{1}{2}tr(\overline{B}'B)$ it is isomorphic to the quadratic space $\mathbb{R}^{6,0}$. Then a matrix $A \in SU(4)$ defines a linear mapping on V by sending an anti-symmetric matrix B to the anti-symmetric matrix $A'BA$. This operation preserves the quadratic form as

$$tr(\overline{A'BA}'A'BA) = tr(\bar{A}'\bar{B}'\bar{A}A'BA) = tr(A^{-1}\bar{B}'BA) = tr(\bar{B}'B).$$

The matrix A defines hence an orthogonal transformation which turns out to inherit the determinant 1 of A after an easy calculation. The resulting homomorphism

$$\pi : SU(4) \to SO(6)$$

has the kernel $\{\pm E\}$ and is onto, so it is a two-sheeted covering map. The center of $SU(4)$ is $\{\pm E, \pm \mathrm{i}E\}$ and $\pm\mathrm{i}E$ are obviously mapped to the only nontrivial central element $-id \in SO(6)$.

Representations of Lie groups

A complex representation of a Lie group G is a homomorphism of Lie groups from G into the general linear (Lie) group of some complex vector space V. This implies that a representation of real Lie groups is supposed to be smooth and a representation of a complex Lie group is supposed to be holomorphic. Correspondingly, a complex representation of a Lie algebra $\mathfrak{g}$ is a homomorphism of Lie algebras into the Lie algebra of endomorphisms of some complex vector space V. Again, it has to be $\mathbb{R}$-linear in the case of a real Lie algebra and $\mathbb{C}$-linear in the case of a complex Lie algebra.

As pointed out above, a representation of Lie groups immediately induces a representation of Lie algebras by taking the differential. The other way round, given a representation ρ of a Lie algebra $\mathfrak{g}$ corresponding to a *simply connected* Lie group G, there is a unique representation $\tilde{\rho}$ of G such that ρ is the differential of $\tilde{\rho}$ by Proposition 4.9. Regarding the third statement of this proposition, the representation theory of Lie algebras and simply connected Lie groups is essentially the same.

Complexifications

As a reference for the following serve the chapters about extension of scalars and complexifications in [Bump].

On the level of Lie algebras, a complexification of a real Lie algebra $\mathfrak{g}$ is defined in a straightforward way by extension of scalars, i.e.

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g} \oplus \mathrm{i}\mathfrak{g}.$$

The isomorphism is an isomorphism of complex Lie algebras. The complexifications of the Lie algebras given above are easy to calculate and well-known:

$$\begin{aligned}
&\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C}) \\
&\mathfrak{sl}(n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C}) \\
&\mathfrak{so}(p,n-p)_{\mathbb{C}} \cong \mathfrak{so}(n,\mathbb{C}) \\
&\mathfrak{sp}(n,\mathbb{H})_{\mathbb{C}} \cong \mathfrak{so}^*(4n)_{\mathbb{C}} \cong \mathfrak{so}(4n,\mathbb{C})
\end{aligned}$$

On the level of Lie groups, a complex Lie group $\hat{G}$ is called complexification of a real Lie group G, if $\hat{\mathfrak{g}} = \mathfrak{g}_{\mathbb{C}}$. However, to define a unique complexification, one has to require more.

Given a real Lie group G, define the analytic complexification as a complex Lie group $G_{\mathbb{C}}$ together with a Lie group homomorphism $i : G \to G_{\mathbb{C}}$ by the following universal property. Given any Lie group homomorphism $f : G \to H$ into a complex Lie group H, there exists a unique analytic homomorphism $F : G_{\mathbb{C}} \to H$ such that $f = F \circ i$. Thus, the analytic complexification is unique up to isomorphism.

The definition immediately implies that any finite-dimensional representation of a real Lie group G extends uniquely to a finite-dimensional holomorphic representation of its analytic complexification $G_{\mathbb{C}}$. In the case of a compact Lie group, the following theorem (Theorem 27.1 in [Bump]) guarantees the existence of an analytic complexification.

Theorem 4.11. *Let K be a compact connected Lie group. Then K has an analytic complexification $K \to K_{\mathbb{C}}$, where $K_{\mathbb{C}}$ is a complex Lie group. The induced map $\pi_1(K) \to \pi_1(K_{\mathbb{C}})$ is an isomorphism. The Lie algebra of $K_{\mathbb{C}}$ is the complexification of the Lie algebra of K. Any holomorphic representation of $K_{\mathbb{C}}$ is completely reducible.*

The analytic complexification of the compact connected group $U(n)$ respectively $SU(n)$ is well-known to be $GL(n,\mathbb{C})$ respectively $SL(n,\mathbb{C})$. Furthermore, the analytic complexification of $SO(n)$ is $SO(n,\mathbb{C})$, especially one has $SO(2)_{\mathbb{C}} \cong S^1_{\mathbb{C}} \cong SO(2,\mathbb{C}) \cong \mathbb{C}^*$. The last isomorphism, similarly to the well-known $SO(2) \cong S^1$, is given by

$$\mathbb{C}^* \to SO(2,\mathbb{C})\,,\ \xi \mapsto \begin{pmatrix} cosh\,\xi & \mathrm{i}\,sinh\,\xi \\ -\mathrm{i}\,sinh\,\xi & cosh\,\xi \end{pmatrix}.$$

The analytic complexification of the Spin groups $Spin(p,q)$ is known to be $Spin(p+q,\mathbb{C})$. Recall the isomorphism $SO(4) \cong Spin(6,\mathbb{R})$ of the previous section which implies immediately that the complexifications

$$SO(4,\mathbb{C}) \cong Spin(6,\mathbb{C})$$

are isomorphic.

4.4 Symmetric Spaces and Hermitian Domains

Nearly all results in this section are rather deep and can be found in the standard references about symmetric spaces which are [Hel] and [Sat]. The reader who is interested in an introduction to the subject to get a general picture rather than complete proofs is referred to the chapter about symmetric spaces in [Bump]. The following survey is kept as short as possible.

Maximal Compact Subgroups

Let G denote a semisimple Lie group. A maximal compact subgroup of G is a compact subgroup which is not contained in any other compact subgroup. If a maximal compact subgroup K exists, all other maximal compact subgroups are conjugate to K and especially isomorphic to K. It turns out that all matrix groups introduced above have a maximal compact subgroup. Since the natural operation of G on the quotient G/K is transitive, the quotient is a homogeneous space by definition.
Is is not hard to see that a covering map of Lie groups induces a covering map of maximal compact subgroups with the same number of sheets. Thus, the isomorphy class of a homogeneous space G/K is completely determined by the Lie algebra $\mathfrak{g}$ of the Lie group G or, in other words, isogeneous Lie groups define isomorphic homogeneous spaces.

Symmetric Spaces and Lie Groups

In differential geometry, a (globally) symmetric space is a Riemanian manifold in which around every point there is an isometry reversing the direction of every geodesic. It is known that the group G of isometries of M becomes a Lie group with the compact-open topology and the stabilizer K of any chosen base point $x_0 \in M$ is a maximal compact subgroup of M. Then the correspondence $g(x_0) \leftrightarrow gK$ gives a G-equivariant diffeomorphism $M \cong G/K$. Now it can be shown that any simply connected Riemanian symmetric space can be decomposed uniquely into the direct product of a Euclidean space M_0 and a finite number of irreducible (i.e. indecomposable) symmetric spaces M_i, that is

$$M = M_0 \times M_1 \times ... \times M_r.$$

Define a symmetric space of non-compact type as a simply connected symmetric space M, such that the Euclidian part M_0 reduces to a point and all other components M_i are non-compact. In this case, the connected component G^0 of the group of isometries is semisimple and decomposes into the simple components G_i^0 of isometries of M_i. It turns out that G^0 has trivial center and hence, it is the adjoint form of a semisimple Lie algebra. Conversely, starting with a connected simple Lie group with trivial center and a maximal compact subgroup K of G, the homogeneous space G/K can be endowed with a Riemanian metric turning it into an symmetric space. Thus, there is a one-to-one correspondence of simple Lie algebras and irreducible symmetric spaces of the non-compact type which was first established by Cartan in 1919. Cartan labeled the classes of symmetric spaces by capital Roman numbers, not to be mixed up with the Roman numbers of Hermitian Symmetric Domains in the following section.

Hermitian Symmetric Domains

Hermitian symmetric spaces (of non-compact type) are symmetric spaces M (of non-compact type) that can be endowed with a an almost complex structure such that there exists a hermitian metric on M which is invariant under the group of isometries of M. Instead of explaining the details of this definition, their most important property shall be taken as definition.

Definition 4.12. Let G be a semisimple Lie group and K a maximal compact subgroup. The pair (G, K) (or the symmetric space G/K) is of the *hermitian type*, if there is a domain $\mathcal{H}$ in $\mathbb{C}^N$ for an adequate N and a diffeomorphism $G/K \to \mathcal{H}$ inducing a differentiable action of G on $\mathcal{H}$ such that $\mathcal{H} \to \mathcal{H}, z \mapsto gz$ is holomorphic for all $g \in G$. The domain $\mathcal{H}$ is called a hermitian (symmetric) domain.

There is an important theorem due to Harish-Chandra that a pair (G, K) is of hermitian type if and only if K is not semisimple.

The classification of irreducible hermitian symmetric spaces comprises four infinite series and two exceptional spaces which can be found e.g. in [Hel] immediately following the table of all irreducible symmetric spaces. All these spaces are of type III in Cartan's denotation because they are of non-compact type and belong to a non-compact real Lie algebra. Unfortunately, Siegel labeled the four series again by capital Roman numbers as exposed in the following table (cf. [Sat], p.188).

<table>
<tr><th>Type</th><th colspan="2">G</th><th>K</th><th>Cartan's class</th></tr>
<tr><td>$I_{p,q}$</td><td colspan="2">$SU(p,q)$</td><td>$S(U(p) \times U(q))$</td><td>AIII/AIV</td></tr>
<tr><td>II_n</td><td colspan="2">$SO^*(2n)$</td><td>$U(n)$</td><td>DIII</td></tr>
<tr><td>III_n</td><td colspan="2">$Sp(n,\mathbb{R})$</td><td>$U(n)$</td><td>CI</td></tr>
<tr><td rowspan="2">IV_q</td><td rowspan="2">$Spin(2,q)$</td><td>q odd</td><td rowspan="2">$Spin(2,0) \cdot Spin(0,q)$</td><td>BI</td></tr>
<tr><td>q even</td><td>DI</td></tr>
</table>

Table 1: *Classification of Hermitian Symmetric Spaces*

Recall that the even members $SO^+(4n)$ of the series II are described in this thesis by the isomorphic groups $Sp(n, \mathbb{H})$. The isogeny

$$\mathfrak{so}(2,6) \cong \mathfrak{sp}(2,\mathbb{H}) \quad (\cong \mathfrak{so}^*(8)).$$

translates to an isometry of symmetric spaces

$$IV_6 \cong II_4$$

which will be examined in detail in the final chapter.

Chapter 5

Automorphic Forms

After establishing the definition of automorphic forms in a very general context this chapter continues to introduce modular forms of quaternions and modular forms of orthogonal groups as a specialization of the general theory. The aim is to provide a common ground for both types of modular forms and prepare the special connection between them that will be described in the next chapter.

5.1 Vector-Valued Automorphic Forms

Most of the material of this section is taken from a lecture of Prof. Freitag on automorphic forms.

Let (G, K) be a pair of semisimple Lie group G with maximal compact subgroup K of the hermitian type. Recall that this means that the homogeneous space G/K can be realized as an open domain $\mathcal{H}$ in some $\mathbb{C}^n$ such that G operates on $\mathcal{H}$ by holomorphic transformations.

Definition 5.1. A function $f : G \to \mathbb{C}$ is called K-finite if the span of $\{f^k,\, k \in K\}$ is a complex finite-dimensional vector space where $f^k(g) := f(gk^{-1})$.

The rule $(f^{k_1})^{k_2} = f^{k_1 k_2}$ is obvious and implies that K operates linearly on V by $(\tilde{k}, f^k) \mapsto (f^k)^{\tilde{k}}$. Thus, any K-finite $f : G \to \mathbb{C}$ function belongs a certain finite-dimensional complex representation $\rho_f : K \to GL(V)$.

Given a discrete subgroup Γ of G, the coset space $\Gamma\backslash G$ can be endowed with the structure of a complex manifold.

Definition 5.2. (Vector-Valued Automorphic Form I) Let (G, K) be a pair of a semisimple Lie group G and a maximal compact subgroup K of the hermitian type, Γ a discrete subgroup of G with the property $vol(\Gamma\backslash G) < \infty$ and $\rho : K \to GL(V)$ a fixed representation of K on a complex finite-dimensional vector space V. A vector-valued automorphic form is defined as a map $F : G \to V$ satisfying

(i) $F(gk) = \rho(k^{-1})F(g) \qquad \forall k \in K$

(ii) $F(\gamma g) = F(g) \qquad \forall \gamma \in \Gamma$

Usually, the representation ρ is given by a K-finite function on G.

Example 5.3. The automorphic forms of the hermitian pair $(Sp(n, \mathbb{R}), U(n))$ of type III are the Siegel modular forms. The automorphic forms of the pair $(Sp(n, \mathbb{H}), U(2n))$ of type II are Krieg's quaternary modular forms. In obvious analogy to Siegel modular forms, their basic properties are introduced in the following section. The pairs $(SO^+(2,n), SO(2) \times SO(n))$ of type IV induce the so-called orthogonal modular forms. The final section of this chapter deals with orthogonal modular forms.

5.2 Factors of Automorphy

When working with holomorphic automorphic forms of hermitian symmetric spaces, the introduction of holomorphic factors of automorphy proved its worth. The following material can be found in [Mur].

For every finite-dimensional complex vector space V, the group $GL(V)$ can be considered as an open submanifold of $\mathbb{C}^{(dim\, V)^2}$. Thus, one can speak of differentiable or holomorphic functions with values in $GL(V)$.

Definition 5.4. (Holomorphic Factor of Automorphy) Let (G, K) be of the hermitian type and $\mathcal{H} \subset \mathbb{C}^n$ a realization of G/K as an open domain. Let V be a finite-dimensional complex vector space V. A *holomorphic factor of automorphy with values in GL(V)* is a differentiable mapping $J : G \times \mathcal{H} \to GL(V)$ such that

(i) it satisfies the cocycle relation

$$J(gh, z) = J(g, h(z))J(h, z) \qquad \forall g, h \in G, z \in \mathcal{H}$$

(ii) and the map $J(g, .) : \mathcal{H} \to GL(V)$ is holomorphic for all $g \in G$.

Two factors of automorphy $J_1,\ J_2 : G \times \mathcal{H} \to GL(V)$ are called (holomorphically) equivalent if there exists a holomorphic mapping $f : \mathcal{H} \to GL(V)$ such that

$$J_2(g, z) = f(gz)^{-1} J_1(g, z) f(z) \qquad \text{for all } g \in G \text{ and } z \in \mathcal{H}.$$

Every factor of automorphy J obviously defines a representation of K by evaluating the second component of J in the chosen base point a and restricting the first component to K. The following result guarantees that every irreducible representation of K extends to a unique equivalence class of holomorphic factors of automorphy. Extension means here that the restriction of this factor of automorphy returns the given representation.

Theorem 5.5. *Let $\mathcal{H}$ be a hermitian symmetric domain considered as a quotient G/K of a semisimple connected Lie group by a maximal compact subgroup K, provided with a base point $a \in K$.*

(i) Every representation $\rho : K \to GL(V)$ extends to a holomorphic factor of automorphy

$$J_\rho : G \times \mathcal{H} \to GL(V).$$

(ii) If the representation ρ of K is irreducible, all holomorphic factors of automorphy on $\mathcal{H}$ which restrict to this representation are equivalent.

The proof has to be omitted as it needs a profound knowledge of the structure theory of real Lie algebras (cf. [Mur] or [Sat]). However, when trying to find a factor of automorphy in practice it is very helpful to know that the proof initially constructs a factor of automorphy

$$J : G \times \mathcal{H} \to K_{\mathbb{C}}$$

with values in the analytic complexification $K_{\mathbb{C}}$ of K. It satisfies both defining properties of a factor of automorphy in the above definition when simply replacing $GL(V)$ by $K_{\mathbb{C}}$. Additionally, it has the property that

$$J(k,a) = k \qquad \text{for all } k \in K.$$

Since every representation ρ of K uniquely extends to a holomorphic representation $\rho_{\mathbb{C}}$ of $K_{\mathbb{C}}$, a factor of automorphy with values in $GL(V)$ is then obtained by composing J with $\rho_{\mathbb{C}}$.

Example 5.6. When dealing with Siegel modular forms it is easy to show that a factor of automorphy with values in $K_{\mathbb{C}} = GL(n,\mathbb{C})$ is given by

$$J(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z) = CZ + D.$$

In the case of elliptic modular forms, this specializes to the ubiquitous "cz + d" with values in $GL(1,\mathbb{C}) = \mathbb{C}^*$. By the second part of the theorem, at least those factors of automorphy that reduce to irreducible representations ρ of K are equivalent to the composition of $CZ + D$ with $\rho_{\mathbb{C}}$.

After having introduced factors of automorphy, vector-valued automorphic forms can be defined again from a different point of view.

Definition 5.7. (Vector-Valued Automorphic Form II) Let (G,K) be of the hermitian type and $\mathcal{H} \subset \mathbb{C}^n$ a realization of G/K as an open domain. Let Γ a discrete subgroup with the property $vol(\Gamma\backslash G) < \infty$, V a finite-dimensional complex vector space and $J : G \times \mathcal{H} \to GL(V)$ a fixed holomorphic factor of automorphy $J : G \times \mathcal{H} \to GL(V)$.
A vector-valued automorphic form is defined as a holomorphic function $f : \mathcal{H} \to V$ satisfying

$$f(\gamma z) = J(\gamma, z) f(z) \qquad \forall \gamma \in \Gamma$$

Proposition 5.8. *The two definitions of automorphic forms are equivalent.*

Proof. Starting with definition 5.7, one can find a maximal compact subgroup of G by taking the stabilizer $K := Stab_G(a)$ of a chosen point $a \in \mathcal{H}$ in G. Then G/K is equivariant and diffeomorphic to $\mathcal{H}$ by

$$gK \mapsto g(a)$$

and (G,K) is obviously of the hermitian type. Define a representation ρ of K on V by evaluating the given factor of automorphy in a and restricting to K. Then all assumptions of definition 5.2 are fulfilled and one can define

$$F(g) := J(g,a)^{-1} f(ga).$$

Applying the cocycle relation several times, the following calculation shows that F is automorphic in the sense of 5.2.

$$\begin{aligned} F(\gamma g k) &= J(\gamma g k, a)^{-1} f(\gamma g k a) = (J(\gamma, gka) J(g, ka) J(k,a))^{-1} J(\gamma, gka) f(gka) \\ &= J(k,a)^{-1} J(g,a)^{-1} f(ga) = \rho(k^{-1}) F(g) \end{aligned}$$

The other way round starting with definition 5.2, one chooses a realization $\mathcal{H}$ of the hermitian domain G/K. The representation ρ extends to a factor of automorphy

$$J = J_\rho : G \times \mathcal{H} \to GL(V)$$

by Theorem 5.5. An automorphic form in the sense of 5.7 is defined by setting

$$f(gK) := J(g^{-1}, gK)^{-1} F(g).$$

This definition is independent of the choice of the representative as for any $k \in K$

$$\begin{aligned} f(gkK) &= \underbrace{J(k^{-1}g^{-1}, gK)}_{J(k^{-1},K)J(g^{-1},gK)}{}^{-1} \underbrace{F(gk)}_{\rho(k^{-1})F(g)} \\ &= J(g^{-1}, gK)^{-1}(\underbrace{J(k^{-1}, K)}_{\rho(k^{-1})})^{-1}\rho(k^{-1})F(g) \\ &= J(g^{-1}, gK)^{-1}F(g) = f(gK). \end{aligned}$$

And the invariance under Γ follows from

$$\begin{aligned} f(\gamma gK) &= \underbrace{J(g^{-1}\gamma^{-1}, \gamma gK)}_{J(g^{-1},gK)J(\gamma^{-1},\gamma gK)}{}^{-1} \underbrace{F(\gamma g)}_{F(g)} \\ &= J(\gamma^{-1}, \gamma gK)^{-1}J(g^{-1}, gK)^{-1}F(g) \\ &= J(\gamma, gK)f(gK). \end{aligned}$$

To see that the last line holds observe that

$$J(\gamma^{-1}, \gamma gK)J(\gamma, gK) = J(1_K, gK) = \rho(1_K) = 1_{GL(V)}.$$

□

5.3 Modular Forms on Quaternary Half-Spaces

In this section, a realization of the hermitian domains $(Sp(n, \mathbb{H}), U(2n))$ of type II_{2n} as an open tube domain is introduced. This realization and the operation of $Sp(n, \mathbb{H})$ are obviously analogous to the theory of Siegel Modular forms. The reference for this section is [Kri].

The Quaternary Half-Space

The *quaternary half-space* $\mathscr{H}_n$ is defined as a subset of $Sym(n, \mathbb{H} \otimes \mathbb{C}) \cong Sym(n, \mathbb{H}) \oplus \mathrm{i}Sym(n, \mathbb{H})$ by

$$\begin{aligned} \mathscr{H}_n &:= Sym(n, \mathbb{H}) + \mathrm{i}Pos(n, \mathbb{H}) \\ &= \{Z = X + \mathrm{i}Y \in M_n(\mathbb{H}) \oplus \mathrm{i}M_n(\mathbb{C}) \mid X = \bar{X}', Y > 0\} \end{aligned}$$

The half-space $\mathscr{H}_n$ is obviously open in the finite-dimensional complex vector space $Sym(n, \mathbb{H}\otimes\mathbb{C})$, hence holomorphic functions on the half-space can be defined in the usual way.

In the following, the action of the symplectic group on the half-space has to be defined.

Definition and Proposition 5.9. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{H})$ *and* $Z \in \mathscr{H}_n$. *The symplectic transformation of* $\mathscr{H}_n$

$$Z \longmapsto M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}$$

is well-defined and holomorphic. It defines a transitive action of $Sp(n, \mathbb{H})$ *on* $\mathscr{H}_n$. *Two matrices* M *and* $N \in Sp(n, \mathbb{H})$ *define the same transformation if and only if* $M = \pm N$.

Proof. Confer Theorem 3.11 in [Kri]. The proof is not difficult and most calculations can be reduced to well-known analogons for Siegel modular forms (cf. [Fr1]) by the embeddings $\widehat{M}$ and $\widehat{Z}$.

□

Corollary 5.10. Thus, $Sp(n,\mathbb{H})/\pm E$ can be identified with a subgroup of the group of biholomorphic transformations of $\mathscr{H}_n$

$$Bih(\mathscr{H}_n) := \{f : \mathscr{H}_n \to \mathscr{H}_n \quad biholomorphic\}.$$

Remark 5.11. The generators of the symplectic group (Lemma 2.9) act on the half-space as follows

$$\begin{array}{rcl} (J, Z) & \mapsto & -Z^{-1} \\ (\begin{pmatrix} E & S \\ 0 & E \end{pmatrix} Z) & \mapsto & Z+S \\ (\begin{pmatrix} \bar{W}' & 0 \\ 0 & W^{-1} \end{pmatrix}, Z) & \mapsto & Z[W] = \bar{W}' Z W \end{array}$$

The holomorphic transformation of $\mathscr{H}_2 = Sym(2,\mathbb{H}) + \mathrm{i}Pos(2,\mathbb{H})$

$$\tau : X + \mathrm{i}Y \longmapsto X' + \mathrm{i}Y'$$

is well-defined because of the characterization of positivity in Lemma 2.7. However, for $n \geq 3$, $Y > 0$ does not imply $Y' > 0$ and τ is no longer defined as a transformation of the half-space.

Actually [Kri] proves that there are no more biholomorphic transformations of $\mathscr{H}_n$ than the symplectic transformations and, in the case $n = 2$, the transformation τ.

Theorem 5.12.

$$Bih(\mathscr{H}_2) \cong \begin{cases} Sp(2,\mathbb{H})/\pm E \,\cup\, \tau(Sp(2,\mathbb{H})/\pm E) & \text{for } n = 2 \\ Sp(n,\mathbb{H})/\pm E & \text{for } n \geq 3 \end{cases}$$

The following lemma establishes an isomorphism of $U(2n)$ and the stabilizer of the point $\mathrm{i}E$ in $Sp(n,\mathbb{H})$. By the transitivity of the action, $Sp(n,\mathbb{H})/U(2n)$ is isomorphic to the half-space $\mathscr{H}_n$ which is consequently an unbounded realization of the hermitian symmetric space of type II_{2n}.

Lemma 5.13. *The mapping*

$$Stab_{Sp(n,\mathbb{H})}(\mathrm{i}E) \tilde{\to} U(2n)\,, \qquad \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto \check{A} + \mathrm{i}\check{B}$$

is an isomorphism.

Proof. For the time being, observe that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Stab_{Sp(n,\mathbb{H})}(\mathrm{i}E) \quad \Leftrightarrow \quad (\mathrm{i}A+B)(\mathrm{i}C+D)^{-1} = \mathrm{i}E \quad \Leftrightarrow \quad A = D \wedge B = -C. \qquad (*)$$

Using $\overline{\check{B}}' = \check{\overline{B}}'$ one calculates

$$\begin{array}{rcl} E & = & (\overline{\check{A} + \mathrm{i}\check{B}})'(\check{A} + \mathrm{i}\check{B}) \\ & \Leftrightarrow & \check{E} = (\check{\overline{A}}' - \mathrm{i}\check{\overline{B}}')(\check{A} + \mathrm{i}\check{B}) \\ & \Leftrightarrow & \check{E} = \check{\overline{A}}'\check{A} + \check{\overline{B}}'\check{B} + \mathrm{i}(\check{\overline{A}}'\check{B} - \check{\overline{B}}'\check{A}) \\ & \Leftrightarrow & -\overline{A}'C + \overline{C}'A = -\overline{B}'D + \overline{D}'B = 0 \ \wedge \quad \overline{A}'D - \overline{C}'B = E \qquad \text{by } (*). \end{array}$$

The relations in the last line are the fundamental relations of symplectic matrices. Hence, the mapping is well-defined and Lemma 2.11 together with $(*)$ implies that the mapping is bijective. The multiplication is the same comparing the complex multiplication on the right with

$$\begin{pmatrix} A_1 & B_1 \\ -B_1 & A_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ -B_2 & A_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 - B_1B_2 & A_1B_2 + A_2B_1 \\ * & * \end{pmatrix}$$

on the left. $\square$

Modular Forms of Quaternions and Factors of Automorphy

The Hurwitz quaternions $\mathcal{O}$ are defined as the set of all quaternions of the form $a_0+a_1\mathrm{i}_1+a_2\mathrm{i}_2+a_3\mathrm{i}_3$ such that all a_i are integral or half integral. As a discrete subgroup Γ of $Sp(n,\mathbb{H})$, one usually chooses the intersection of $Sp(n,\mathbb{H})$ with the set of matrices with entries in $\mathcal{O}$. The demanded properties of Γ are shown in [Kri]. Then vector-valued automorphic forms can be defined as in the previous section and are called Modular Forms of Quaternions.

Once more analogously to Siegel modular forms, a holomorphic factor of automorphy with values in $K_{\mathbb{C}} = GL(2n,\mathbb{C})$ can be defined by

$$J:\, Sp(n,\mathbb{H})\times\mathscr{H}_n \to GL(2n,\mathbb{C})\,,\quad (\begin{pmatrix} A & B\\ C & D\end{pmatrix}, Z)\mapsto \check{C}\check{Z}+\check{D}.$$

The cocycle relation can be verified by a simple calculation and the term is invertible by Proposition 5.9. Hence, the equivalence classes of factors of automorphy corresponding to irreducible representations ρ of $K = U(2n)$ are obtained by composition of their extensions $\rho_{\mathbb{C}}$ to the complexification with this $K_{\mathbb{C}}$-valued factor of automorphy.

However, since $Sp(n,\mathbb{H})$ is not simply connected, there are more factors of automorphy to be found when replacing $Sp(n,\mathbb{H})$ by its simply connected form and finding a factor of automorphy in the larger maximal compact subgroup of this form. This will be accomplished in the final section of this thesis.

5.4 Modular Forms of Orthogonal Groups

This section introduces a realization of the hermitian symmetric spaces of type IV_n corresponding to the group $SO^+(2,n)$ or its two-fold covering group $Spin(2,n)$. As a reference serve the (so far unpublished) notes [Fr2] of a lecture of Prof. Freitag.

Orthogonal modular groups

Lattices and modular groups are not needed in this thesis, but they are introduced rapidly to write down the definition of modular forms of orthogonal groups.

Let Γ be an even lattice of signature $(2,n)$. A lattice is a free Abelian group together with a non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ with values in $\mathbb{Q}$. The lattice is even if the quadratic form $q(x) := \frac{1}{2}\langle x,x\rangle$ is integral and even for all x. By the signature of the lattice one means the signature of the corresponding quadratic space $V = L\otimes_{\mathbb{Z}}\mathbb{R}$.

$$O(V)\cong O(2,n)$$

denotes the orthogonal group of V and the integral orthogonal subgroup is defined as

$$O(L) := \{g\in O(V)\mid g(L) = L\}.$$

Let $O^+(V)$ denote the subgroup of index 2 of G defined in section 3.3. Then any subgroup Γ of $L\otimes_{\mathbb{Z}}\mathbb{Q}$ commensurable with

$$O^+(L) = O(L)\cap O^+(V)$$

is called an orthogonal modular group. Two subgroups A and B are called commensurable if their intersection $A\cap B$ has finite index in both A and B.

The Orthogonal Half-Space

Let (V, q) denote the real quadratic space of signature $(2, n)$ coming from a fixed lattice. When $n \geq 2$, the quadratic space V can be decomposed into the sum of two hyperbolic planes and a negative definite quadratic space V_0

$$V = H_1 \perp H_2 \perp V_0$$

according to Lemma 3.7. Let $f_1, ..., f_4, e_1, ..., e_{n-2}$ denote the standard basis of V corresponding to this decomposition. The bilinear form $\langle \cdot, \cdot \rangle$ of V can be extended uniquely to a $\mathbb{C}$-bilinear form on the complexification of V

$$V(\mathbb{C}) := V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \mathrm{i}V.$$

Consider the complex projective space $\mathbb{P}(V(\mathbb{C}))$ and the projective coordinates with respect to the given basis via the natural projection

$$\pi : V(\mathbb{C}) - \{0\} \to \mathbb{P}(V(\mathbb{C})), \qquad z \mapsto [z].$$

In these coordinates, an element of $\mathbb{P}(V(\mathbb{C}))$ will be written

$$[z_1, z_2, z_3, z_4; Z] \quad \text{where } z_i \in \mathbb{C} \text{ and } Z \in V_0(\mathbb{C}) \cong \mathbb{C}^{n-2}$$

It is easy to show that none of the first four coordinates of a point $z \in \mathcal{K}$ can vanish. Thus, the orthogonal half-space $\mathcal{H}_n$

$$\mathcal{H}_n := \{[z_1, z_2, -z_1 z_2 - q(Z), 1; Z] \mid y_1 y_2 + q(Y) > 0\,,\ y_1 > 0\}$$

is well-defined. The subgroup $O^+(V)$ of index 2 in the orthogonal group $O(V)$, which has been introduced as the image of $Pin(V)$ in section 3.3, acts on $\mathcal{H}_n$ by

$$O^+(V) \times \mathcal{K} \to \mathcal{K}\,, \quad (\sigma, [x + \mathrm{i}y]) \mapsto [\sigma(x) + \mathrm{i}\sigma(y)].$$

The following properties of the orthogonal half-space and the operation of the group $O^+(V)$ are shown in [Fr2]. Regard $K := SO(2) \times SO(6)$ as a subgroup of $O(V)$ by decomposing V orthogonally in

$$V \cong \mathbb{R}^{2,n} \cong \mathbb{R}^{2,0} \perp \mathbb{R}^{0,n}$$

and letting $SO(2)$ and $SO(6)$ operate on the respective summand.

Proposition 5.14. *Both $O^+(V)$ and its connected component $SO^+(V)$ operate transitively on $\mathcal{H}_n$ by holomorphic transformations. Let $Stab^+$ and $Stab$ denote the stabilizers of some point of $\mathcal{H}_n$ in $O^+(V)$ respectively in $SO^+(V)$. Then, one has*

$$\mathcal{H}_n \cong O^+(V)/Stab^+ \cong SO^+(V)/Stab \cong SO^+(V)/K$$

The group of biholomorphic transformations $Bih(\mathcal{H}_n)$ is isomorphic to $O^+(V)/\pm id_V$.

Obviously, the half-space $\mathcal{H}_n$ can be embedded as an open domain in $\mathbb{C}^n$ and is thus a hermitian domain of type IV_n.

Orthogonal Modular Forms and Factor of Automorphy

Let $\tilde{\mathcal{H}}_n$ denote the non-projective inverse image $\pi^{-1}(\mathcal{H}_n)$ in $V(\mathbb{C}) - \{0\}$. An element $g \in O^+(V)$ acts on $\tilde{\mathcal{H}}_n$ by

$$(z_1, z_2, *, 1, Z) \mapsto (\tilde{z}_1, \tilde{z}_2, *, a; \tilde{Z}).$$

The fourth coordinate $a{=}a(z_1, z_2, Z)$ has to be normalized to 1 in the projective space in order to match the given characterization of $\mathcal{H}_n$ and to extend the action of $O^+(V)$ to the tube domain in $\mathbb{C}^n$.

Lemma 5.15. *The mapping*

$$J : O(V) \times \tilde{\mathcal{H}}_n \to \mathbb{C}^* \ , \ (g, (z_1, z_2, *, 1, Z)) \mapsto a(z_1, z_2, Z),$$

is a one-dimensional holomorphic factor of automorphy.

There is an alternative way to calculate this factor of automorphy considering the holomorphic transformation of $\mathcal{H}_n \hookrightarrow \mathbb{C}^n$ induced by an element $g \in O(V)$.

Lemma 5.16. *Let $j(g, (z_1, z_2, Z))$ denote the Jacobian determinant of $g \in O(V)$ as transformation of $\mathcal{H}$. Then*

$$j(g, (z_1, z_2, Z)) = det\,(g) J(g, Z)^n.$$

Now all prerequisites for defining one-dimensional modular forms are set up.

Definition 5.17. Let Γ be an orthogonal modular group corresponding to a lattice L. A *modular form of weight k and with respect to some character $v : \Gamma \to \mathbb{C}^*$* is a holomorphic function $f : \tilde{\mathcal{H}}_n \to \mathbb{C}$ with the properties

- (i) $f(\gamma(z)) = v(\gamma) f(z) \qquad \forall \gamma \in \Gamma$
- (ii) $f(tz) = t^{-k} f(z) \qquad \forall t \in \mathbb{C}$

In order to see that this definition given in [Fr2] is again a special case of the general definition of an automorphic form one easily proves the following lemma where $\mathcal{H}_n$ is supposed to be embedded into $\mathbb{C}^n$.

Lemma 5.18. *Modular forms of weight k and with respect to a character v are in one-to-one correspondence to holomorphic functions $F : \mathcal{H}_n \to \mathbb{C}$ with the transformation property*

$$F(\gamma(z)) = J(\gamma, z)^{-k} v(\gamma) f(z) \qquad \forall \gamma \in \Gamma.$$

Vector-valued automorphic forms in the sense of the first definition of this chapter can be defined when choosing a representation of the maximal compact subgroup K. In order to find all automorphic forms, one had to replace $O^+(V)$ and $SO^+(V)$ by their (algebraically) simply connected coverings $Pin(V)$ and $Spin(V)$. However, a description of modular forms in this context has not been published so far. In addition, a factor of automorphy with values in $K_{\mathbb{C}}$ that would characterize all possible factors of automorphy of orthogonal modular forms is not known to the author. In the special case $(2, 6)$, a factor of automorphy has been found with the help of a new description of the corresponding Spin group. This is one of the main results of this thesis and is presented in detail in the following chapter.

Chapter 6

An Exceptional Homomorphism for the Signature (2,6)

This is the crucial chapter of this thesis. A correspondence of quaternary and orthogonal modular forms is described in detail. As the main result of this thesis, a description of $Spin(2,6)$ independent of the Clifford algebra is established. A consequence is that the operation of $Spin(2,6)$ on its hermitian domain of type IV_6 can be described by Moebius Transformation on an isomorphic domain of type II_2. Furthermore, the possible factors of automorphy are examined rigorously in the final section.

6.1 The Definition of the Homomorphism

Let V be an eight-dimensional real vector space provided with a non-degenerate symmetric bilinear form of signature (2,6). As described in section 3.7, a basis $f_1, ..., f_4, e_5, ..., e_8$ can be chosen such that V decomposes into two hyperbolic planes and a four-dimensional negative definite space V_0, that is

$$V = H_1 \perp H_2 \perp V_0 = \mathbb{R}f_1 \oplus \mathbb{R}f_2 \perp \mathbb{R}f_3 \oplus \mathbb{R}f_4 \perp \mathbb{R}e_5 \perp \mathbb{R}e_6 \perp \mathbb{R}e_7 \perp \mathbb{R}e_8.$$

In these coordinates the quadratic form is

$$q(x) = x_1x_2 + x_3x_4 - x_5^2 - x_6^2 - x_7^2 - x_8^2$$

and the evaluation of the bilinear form on the basis vectors returns

$$\begin{aligned}&\langle f_1, f_2\rangle = 1\ , \quad \langle f_3, f_4\rangle = 1\ , \quad \langle f_i, f_i\rangle = 0 \quad \text{for } 1 \le i \le 4\\ &\langle e_j, e_j\rangle = -2 \quad \text{for } 5 \le j \le 8\end{aligned}$$

and all other combinations are zero.

An orthonormal basis will be needed as well to calculate the chiral element of the Clifford algebra. The basis

$$e_1 := f_1 + f_2\ ,\ e_2 := f_3 + f_4\ ,\ e_3 := f_1 - f_2\ ,\ e_4 := f_3 - f_4\ ,\ e_5,\ e_6,\ e_7,\ e_8$$

is obviously orthonormal and the quadratic form is that of the standard quadratic space $\mathbb{R}^{2,6}$

$$q(x) = {x_1}^2 + {x_2}^2 - {x_3}^2 - ... - x_8^2$$

This orthonormal basis defines a chiral element

$$\chi = e_1 \dots e_8$$

of the Clifford algebra $\mathscr{C}(V)$ which is always meant when referring to *the* chiral element in this chapter.

The Spin Representation of the Clifford Algebra

The following two isomorphisms explicitly establish the spin representation of the even part $\mathscr{C}^+(V)$ of the Clifford algebra of V.

Lemma 6.1. *There is an isomorphism of algebras*

$$\mathscr{C}(V) \to M_4(\mathscr{C}(V_0))$$

with the following properties:

(i) By restriction one obtains an isomorphism

$$\mathscr{C}^+(V) \to M_4(\mathscr{C}^+(V_0)).$$

(ii) The main involution of $\mathscr{C}(V)$ corresponds to the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d'' & -b'' \\ -c'' & a'' \end{pmatrix}$$

where a,b,c,d denote 2×2-matrices and a'' is supposed to be the main involution of $\mathscr{C}(V_0)$ applied entry-wise to the transpose of a. The main automorphism * of course corresponds to applying the main automorphism of $\mathscr{C}(V_0)$ element by element.

Proof. Define a linear mapping $V \to M_4(\mathscr{C}(V_0))$ by

$$\begin{aligned}
f_1 &\longmapsto e_5 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
f_2 &\longmapsto e_5 \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
f_3 &\longmapsto e_5 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
f_4 &\longmapsto e_5 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\
e_5 &\longmapsto e_5 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \\
e_i &\longmapsto e_i E \qquad\qquad\qquad \text{for } 6 \leq \mathrm{i} \leq 8
\end{aligned}$$

As the defining relation of the Clifford algebra is obviously satisfied for all basis vectors x, the universal property of the Clifford algebra guarantees a unique extension of this linear map to a homomorphism of the whole Clifford algebra $\mathscr{C}(V)$. The map is easily observed to be onto as certain products of the f_i are mapped to all matrices with only one nonzero entry, for instance

$$f_1 f_3 \mapsto (\delta_{13})\,,\ f_2 f_3 \mapsto (\delta_{24})\,,\ f_4 f_1 \mapsto (\delta_{42})\,,\ f_4 f_2 \mapsto (\delta_{31}).$$

Since the dimension of both algebras is 2^n the induced mapping is an isomorphism. The image of $\mathscr{C}^+(V)$ is obviously contained in $M_4(\mathscr{C}^+(V_0))$ and the restricted map is one-to-one again for reasons of dimensions. And observing that $ab'' = b''a''$ implies that the mapping given in (ii) is an involution. The images of the basis vectors are invariant under this involution which proves the last property. □

As $V_0 = span(e_5, ..., e_8)$ corresponds to the standard quadratic space $\mathbb{R}^{0,4}$, the even part of the Clifford algebra $\mathscr{C}^+(V_0) \cong \mathscr{C}^+(0,4) \cong \mathscr{C}(0,3)$ (by Lemma 3.16) is isomorphic to $\mathbb{H} \times \mathbb{H}$ by the classification in [Law]. The isomorphism chosen here promises to be most convenient for later computations.

Lemma 6.2. *The linear mapping*

$$\mathscr{C}^+(V_0) \longrightarrow \mathbb{H} \times \mathbb{H}$$

defined on generators by

$$e_5 e_6 \mapsto (\mathrm{i}_1, -\mathrm{i}_1), \quad e_5 e_7 \mapsto (\mathrm{i}_2, -\mathrm{i}_2), \quad e_5 e_8 \mapsto (\mathrm{i}_3, -\mathrm{i}_3)$$

is an isomorphism of algebras. The main involution of $\mathscr{C}^+(V_0)$ corresponds to the standard involution $(a,b) \mapsto (\bar{a}, \bar{b})$ on $\mathbb{H} \times \mathbb{H}$.

Proof. Follows easily by calculating the images of the other basis vectors which are

$$\begin{array}{rcl}
1_{\mathscr{C}^+(V_0)} = e_5^2 e_6^2 = -(e_5 e_6)(e_5 e_6) & \mapsto & -(\mathrm{i}_1^2, (-\mathrm{i}_1)^2) = (1,1) \\
e_6 e_7 = e_5 e_6 e_5 e_7 & \mapsto & (\mathrm{i}_1 \mathrm{i}_2, (-\mathrm{i}_1)(-\mathrm{i}_2)) = (\mathrm{i}_3, \mathrm{i}_3) \\
e_6 e_8 & \mapsto & (-\mathrm{i}_2, -\mathrm{i}_2) \\
e_7 e_8 & \mapsto & (\mathrm{i}_1, \mathrm{i}_1) \\
e_5 e_6 e_7 e_8 & \mapsto & (\mathrm{i}_1 \mathrm{i}_2 \mathrm{i}_3, -\mathrm{i}_1 \mathrm{i}_2 \mathrm{i}_3) = (-1, 1)
\end{array}$$

The linear independence of the images implies injectivity and since the dimensions are equal the map is onto as well. The involution on the left hand side changes the sign of all two-products and leaves 1 and the four-product invariant . This corresponds obviously to the standard involution on the right hand side. □

Concatenating the two isomorphisms delivers the spin representation of $\mathscr{C}^+(V)$ on the complex vector space $\mathbb{H}^4 \oplus \mathbb{H}^4$ denoted by ρ. The half-spin representations $\rho^\pm$ are are the projections of ρ to one of the factors.

The Spin Representation of Spin(2,6)

Let $-\mathbb{H}$ denote the real vector space $\mathbb{H}$ endowed with the quadratic form $q(h) = -\bar{h}h$, The quadratic spaces V_0 will often be identified with this quadratic space via the mapping

$$e_5 \mapsto 1\,,\ e_6 \mapsto \mathrm{i}_1\,,\ e_7 \mapsto \mathrm{i}_2\,,\ e_8 \mapsto \mathrm{i}_3.$$

Proposition 6.3. *The restriction of the spin representation $\rho : \mathscr{C}^+(V) \to M_4(\mathbb{H}) \times M_4(\mathbb{H})$ embeds $Spin(V)$ into $Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H})$. The resulting faithful spin representation*

$$\rho : Spin(V) \to Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H})$$

acts on the Eichler transformations generating $Spin(V)$ and on the chiral element as follows. For later calculations, the corresponding transformations of $SO^+(V)$ have been added to the table.

Element of Spin(V)	*Image in $SO^+(V)$* $z = (z_1, z_2, z_3, z_4; Z) \mapsto$	*Image in $Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H})$*
-1	z	$(-E, -E)$
χ	$-z$	$(E, -E)$
$1 + f_1 h$	$(z_1 + \langle h, Z\rangle - q(h) z_2, z_2, z_3, z_4; Z - z_2 h)$	$\left(\begin{pmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\bar{h} & 1 \end{pmatrix}, \begin{pmatrix} 1 & \bar{h} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -h & 1 \end{pmatrix}\right)$
$1 + f_2 h$	$(z_1, z_2 + \langle h, Z\rangle - q(h) z_1, z_3, z_4; Z - z_1 h)$	$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ \bar{h} & 1 & 0 & 0 \\ 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ h & 1 & 0 & 0 \\ 0 & 0 & 1 & -\bar{h} \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$
$1 + f_3 h$	$(z_1, z_2, z_3 + \langle h, Z\rangle - q(h) z_4, z_4; Z - z_4 h)$	$\left(\begin{pmatrix} 1 & 0 & 0 & h \\ 0 & 1 & \bar{h} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & \bar{h} \\ 0 & 1 & h & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$
$1 + f_4 h$	$(z_1, z_2, z_3, z_4 + \langle h, Z\rangle - q(h) z_3; Z - z_3 h)$	$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & h & 1 & 0 \\ \bar{h} & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \bar{h} & 1 & 0 \\ h & 0 & 0 & 1 \end{pmatrix}\right)$
$1 + t f_1 f_3$	$(z_1 + t z_4, z_2, z_3 - t z_2, z_4; Z)$	$\left(\begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$
$1 + t f_2 f_3$	$(z_1, z_2 + t z_4, z_3 - t z_1, z_4; Z)$	$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$
$1 + t f_4 f_2$	$(z_1, z_2 - t z_3, z_3, z_4 + t z_1; Z)$	$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$
$1 + t f_4 f_1$	$(z_1 - t z_3, z_2, z_3, z_4 + t z_2; Z)$	$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix}\right)$

Remark 6.4. The generators of the form $1 + f_4 h$ are redundant. This will be proved after Proposition 6.5.

Proof. The image of $Spin(V)$ lies in the product of the symplectic groups because $N(g) = gg' = 1$ implies in both components

$$\begin{pmatrix} \bar{d}' & -\bar{b}' \\ -\bar{c}' & \bar{a}' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = E \Leftrightarrow \begin{pmatrix} \bar{a}' & \bar{c}' \\ \bar{b}' & \bar{d}' \end{pmatrix} J \begin{pmatrix} a & b \\ c & d \end{pmatrix} = J \Leftrightarrow \bar{M}'JM = J.$$

Concatenating the two homomorphisms for the Eichler transformations is easy, for example

$$\begin{aligned} 1 + f_1 h \quad &\mapsto \quad E + h_0 e_5^2 \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix} + \sum_{i=6}^{8} h_{i-5} e_5 e_i \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix} \\ &\mapsto \left(\begin{pmatrix} 1&h&0&0\\0&1&0&0\\0&0&1&0\\0&0&-\bar{h}&1 \end{pmatrix}, \begin{pmatrix} 0&\bar{h}&0&0\\0&0&0&0\\0&0&0&0\\0&0&-h&0 \end{pmatrix} \right) \end{aligned}$$

The calculation for the chiral element is still straightforward, but slightly more laborious.

$$\begin{aligned} \chi \quad &= \quad (f_1 + f_2)\,(f_3 + f_4)\,(f_1 - f_2)\,(f_3 - f_4)\, e_5\, e_6\, e_7\, e_8 \\ &\mapsto \quad {e_5}^4 e_5 e_6 e_7 e_8 \\ &\qquad \underbrace{\begin{pmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix}\begin{pmatrix} 0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0 \end{pmatrix}}_{\begin{pmatrix} 0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0 \end{pmatrix}}\underbrace{\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}\begin{pmatrix} 0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0 \end{pmatrix}}_{\begin{pmatrix} 0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0 \end{pmatrix}}\begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix} \\ &= \quad e_5 e_6\, e_5 e_7\, e_5 e_8 \begin{pmatrix} -1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix} \\ &\mapsto \quad -(\mathrm{i}_1\mathrm{i}_2\mathrm{i}_3 E, -\mathrm{i}_1\mathrm{i}_2\mathrm{i}_3 E) \\ &= \quad (E, -E) \end{aligned}$$

The image of the chiral element in the spinorial kernel has been calculated in section 3.3 and the images of the Eichler transformations are easily verified recalling that $1 + f_i v$ with $\langle f_i\,,\, v\rangle = 0$ is mapped to

$$a \mapsto E(f_i, v)(a) = a - \langle a, f_i\rangle v + \langle a, v\rangle f_i - q(v)\langle a, f_i\rangle f_i$$

in $SO^+(V)$. □

6.2 The Equivariance of the Hermitian Domains

This section constructs explicitly an equivariant isomorphism of the quaternary half-space and the orthogonal half-space that allows the comparison of the corresponding modular forms. Whereas the isomorphism itself is already given in [F-H], this thesis adds some details and interpretations concerning Lie theory and spin representations.

Recall that $\mathcal{Z}(Spin(V)) = \{\pm E, \pm\chi\}$ and $\mathcal{Z}(Sp(2,\mathbb{H})) = \{\pm E\}$ and both groups are connected. Hence, the isogeny

$$\mathfrak{so}(2,6) \cong \mathfrak{sp}(2,\mathbb{H})$$

implies by Proposition 4.9 that $Spin(2,6)$ must be a a two-sheeted covering of $Sp(2,\mathbb{H})$. The considerations about connected isogenous Lie groups in section 4.3 imply furthermore that the adjoint forms

$$Spin(2,6)/\{\pm E, \pm\chi\} \cong Sp(2,\mathbb{H})/\{\pm E\} \cong SO^{+}(2,6)/\{\pm id\} \tag{6.1}$$

have to be isomorphic.

The Half-Spin Representations of Spin(2,6)

In the following proposition, both half-spin representations of $Spin(V)$ are shown to be two-sheeted covering maps from $Spin(V)$ to $Sp(2,\mathbb{H})$ without using Lie theory.

Proposition 6.5. *The half-spin representations*

$$\rho^{\pm} : Spin(V) \to Sp(2,\mathbb{H})$$

obtained by projecting the spin representation ρ to both factors are irreducible inequivalent representations of $Spin(V)$. Both are two-sheeted covering maps of connected Lie groups with kernel $\{1,\chi\}$ respectively $\{1,-\chi\}$.

Proof. The irreducibility of both projections is an immediate consequence of the structure of $\mathscr{C}^{+}$ as shown in section 3.2. The inequivalence can be shown easily considering the images $\rho^{+}(\chi) = E$ and $\rho^{-}(\chi) = -E$ of the chiral element χ. Given any $\mathbb{C}$-linear isomorphism $\phi : \mathbb{H}^4 \to \mathbb{H}^4$, the obvious relation $\phi\rho^{+}(\chi)\phi^{-1} = E \neq \rho^{-}(\chi)$ prevents it from being equivariant. Furthermore, χ and $-\chi$ are obviously contained in the kernel of ρ^{+} respectively ρ^{-}.
If the differential of the mapping was shown to be an isomorphism of Lie algebras, the remaining statement would immediately follow from Lie theory. However, an elementary proof will be given here which does not rely on any results from Lie theory.

In the next section will be shown explicitly, that there are no more than two pairs $(A,B) \in \rho(Spin(V))$ with the same A. Hence, the statement about the kernel of the first projection is proven because ρ is injective. Surjectivity can be seen directly, writing down the preimages of the generators of $Sp(2,\mathbb{H})$ (Lemmata 2.9 and 2.5) under ρ^{+}. As above, V_0 is identified with $-\mathbb{H}$.

$$\begin{aligned}
e_1e_2 = f_1f_3 + f_2f_3 + f_2f_4 + f_1f_4 &\mapsto J \\
(1+af_1f_3)(1+bf_2f_3)(1+f_3h) &\mapsto \begin{pmatrix} E & S \\ 0 & E \end{pmatrix} \text{ with } S = \begin{pmatrix} a & h \\ \bar{h} & b \end{pmatrix} \in Sym(2,\mathbb{H}) \\
f_3f_4f_2e_5 + f_3f_4f_1e_5 - f_4f_3f_2e_5 - f_4f_3f_1e_5 &\mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
1 + f_1e_5 &\mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \\
f_1f_2f_3f_4h + f_2f_1f_3f_4 + f_2f_1f_4f_3\bar{h}^{-1} + f_1f_2f_4f_3 &\mapsto \begin{pmatrix} h & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{h}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

All calculations are straightforward, the first one has been already carried out in the proof of Lemma 6.1. The proof that all elements on the left hand side are really contained in the Spin

group is easy.
The arguments for the second projection ρ^- are completely analogous. $\square$

Corollary 6.6. The equation

$$J^{-1}\begin{pmatrix}1&&&\\&1&&\\&h&1&\\\bar h&&&1\end{pmatrix}J=\begin{pmatrix}1&&&-h\\&1&-\bar h&\\&&1&\\&&&1\end{pmatrix}$$

shows the redundancy of the generators of the form $1+f_4h$ regarding that the two preimages of of J must be a combination of some $\pm(1+f_if_j)$, $1\leq i,j\leq 4$.

As a corollary, all quotients of the group $Spin(V)$ by subgroups of its center $\mathcal{Z}=\{\pm1,\pm\chi\}$ are identified.

$$\begin{tikzcd}
 & Spin(V) \arrow[dl, "2"'] \arrow[d, "2"] \arrow[dr, "2"] & \\
Spin(V)/\pm1 \arrow[d, Rightarrow, "\pi"', "\sim"] & Spin(V)/\{1,\chi\} \arrow[d, Rightarrow, "\rho^+"', "\sim"] & Spin(V)/\{1,-\chi\} \arrow[d, Rightarrow, "\rho^-"', "\sim"] \\
SO^+(V) \arrow[d, "2"'] & Sp(2,\mathbb{H}) \arrow[d, "2"'] & Sp(2,\mathbb{H}) \arrow[dl, "2"'] \\
SO^+(V)/\pm id \arrow[r, Leftrightarrow, "\sim"] & Sp(2,\mathbb{H})/\pm E \arrow[r, Leftrightarrow, "\sim"] & Spin(V)/\{\pm1,\pm\chi\}
\end{tikzcd} \tag{6.2}$$

The images of the Eichler transformations under the isomorphism $SO^+(V)/\pm id\cong Sp(2,\mathbb{H})/\pm E$ can be read off the table in Proposition 6.3 when projecting to one of the factors. In the following, the first projection will always be preferred.

Orthogonal Half-Space versus Quaternary Half-Space

The isogeny $\mathfrak{so}(2,6)\cong\mathfrak{sp}(2,\mathbb{H})$ and the general theory about symmetric spaces already imply the isomorphism of the corresponding hermitian domains. Again, an explicit construction of the correspondence will be achieved without using the general theory.

The conjugation on $\mathbb{H}$ can be extended to a $\mathbb{C}$-linear conjugation

$$z'=(x+\mathrm{i}y)'=\bar x+\mathrm{i}\bar y$$

on $\mathbb{H}\otimes\mathbb{C}$. Applying this denotation and the criterion for positivity in Lemma 2.7, the quaternary tube domain of degree 2 can be written as

$$\begin{aligned}&\mathscr{H}_2=Sym(2,\mathbb{H})+iPos(2,\mathbb{H})\\ =\;&\{\begin{pmatrix}z_0&z_1\\z_1'&z_2\end{pmatrix}\mid z_0,z_2\in\mathbb{C}\,,\;z_1\in\mathbb{H}\otimes\mathbb{C}\,,\;y_0y_2-\bar y_1y_1>0\,,\;y_0>0\}.\end{aligned}$$

The orthogonal tube domain (Proposition 5.14) is

$$\mathcal{H}_6=\{[z_0,z_2,-z_0z_2-q(Z),1;Z]\in\mathbb{P}(V(\mathbb{C}))\mid y_0y_2+q(Y)>0\,,\;y_0>0\}.$$

The identification of V_0 and $-\mathbb{H}$

$$e_5\mapsto1\,,\;e_6\mapsto\mathrm{i}_1\,,\;e_7\mapsto\mathrm{i}_2\,,\;e_8\mapsto\mathrm{i}_3$$

still holds when extending the $\mathbb{R}$-bilinear forms to $\mathbb{C}$-bilinear forms on the complexifications $V_0(\mathbb{C})$ and $\mathbb{H}\otimes\mathbb{C}$. Explicitly, one easily verifies the formula (keeping in mind Lemma 2.3)

$$\begin{aligned}\langle z_1, z_2\rangle &= \langle x_1, x_2\rangle - \langle y_1, y_2\rangle + \mathrm{i}\langle x_1, y_2\rangle + \mathrm{i}\langle y_1, x_2\rangle\\ &= -(z_1 z_2' + z_2 z_1') = -(z_1' z_2 + z_2' z_1)\end{aligned}$$

for the bilinear form on $\mathbb{H}\otimes\mathbb{C}$, hence

$$q(z) = -zz' = -z'z$$

for the quadratic form. Furthermore, a determinant on $Sym(2, \mathbb{H}\otimes\mathbb{C})$ can be introduced by

$$det\, Z := z_0 z_2 - z_1' z_1 = z_0 z_2 + q(z_1).$$

A more general definition respectively references can be found in [Kri].

Proposition 6.7. *The quaternionic half-space $\mathscr{H}_2$ and the orthogonal half-space $\mathscr{H}_6$ are biholomorphically equivalent via the map*

$$Z = \begin{pmatrix} z_0 & z_1 \\ z_1' & z_2 \end{pmatrix} \mapsto [z_0, z_2, -det\, Z, 1; -z_1].$$

where $z_1 \in \mathbb{H}\otimes\mathbb{C}$ on the left hand side and $z_1 \in V_0(\mathbb{C})$ on the right hand side. This map is equivariant in the sense that the diagram

$$\begin{array}{ccc} (Sp(2,\mathbb{H})/\pm E)\times \mathscr{H}_2 & \longrightarrow & \mathscr{H}_2 \\ \downarrow & & \downarrow \\ (SO^+(V)/\pm id)\times \mathcal{H}_6 & \longrightarrow & \mathcal{H}_6 \end{array}$$

commutes.

Remark 6.8. This isomorphism does not lift to an equivariant isomorphism from $SO^+(V)$ to $Sp(2, \mathbb{H})$ regarding the diagram 6.2.

Proof. The equivalence of the half-spaces is obvious by the preliminary remarks. In order to prove the equivariance it suffices to explicitly show the commutativity of the diagram for the Eichler transformations generating $Spin(V)$. One calculates for the generator $1 + f_1 h$ (obviously $h' = \bar{h}$ for a quaternion h)

$$\begin{aligned} &\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_0 & z_1 \\ z_1' & z_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ h' & 1 \end{pmatrix} = \begin{pmatrix} z_0 + hz_1' + z_1 h' + hh' z_2 & z_1 + hz_2 \\ (z_1 + hz_2)' & z_2 \end{pmatrix}\\ \mapsto\ & [z_0 - \langle h, z_1\rangle - q(h) z_2, z_2, \underbrace{-(z_0 z_2 + hz_1' z_2 + z_1 h' z_2 + hh' z_2) + (z_1 + hz_2)'(z_1 + hz_2)}_{-z_0 z_2 + z_1' z_1 = -det\, Z}, 1; -z_1 - hz_2] \end{aligned}$$

which is exactly the image of $[z_0, z_2, -det\, Z, 1; -z_1]$ under the corresponding Eichler transformation as given in Proposition 6.3.

The calculation for the generator $1 + tf_4 f_2$ yields

$$
\begin{aligned}
& \begin{pmatrix} z_0 & z_1 \\ z_1' & z_2 \end{pmatrix} \begin{pmatrix} tz_0+1 & tz_1 \\ 0 & 1 \end{pmatrix}^{-1} \\
= & \begin{pmatrix} z_0 & z_1 \\ z_1' & z_2 \end{pmatrix} \frac{1}{tz_0+1} \begin{pmatrix} 1 & -tz_1 \\ 0 & tz_0+1 \end{pmatrix} \\
= & \frac{1}{tz_0+1} \begin{pmatrix} z_0 & z_1 \\ z_1' & -tz_1' z_1 + z_2 + tz_0 z_2 \end{pmatrix} \\
\mapsto & \left[\frac{z_0}{tz_0+1}, \frac{z_2 + t(z_0 z_2 - z_1' z_1)}{tz_0+1}, \frac{-1}{(tz_0+1)^2} \underbrace{(z_0 z_2 - tz_0 z_1' z_1 + tz_0 z_2 - z_1' z_1)}_{=(1+tz_0) det\, Z}, 1; \frac{-z_1}{tz_0+1}\right] \\
= & [z_0, z_2 + t det\, Z, -det\, Z, 1 + tz_0; -z_1]
\end{aligned}
$$

which is again the sought-after image of $[z_0, z_2, -det\, Z, 1; -z_1]$. The remaining calculations are either very similar or much easier. $\square$

Lemma 5.12 and Proposition 5.14 stated that

$$
\begin{aligned}
& Bih(\mathscr{H}_2) = Sp(2, \mathbb{H})/\pm E \,\cup\, \sigma(Sp(2, \mathbb{H})/\pm E) \\
\cong\; & Bih(\mathcal{H}_6) = O^+(V)/\pm id \cong O^+(2, 6)/\pm id
\end{aligned}
$$

where σ denoted the biholomorphic mapping $Z \mapsto Z'$. On the orthogonal side, $Z \mapsto Z'$ corresponds to the mapping

$$[z_0, z_2, *, 1; z_5, z_6, z_7, z_8] \mapsto [z_0, z_2, *, 1; z_5, -z_6, -z_7, -z_8]$$

which has obviously determinant -1. As $SO^+(2, 6)$ has index 2 in $O^+(2, 6)$, one has another proof of

$$SO^+(2, 6)/\pm id \cong Sp(2, \mathbb{H})/\pm E$$

that does not use any Lie theory.

6.3 A Description of Spin(2,6) independent of the Clifford Algebra

In this section, the image of $Spin(2, 6)$ in the product of the symplectic groups under the spin representation is computed. As a preparation, a certain exceptional automorphism of $Sp(2, \mathbb{H})/\pm E$ is needed.

An Auxiliary Automorphism of $GL(2, \mathbb{H})$

The above automorphism originates mainly in an outer automorphism of $GL(2, \mathbb{H})$.

Lemma 6.9. *The homomorphism $\phi : GL(2, \mathbb{H}) \to GL(2, \mathbb{H})/\pm E$ defined on generators by*

$$
\begin{aligned}
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &\mapsto \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right] \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &\mapsto \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &\mapsto \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] \qquad a \in \mathbb{R}_{>0} \\
\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} &\mapsto \left[\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}\right] \qquad h \in \mathbb{H},\ |h| = 1
\end{aligned}
$$

satisfies the following equation

$$(\bar{A}'YA)' = \bar{\tilde{A}}'Y'\tilde{A} \qquad \text{for } \tilde{A} \in \phi(A) \text{ and for all } Y \in Sym(2,\mathbb{H}). \tag{6.3}$$

Proof. For real matrices the equation is obvious as $\mathbb{R}$ is the center of $\mathbb{H}$ and quaternary conjugation acts trivially on $\mathbb{R}$. For the last generator let $Y = \begin{pmatrix} y_0 & y_1 \\ \bar{y}_1 & y_2 \end{pmatrix} \in Sym(2,\mathbb{H})$, that is $y_0, y_2 \in \mathbb{R}$ and $y_1 \in \mathbb{H}$. Then

$$\left(\begin{pmatrix} \bar{h} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_0 & y_1 \\ \bar{y}_1 & y_2 \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right)' = \begin{pmatrix} y_0 & \bar{y}_1 h \\ \bar{h}y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{h} \end{pmatrix}\begin{pmatrix} y_0 & \bar{y}_1 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$$

□

With the knowledge that the only outer homomorphism of $GL(4,\mathbb{C})$ is given by transposition followed by inversion, it is possible to find a lifting of the previous homomorphism given by a closed formula.

Lemma 6.10. *The homomorphism $\phi : GL(2,\mathbb{H}) \to GL(2,\mathbb{H})/\pm E$ of the previous lemma lifts to an automorphism of $GL(2,\mathbb{H})$ when setting*

$$\begin{aligned}
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &\mapsto -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &\mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \qquad a \in \mathbb{R}_{>0} \\
\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \qquad h \in \mathbb{H},\ |h| = 1
\end{aligned}$$

After embedding $GL(2,\mathbb{H})$ into $GL(4,\mathbb{C})$ the automorphism is described by the closed formula

$$\boxed{\check{A} \mapsto \sqrt{det\,(\check{A})}B^{-1}\check{A}'^{-1}B} \qquad \in Image(GL(2,\mathbb{H})) \tag{6.4}$$

where $B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$. *The determinant of $\check{A}$ is always real and positive.*

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \in \mathbb{R}_{>0}\right\} \subset GL(2,\mathbb{R})$, and σ denote the sign of $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$. With $det\begin{pmatrix} aE & bE \\ cE & dE \end{pmatrix} = (ad-bc)^2$ follows

$$\begin{aligned}
\begin{pmatrix} aE & bE \\ cE & dE \end{pmatrix} &\mapsto \sqrt{(ad-bc)^2}B^{-1}\begin{pmatrix} aE & bE \\ cE & dE \end{pmatrix}'^{-1} B \\
&= \frac{|ad-bc|}{(ad-bc)}B^{-1}\begin{pmatrix} d & 0 & -c & 0 \\ 0 & d & 0 & -c \\ -b & 0 & a & 0 \\ 0 & -b & 0 & a \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \sigma\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -c & 0 & -d \\ c & 0 & d & 0 \\ 0 & a & 0 & b \\ -a & 0 & -b & 0 \end{pmatrix} = \sigma\begin{pmatrix} aE & bE \\ cE & dE \end{pmatrix}
\end{aligned}$$

For the last generator $det\,(\check{A}) = det\,(\check{h}) = h_1^2 + h_2^2 + h_3^2 + h_4^2 = |h|^2$ is 1, so one receives

$$\begin{aligned}
\begin{pmatrix} \check{h} & 0 \\ 0 & E \end{pmatrix} &\mapsto B^{-1} \begin{pmatrix} \check{h} & 0 \\ 0 & E \end{pmatrix}^{\prime -1} B \\
&= B^{-1} \begin{pmatrix} \begin{pmatrix} h_1 + ih_2 & -h_3 + ih_4 \\ h_3 + ih_4 & h_1 - ih_2 \end{pmatrix}^{-1} & 0 \\ 0 & E \end{pmatrix} B \\
&= B^{-1} \begin{pmatrix} h_1 - ih_2 & h_3 - ih_4 & 0 & 0 \\ -h_3 - ih_4 & h_1 + ih_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & h_3 - ih_4 & -h_1 + ih_2 \\ 0 & 0 & h_1 + ih_2 & h_3 + ih_4 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} E & 0 \\ 0 & \begin{pmatrix} h_1 + ih_2 & h_3 + ih_4 \\ -h_3 + ih_4 & h_1 - ih_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & \check{h} \end{pmatrix}
\end{aligned}$$

Det$(\check{A})$ is always real and positive, as it is for all the generators. □

The Main Result

The key to explicitly describing the image of the Spin group under the spin representation is the following homomorphism from $Sp(2,\mathbb{H})$ to $Sp(2,\mathbb{H})/\pm E$.

Proposition 6.11. *There is an epimorphism*

$$\psi : Sp(2,\mathbb{H}) \to Sp(2,\mathbb{H})/\pm E$$

with kernel $\pm E$ *satisfying*

$$M\langle Z\rangle' = \tilde{M}\langle Z'\rangle \qquad \text{for } \tilde{M} \in \psi(M) \text{ and for all } Z \in H(2,\mathbb{H}). \tag{6.5}$$

This homomorphism does not lift to an automorphism of $Sp(2,\mathbb{H})$ *as shown after the next proposition.*

Proof. Defining the images of the generators by

$$J \mapsto \pm J \tag{6.6}$$

$$\begin{pmatrix} E & S \\ 0 & E \end{pmatrix} \mapsto \pm \begin{pmatrix} E & S' \\ 0 & E \end{pmatrix} \tag{6.7}$$

$$\begin{pmatrix} \bar{W}' & 0 \\ 0 & W^{-1} \end{pmatrix} \mapsto \pm \begin{pmatrix} \overline{\phi(W)}' & 0 \\ 0 & \phi(W)^{-1} \end{pmatrix} \tag{6.8}$$

one easily proves the formula. In the last case, ϕ is the automorphism of Lemma 6.10. As the images of the generators obviously form a set of generators for $Sp(2,\mathbb{H})$, the map is onto. The fact that the action of two matrices M and $N \in Sp(n,\mathbb{H})$ define the same transformation of $\mathscr{H}_2$ if and only if $M = \pm N$ (Proposition 5.9) implies the statement about the kernel. □

The following theorem can be viewed as the main result of this thesis and describes explicitly the image of $Spin(2,6)$ in the product of the symplectic groups.

Theorem 6.12. *Let V be the real quadratic space of signature $(2,6)$ and $Spin(V)$ the corresponding Spin group. The spin representation ρ of Proposition 6.3*

$$\rho : Spin(V) \to Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H})$$

induces an isomorphism

$$Spin(V) \cong \Omega := \{\ (M,\ \tilde{M}\ ) \in Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H}) \quad | \quad \tilde{M} \in \psi(M)\ \}.$$

Here, $\psi : Sp(2,\mathbb{H}) \to Sp(2,\mathbb{H})/\pm E$ is the homomorphism of the previous lemma which is defined by the relation

$$M\langle Z\rangle' = \tilde{M}\langle Z'\rangle.$$

Corollary 6.13. As $Spin(V)$ is connected by Lemma 4.2, $\Omega \subset Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H})$ is also connected.

Corollary 6.14. The homomorphism ψ does not lift to an automorphism of $Sp(2,\mathbb{H})$ because both (E,E) and $(E,-E)$ are contained in Ω.

Proof. If the relation $M\langle Z\rangle' = \tilde{M}\langle Z'\rangle$ can be verified for the images of the Eichler transformations generating the Spin group, the image of the Spin group has to be contained in Ω. By Proposition 6.5, the projection on the first factor is surjective as well as the homomorphism $\psi : Sp(2,\mathbb{H}) \to Sp(2,\mathbb{H})/\pm E$. Then a possible multiplication by $\rho(\chi) = (E,-E)$ guarantees that Ω is contained in the image of the Spin group and the proposition is proved.

In the following, the formula defining ψ is verified for the images of all generators given in Proposition 6.3. For the generator $1 + f_1 h$, it suffices to show that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ is mapped to $\begin{pmatrix} 1 & \bar{h} \\ 0 & 1 \end{pmatrix}$ by the homomorphism ϕ of Lemma 6.10. This is true because of

$$\begin{aligned} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &\mapsto \begin{pmatrix} |h| & 0 \\ 0 & h|h|^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |h|^{-1} & 0 \\ 0 & |h|h^{-1} \end{pmatrix} = \begin{pmatrix} 1 & |h|^2 h^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \bar{h} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The calculation for the generator $1 + f_2 h$ is completely analogous. The image of $1 + f_3 h$ under the first projection operates as $Z' \mapsto Z' + \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}$ which is obviously the transpose of the operation of the other projection on Z. The same argument applies to $1 + f_1 f_3$ and $1 + f_2 f_3$. As shown in corollary 6.6, the generator $1 + f_4 h$ is (fortunately) redundant. The calculations for the remaining cases $1 + f_4 f_1$ and $1 + f_4 f_2$ are again analogous, so only the first is treated as follows. Both projections are the same and the action has already been calculated in the proof of Proposition 6.7 as

$$Z \mapsto \frac{1}{tz_0 + 1} \begin{pmatrix} z_0 & z_1 \\ z_1' & -tz_1' z_1 + z_2 + tz_0 z_2 \end{pmatrix}$$

which is obviously the same when interchanging z_1 and z_1' before and transposing afterwards. □

The Operation of Spin(V) on the Quaternary Half-Space

With the help of the new description of the Spin group, the operation on the half space $\mathscr{H}_2$ can be written down explicitly as Moebius transformation. Consider the embedding

$$\mathscr{H}_2 \hookrightarrow \mathscr{H}_2 \times \mathscr{H}_2\,, \quad Z \mapsto (Z, Z').$$

This embedding allows to write the natural operation of $Spin(V) \subset Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H})$ on the Half-Space as Moebius transformation on both factors, that is

$$Spin(V) \times \mathscr{H}_2 \to \mathscr{H}_2 \ , \ (\,(M,\tilde{M})\,,\,(Z,Z')\,) \mapsto (\,M\langle Z\rangle, \tilde{M}\langle Z'\rangle\,).$$

The defining relation of the Spin group given in the main theorem guarantees that this operation is well-defined.

The given operation can be generalized similarly for all Spin groups isomorphic to some $Spin(2,n)$ as done by Prof. Freitag in a paper which is unfortunately not published so far. However, the convenient description of the image of the group $Spin(2,6)$ involving the quaternions and the quaternary half-space is not available in the general context.

6.4 Factors of Automorphy of Spin(2,6)

In this section, the maximal compact subgroup of the Spin group shall be examined in detail. The resulting description of its complexification together with the explicit description of a factor of automorphy with values in $K_{\mathbb{C}}$ leads to the knowledge of all possible factors of automorphy of the Spin group.

One-Dimensional Factors of Automorphy

To begin with, the one-dimensional factors of automorphy of the symplectic and the orthogonal world are compared. In turns out, that the square of the orthogonal factor of automorphy "a" (definition 5.15), corresponds to the quaternary factor of automorphy given at the end of section 5.3.

Proposition 6.15. *The equivariant isomorphism*

$$(Sp(2,\mathbb{H})/\pm E) \times \mathscr{H}_2 \quad \cong \quad (SO^+(V)/\pm id) \times \mathcal{H}_6$$

induces the following relation of the quaternary factor of automorphy and the orthogonal factor of automorphy.

$$J(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z) = det\,(\check{C}\check{Z} + \check{D}) = J(g, (z_0, z_2, *, 1, -z_1))^2$$

Proof. Using the table in Proposition 6.3, the assertion is easily verified for the Eichler transformations. Recall that the image of $Z = \begin{pmatrix} z_0 & z_1 \\ z_1' & z_2 \end{pmatrix} \in \mathscr{H}_2$ is $z = (z_0, z_2, -det\,Z, 1; -z_1) \in \mathcal{H}_6$. Then the orthogonal factor of automorphy can immediately be read off the table as the fourth coordinate of the image of z. The matrix $\check{C}\check{Z} + \check{D}$ is triangular in all cases except for the redundant $1 + f_4 h$ and the relation is obvious for all generators. □

Maximal Compact Subgroups of Spin(2,6)

A maximal compact subgroup of Spin(2,6) may be determined by very different means.

Description as Stabilizer

The description of the Spin group and its operation on the half plane of the previous section immediately yield a maximal compact subgroup as the stabilizer of the point iE, that is

$$\begin{aligned}
K &:= Stab_{Spin(V)}(\mathrm{i}E)\\
&= \{ \left(M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tilde{M} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ -\tilde{B} & \tilde{A} \end{pmatrix} \right) \in Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H}) \quad | \quad M\langle Z' \rangle = \tilde{M}\langle Z \rangle' \,\}\\
&= \{ \, (M,\tilde{M}) \in Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H}) \quad | \quad MJ = JM \,,\; \tilde{M}J = J\tilde{M} \,,\; M\langle Z' \rangle = \tilde{M}\langle Z \rangle' \,\}\\
&\cong \{ \, (N,\tilde{N}) \in U(4) \times U(4) \quad | \quad \tilde{N} \in \tilde{\psi}(N) \, \}.
\end{aligned}$$

The isomorphism to the last line is given by the mapping

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto \check{A} + \mathrm{i}\check{B}$$

on both factors (confer Proposition 5.13). Since this isomorphism maps $-E$ to $-E$, the following diagram defines the mapping $\tilde{\psi}$.

$$\begin{tikzcd}
Stab_{Sp(2,\mathbb{H})}(\mathrm{i}E) \arrow[r, "\psi"] \arrow[d, "\sim"'] & Stab_{Sp(2,\mathbb{H})}(\mathrm{i}E)/\pm E \arrow[d, "\sim"] \\
U(4) \arrow[r, "\tilde{\psi}"'] & U(4)/\pm E
\end{tikzcd}$$

Description within the Spin group

Another description within the Clifford algebra can be found in paragraph 5 in the appendix of [Sat]. Define $V^+ := span(e_1, e_2) \cong \mathbb{R}^{2,0}$ and $V^- := span(e_3, ..., e_8) \cong \mathbb{R}^{0,6}$ where $\{e_1, ..., e_8\}$ denotes the standard orthonormal basis of V introduced at the beginning of this chapter. Then, one has $Spin(V^+) \cong Spin(2,0) \cong S^1$ and $Spin(V^-) \cong Spin(0,6) \cong SU(4)$. As Satake explicitly calculates the Cartan involution he easily deduces a maximal compact subgroup as the set of fix points of this involution which turns out to be

$$Spin(V^+) \cdot Spin(V^-) \cong K.$$

The spin representation ρ establishes an isomorphism to the first description within $Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H})$.

Coverings of Maximal Compact Subgroups

A third description of the maximal compact subgroup can be obtained independently of the previous descriptions by applying the theory of covering spaces of Lie groups as introduced in section 4.3.

It has been pointed out already in that section that a covering map of a semisimple Lie group induces a covering map of maximal compact subgroups (assuming they exist) with the same number of sheets. As $\pi_1(U(4)) = \mathbb{Z}$, there is only one two-sheeted covering of the maximal compact subgroup $U(4)$ of $Sp(2,\mathbb{H})$ up to isomorphism. This covering has to be a maximal compact subgroup of $Spin(V)$ considering the two-sheeted covering ρ^+ of the final section. Furthermore, this subgroup has to be a two-fold covering of $S^1 \times SO(6)$, the maximal compact subgroup of $SO^+(V)$.

As the fundamental group of a maximal compact subgroup is the same as that of the whole group, $\pi_1(Spin(V))$ has to be a subgroup of index 2 in $\mathbb{Z}$. There is only $2\mathbb{Z} \cong \mathbb{Z}$, hence one has

Lemma 6.16.

$$\pi_1(Spin(V)) \cong \mathbb{Z}.$$

Now consider the obviously well-defined mapping

$$\kappa : S^1 \times SU(4) \to U(4)\,,\ (z, A) \mapsto zA.$$

This mapping is onto (simply write $B = \sqrt[4]{det\, B}\frac{B}{\sqrt[4]{det\, B}}$ for any $B \in U(4)$) and the kernel turns out to be $\{\pm(1, E), \pm(-\mathrm{i}, \mathrm{i}E)\}$ as the following easy calculation shows.

$$\begin{aligned} (z, A) \in Ker(\kappa) \quad &\Leftrightarrow \quad zA = E,\ |z| = 1,\ \bar{A}'A = E,\ det\,(A) = 1 \\ &\Leftrightarrow \quad A = \lambda E,\ z\lambda = 1,\ |\lambda| = |z| = 1,\ \lambda^4 = 1 \end{aligned}$$

Hence, the quotient of $S^1 \times SU(4)$ by the only subgroup of the kernel of order 2 must be the sought-after two-sheeted covering of $U(4)$, that is

$$K \cong S^1 \times SU(4)/\pm(1, E).$$

The isomorphism to Satake's description is obvious considering that $Spin(V^+) \cong S^1$ and $Spin(V^-) \cong SU(4)$ as remarked above.

To see the whole picture the examination of the covering relations is also applied to the orthogonal world. Recall that a two-sheeted covering $\pi : SU(4) \to SO(6)$ with kernel $\pm E$ has been constructed in section 4.10. The mapping

$$S^1 \times SU(4) \to S^1 \times SO(6)\,,\ (z, A) \mapsto (z^2, \pi(A))$$

is then obviously a four-sheeted covering with kernel $\{\pm(1, E), \pm(1, -E)\}$. Summarized in a diagram, the interrelations between maximal compact subgroups are as follows.

$$\begin{tikzcd}
 & S^1 \times SU(4) \arrow[dl, "z\cdot A"', "4:1"] \arrow[d, "2:1"'] \arrow[dr, "(z^2,\pi(A))", "4:1"'] & \\
U(4) \arrow[dr, "2:1"'] & K \arrow[l, "2:1"] \arrow[r, "2:1"'] \arrow[d, "4:1"'] & S^1 \times SO(6) \arrow[dl, "2:1"] \\
 & \tilde{K} &
\end{tikzcd}$$

Here, $\tilde{K}$ denotes the maximal compact subgroup of the adjoint form, that is

$$\tilde{K} \ \cong\ U(4)/\pm E \ \cong\ S^1 \times SO(6)/\pm(1, E) \ \cong\ S^1 \times SU(4)/\{\pm(1, \pm E), \pm(i, \pm\mathrm{i}E)\}$$

Summary of Maximal Compact Subgroups of Spin(2,6)

Proposition 6.17. *The following descriptions of the maximal compact subgroup K of $Spin(V)$ are isomorphic.*

$$\begin{aligned} K &= \{\left(M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tilde{M} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ -\tilde{B} & \tilde{A} \end{pmatrix}\right) \in Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H}) \quad | \quad M\langle Z' \rangle = \tilde{M}\langle Z \rangle' \} \\ &\cong \{(N, \tilde{N}) \in U(4) \times U(4) \quad | \quad \tilde{N} \in \tilde{\psi}(N)\} \\ &\cong Spin(V^+) \cdot Spin(V^-) \\ &\cong S^1 \times SU(4)/\pm(1,E) \end{aligned}$$

The Complexification of the Maximal Compact Subgroup

In the following, the analytic complexification of the maximal compact subgroup K will be determined in order to write down the factors of automorphy.

A complexification of the stabilizer of iE is

$$Stab_{Sp(n,\mathbb{H})}(\mathrm{i}E)_{\mathbb{C}} := \{M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in GL(2n, \mathbb{H}\otimes\mathbb{C}) \mid \bar{M}'JM = J\}$$

when tensoring the coefficients with $\mathbb{C}$.

Lemma 6.18. *The mapping*

$$Stab_{Sp(n,\mathbb{H})}(\mathrm{i}E)_{\mathbb{C}} \tilde{\to} GL(2n,\mathbb{C})\ , \ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto \check{A} + \mathrm{i}\check{B}$$

is a well-defined isomorphism.

Proof. As calculated after Lemma 2.13, the symplectic relation after applying the isomorphism $\check{}$ is

$$\begin{aligned} & \check{M}'\tilde{I}\check{M} = \tilde{I} \\ \Leftrightarrow\ & \begin{pmatrix} \check{A}' & -\check{B}' \\ \check{B}' & \check{A}' \end{pmatrix} \begin{pmatrix} 0 & \tilde{J} \\ -\tilde{J} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} 0 & \tilde{J} \\ -\tilde{J} & 0 \end{pmatrix} \\ \Leftrightarrow\ & \check{B}'\tilde{J}\check{A} - \check{A}'\tilde{J}\check{B} = 0\ ,\ \check{B}'\tilde{J}\check{B} + A'\tilde{J}\check{A} = \tilde{J}. \end{aligned}$$

with $\tilde{J} = \begin{pmatrix} A & & & \\ & A & & \\ & & \ldots & \\ & & & A \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

These two conditions guarantee that the inverse of

$$g := \check{A} + \mathrm{i}\check{B}$$

is $g^{-1} = \tilde{J}^{-1}\check{A}'\tilde{J} - i\tilde{J}^{-1}\check{B}\tilde{J}$. Hence,

$$\tilde{J}g^{-1\prime}\tilde{J}^{-1} = \check{A} - \mathrm{i}\check{B}$$

and solving for A and B yields

$$\check{A} = \frac{1}{2}(g + \tilde{J}g^{-1\prime}\tilde{J}^{-1})\ , \quad \check{B} = \frac{1}{2i}(g - \tilde{J}g^{-1\prime}\tilde{J}^{-1}).$$

These formulas explicitly define an inverse mapping. Together with the fact that $GL(2n, \mathbb{H}\otimes\mathbb{C}) \cong GL(4n, \mathbb{C})$, this implies that the mapping is one-to-one. $\square$

Since $GL(n,\mathbb{C})$ is the analytic complexification of $U(n)$, which is unique up to isomorphism, this lemma implies that $Stab_{Sp(n,\mathbb{H})}(\mathrm{i}E)_{\mathbb{C}}$ is the analytic complexification of $Stab_{Sp(n,\mathbb{H})}(\mathrm{i}E) \cong U(n)$.

Now define

$$K_{\mathbb{C}} := \{(M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tilde{M}) \in Sp(2, \mathbb{H}\otimes\mathbb{C}) \times Sp(2, \mathbb{H}\otimes\mathbb{C}) \mid \bar{M}'JM = J\,,\ \tilde{M} \in \psi_{\mathbb{C}}(M)\}$$

where $\psi_{\mathbb{C}}$ denotes the unique extension of the map $\psi : Sp(2,\mathbb{H}) \to Sp(2,\mathbb{H})/\pm E$ to the complexification $Sp(n, \mathbb{H}\otimes\mathbb{C})$. With the help of the defining relation of ψ, it is easy to show that $\psi_{\mathbb{C}}$ is holomorphic and thus $K_{\mathbb{C}}$ is an analytic complexification of K. Analogously to the case of $Stab_{Sp(n,\mathbb{H})}(\mathrm{i}E)$ and $U(2n)$, the diagram

$$\begin{tikzcd}
Stab_{Sp(2,\mathbb{H}\otimes\mathbb{C})}(\mathrm{i}E) \arrow[r, "\psi_{\mathbb{C}}"] \arrow[d, "\sim"'] & Stab_{Sp(2,\mathbb{H}\otimes\mathbb{C})}(\mathrm{i}E)/\pm E \arrow[d, "\sim"] \\
GL(4,\mathbb{C}) \arrow[r, "\tilde{\psi_{\mathbb{C}}}"'] & GL(4,\mathbb{C})/\pm E
\end{tikzcd}$$

defines a mapping characterizing the image of $K_{\mathbb{C}}$ in $GL(4,\mathbb{C}) \times GL(4,\mathbb{C})$.

It is not hard to extend the remaining descriptions of the maximal compact subgroup K to the complexified case. Recall that $V(\mathbb{C})$ denotes the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ of a real vector space V.

Proposition 6.19. *The following descriptions of the analytic complexification of the maximal compact subgroup of the Spin group are isomorphic.*

$$\begin{aligned}
K_{\mathbb{C}} &= \{(M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tilde{M}) \in Sp(2, \mathbb{H}\otimes\mathbb{C}) \times Sp(2, \mathbb{H}\otimes\mathbb{C}) \mid \bar{M}'JM = J\,,\ \tilde{M} \in \psi_{\mathbb{C}}(M)\} \\
&\cong \{(N, \tilde{N}) \in GL(4,\mathbb{C}) \times GL(4,\mathbb{C}) \mid \tilde{N} \in \tilde{\psi_{\mathbb{C}}}(N)\} \\
&\cong Spin(V^+(\mathbb{C})) \cdot Spin(V^-(\mathbb{C})) \qquad\qquad [\subset Spin(8,\mathbb{C}) \cong Spin(2,6)_{\mathbb{C}}\ ] \\
&\cong \mathbb{C}^* \times SL(4,\mathbb{C})/\pm(1,E)\,.
\end{aligned}$$

Recalling the examples of analytic complexifications in section 4.3, it is clear that $\mathbb{C}^* \times SL(4,\mathbb{C})/\pm(1,E)$ is an analytic complexification of $S^1 \times SU(4)/\pm(1,E)$. Furthermore, the isomorphisms

$$Spin(V^+(\mathbb{C})) \cong Spin(2,\mathbb{C}) \cong SO(2,\mathbb{C}) \cong \mathbb{C}^*$$

and

$$Spin(V^-(\mathbb{C})) \cong Spin(6,\mathbb{C}) \cong SL(4,\mathbb{C})$$

which are described in section 4.3 ensure that the third line is another description of the analytic complexification.

Factors of Automorphy

With the help of the new description of the Spin group within $Sp(2,\mathbb{H}) \times Sp(2,\mathbb{H})$ and its operation on $\mathscr{H}_2$ as Moebius transformation, a $K_{\mathbb{C}}$-valued factor of automorphy can be found in analogy to the well-known "CZ+D" in the theory of Siegel modular forms.

Proposition 6.20. *The mapping*

$$\begin{aligned}
J : Spin(V) \times \mathscr{H}_2 &\to K_{\mathbb{C}}, \\
(\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}\right), Z) &\mapsto \overbrace{(C,\tilde{C})(Z,Z') + (D,\tilde{D})}^{\check{}} = (\check{C}\check{Z} + \check{D},\, \check{\tilde{C}}\check{Z}' + \check{\tilde{D}}\,).
\end{aligned}$$

defines a factor of automorphy of $Spin(V)$ with values in $K_{\mathbb{C}}$.

Proof. The calculation that the cocycle relation is fulfilled is easy and exactly the same as in the case of the factor of automorphy $CZ + D$ in the theory of Siegel modular forms. The claim that the factor of automorphy has values in $K_{\mathbb{C}}$ has to be verified for the generators of $Sp(2, \mathbb{H})$. For the generators of the form $\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$, the factor of automorphy (E, E) respectively $(E, -E)$ is obviously in $K_{\mathbb{C}}$. For the generators of the form $\begin{pmatrix} \bar{W}'^{-1} & 0 \\ 0 & W \end{pmatrix}$ with invertible W, the factor $(\check{W}, \check{\check{W}})$ is in $K_{\mathbb{C}}$ by definition of the homomorphism $\tilde{\psi}_{\mathbb{C}} : GL(4, \mathbb{C}) \to GL(4, \mathbb{C})/\pm E$. To cover the generator J, it remains to show that for an invertible matrix $Z \in Sym(2, \mathbb{H} \otimes \mathbb{C})$, the factor $(\check{Z}, \check{Z}')$ is in $K_{\mathbb{C}}$.

The preimage of $\check{Z}$ in $Sp(2, \mathbb{H} \otimes \mathbb{C})$ is $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with

$$A = \frac{1}{2}(Z + Z^{-1}) \text{ and } B = \frac{1}{2i}(Z - Z^{-1}).$$

The inverse of Z is hermitian by Formula 2.1, so A and B are in $Sym(2, \mathbb{H} \otimes \mathbb{C})$. The symplectic relations reduce to

$$AB = BA \qquad \text{and} \qquad A^2 + B^2 = E \tag{6.9}$$

which is easily verified. Another short calculation yields

$$Z'^{-1} = (Z^{-1})'$$

for $Z \in Sym(2, \mathbb{H} \otimes \mathbb{C})$. Therefore, it has to be verified that the image of $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ under the homomorphism ψ is $\begin{pmatrix} A' & B' \\ -B' & A' \end{pmatrix}$. The necessary calculation greatly simplifies when expressing $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ as a product of generators. Looking up the proof of Lemma 2.9 and remembering the relations 6.9, one finds that

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} E & -BA \\ 0 & E \end{pmatrix} J \begin{pmatrix} E & -B^{-1}A \\ 0 & E \end{pmatrix}.$$

The commuting hermitian 2 times 2 matrices A and B satisfy

$$(BA)' = B'A' \qquad\qquad (\Leftrightarrow\ (B^{-1}A)' = (B^{-1})'A'\ )$$

and

$$(BZB)' = B'Z'B' \text{ for all } Z \in Sym(2, \mathbb{H} \otimes \mathbb{C}).$$

Both calculations have to be carried out explicitly using several properties of quaternions and hermitian matrices. Nevertheless, they are so easy that they have been omitted. Lastly recall (6.6) that $\psi_{\mathbb{C}}$ maps

$$\begin{array}{rcll} J & \mapsto & \pm J & \\ \begin{pmatrix} E & S \\ 0 & E \end{pmatrix} & \mapsto & \pm \begin{pmatrix} E & S' \\ 0 & E \end{pmatrix} & \\ \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} & \mapsto & \pm \begin{pmatrix} F & 0 \\ 0 & F^{-1} \end{pmatrix} & \qquad \text{with } (BZB)' = FZ'F \end{array}$$

for a S and $B \in Sym(2, \mathbb{H} \otimes \mathbb{C})$. Then it is obvious that the product of the images of the four generators yields $\begin{pmatrix} A' & B' \\ -B' & A' \end{pmatrix}$ as needed. $\square$

As described in detail in section 5.2, this factor of automorphy can be composed with complex finite-dimensional representations of $K_{\mathbb{C}}$ to obtain factors of automorphy with values in $GL(V)$. In this way, one finds a representative of each equivalence class of factors of automorphy belonging to irreducible representation of K. Thus, to complete the analysis of possible factors of automorphy, the well-known representation theory of $K_{\mathbb{C}}$ in the form $\mathbb{C}^* \times SL(4,\mathbb{C})/\pm(1,E)$ is explained briefly. Details and proofs can be found in [Ful].

In the following, all representations are supposed to be complex and finite-dimensional. To begin with, representations of quotients of a group A by a normal subgroup B are representations of A which map all elements of B to the trivial transformation. It is enough to describe all irreducible representations of $K_{\mathbb{C}}$, because any holomorphic representation of $K_{\mathbb{C}}$ is completely reducible by Theorem 4.11. Further on, irreducible representations of direct products of groups A and B are tensor products of an irreducible representation of A and an irreducible representation of B. However, irreducible representations of the abelian group $\mathbb{C}^*$ are one-dimensional, more precisely, there are only the characters $z \mapsto z^n$ for all $n \in \mathbb{Z}$. Thus, the irreducible representations of $(\,\mathbb{C}^* \times SL(4,\mathbb{C})\,)/\pm(1,E)$ are of the form

$$[(z,A)] \mapsto z^n \tau(A)$$

where $n \in \mathbb{Z}$ and τ is an irreducible representation of $SL(4,\mathbb{C})$ such that

$$\tau(-E) = \begin{cases} id_V & \text{if n is even,} \\ -id_V & \text{if n is odd.} \end{cases}$$

Lastly, the irreducible representations of the simply connected simple Lie group $SL(4,\mathbb{C})$ are in one-to-one correspondence to irreducible representations of its simple Lie algebra $\mathfrak{sl}(4,\mathbb{C})$. The theory of weights reveals that the irreducible representations of $\mathfrak{sl}(4,\mathbb{C})$ are parameterized by three nonnegative integers a_1, a_2 and a_3 which are the coefficients of the highest weight written as a linear combinations of the fundamental weights. The corresponding irreducible representation is denoted by $\Gamma_{(a_1,a_2,a_3)}$. The trivial one-dimensional representation turns out to be $\Gamma_{(0,0,0)}$, the four-dimensional standard representation $V = \mathbb{C}^4$ is $\Gamma_{(1,0,0)}$ and the 15-dimensional adjoint representation of $\mathfrak{sl}(4,\mathbb{C})$ on itself is $\Gamma_{(1,0,1)}$. Furthermore, the six-dimensional representation $\Lambda^2 V$ corresponds to $\Gamma_{(0,1,0)}$ and the four-dimensional dual representation V^* is isomorphic to $\Lambda^3 V$ and corresponds to $\Gamma_{(0,0,1)}$. The symmetric powers $Sym^k(V)$ of the standard representation are also irreducible and correspond to $\Gamma_{(k,0,0)}$. There is a generalization of exterior and symmetric powers known as Weyl's construction or Schur functor that assigns to any partition λ of d an irreducible representation of $GL(V)$ as a subrepresentation of $V^{\otimes d}$ when V is any complex finite-dimensional vector space. Inserting the standard representation $V = \mathbb{C}^4$ and restricting to $SL(4,\mathbb{C})$ delivers an irreducible representation of $SL(4,\mathbb{C})$ which defines an irreducible representation of the Lie algebra $\mathfrak{sl}(4,\mathbb{C})$ by differentiation. In this way, the irreducible representation $\Gamma_{(a_1,a_2,a_3)}$ can be constructed explicitly when starting with the partition $\lambda = (a_1+a_2+a_3,\, a_2+a_3,\, a_3,\, 0)$ and one finds the dimension formula

$$\begin{aligned} dim(\Gamma_{(a_1,a_2,a_3)}) &= \prod_{1\le i,j\le 4} \frac{a_i + \ldots + a_{j-1} + j - i}{j-i} \\ &= \frac{1}{12}(a_1+1)(a_1+a_2+2)(a_1+a_2+a_3+3)(a_2+1)(a_2+a_3+2)(a_3+1). \end{aligned}$$

The so-called "Littlewood-Richardson rule" delivers remarkable formulas describing explicitly the decomposition of the tensor product of two arbitrary irreducible representations into irreducible representations.

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