On weak cokernels

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Abstract

We show that the weak cokernels of morphisms of 2-groups studied in [CarrascoGarzónVitale:2006] are isomorphic to the corresponding construction in [RobertsSchreiber:2007] which can be thought of in terms of mapping cones. Motivated by this we adopt the discussion of weak cokernels to Lie n-algebras, for arbitrary n, following [SchreiberStasheff].

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1 Introduction

Given two 2-groups $G_{(2)}$ and $H_{(2)}$ and a strictly injective morphism of 2-groups

$$t: H_{(2)} \to G_{(2)}$$

[CarrascoGarzónVitale:2006] showed how to construct a weak cokernel weaker(t)

$$H_{(2)} \xrightarrow{t} G_{(2)} \longrightarrow \operatorname{wcoker}(t)$$

which is a Gray 3-group. We demonstrate that one may think of this as the mapping cone of t in a generalization of the construction considered in [RobertsSchreiber:2007] and write

$$\operatorname{wcoker}(t) := (H_{(2)} \xrightarrow{t} G_{(2)}) \,.$$

It follows that for any given short exact sequence of strict 2-groups

$$K_{(2)} \xrightarrow{t} G_{(2)} \longrightarrow B_{(2)}$$

one obtains the setup

$$K_{(2)} \xrightarrow{t} G_{(2)} \xrightarrow{f} (H_{(2)} \xrightarrow{t} G_{(2)})$$
.

We want to eventually understand the obstruction to lifting a $\Sigma B_{(2)}$ -valued 2-functor

$$\mathcal{P} \longrightarrow \Sigma B_{(2)}$$

through the exact sequence

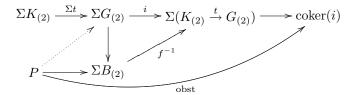
$$\Sigma K_{(2)} \xrightarrow{\Sigma t} \Sigma G_{(2)}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\Sigma B_{(2)}$$

i.e. to construct

The obstruction to this should be the composite denoted obst in



with f^{-1} some suitable "local inverse" to f. This should exist in the context of ana-2-functors.

While we do not try to make this more precise at the level of 2-groups, we can study the analogous situation in the context of (semistrict) Lie *n*-algebras.

Following [StasheffSchreiber] these we can conceives as quasi-free differential graded commutative algebras living in the obious 2-category of algebra chain maps and homotopies. This allows us to work with arbitrary n.

We reproduce the construction analogous to the above one for sequences

$$\mathfrak{k}_{(n)}^* \overset{t^*}{\lessdot} \mathfrak{g}_{(n)}^* \overset{\bullet}{\lessdot} \mathfrak{b}_{(n)}^*$$

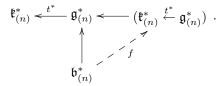
with t^* assumed to be particularly well behaved. (A generalization away from this assumption is certainly expected to exists, but not studied here.)

Thinking of the weak cokernel of 2-groups as a mapping cone proves to be useful for the generalization to Lie n-algebras:

we define the mapping cone Lie (n+1)-algebra

$$\big(\mathfrak{k}_{(n)}\stackrel{t}{\to}\mathfrak{g}_{(n)}\big)$$

and show that it does fit into



Moreover, we show that in this context now the map f does have a weak inverse

$$f^{-1}: (\mathfrak{k}_{(n)}^* \stackrel{t^*}{\leftarrow} \mathfrak{g}_{(n)}^*) \to \mathfrak{b}_{(n)}^*.$$

2 Mapping cones and weak cokernels of Lie nalgebras

We conceive semistrict Lie n-algebras dually as differential graded commutative algebras which are freely generated, as graded commutative algebras, in degree $1 \le d \le n$. We refer to them as quasi-free differential graded commutative algebras (qDGCAs).

These we take here to live in the 2-category whose morphisms are chain maps that are at the same time algebra homomorphisms, and whose 2-morphisms are chain homotopies.

Definition 1 (mapping cone of qDGCAs) Let

$$\mathfrak{h}_{(n)}^* \lessdot^{t^*} \mathfrak{g}_{(n)}^*$$

be a morphism of qDGCAs such that t^* restricts to a surjective morphism on the underlying vector spaces, hence that it surjectively maps generators to generators.

The mapping cone of t^* is the qDGCA whose underlying graded algebra is

$$\bigwedge^{\bullet}(s\mathfrak{g}_{(n)}^* \oplus ss\mathfrak{f}_{(n)}^*)$$

and whose differential d_{t^*} is such that it acts on generators schematically as

$$d_{t^*} = \left(\begin{array}{cc} d_{\mathfrak{g}_{(n)}} & 0 \\ t^* & d_{\mathfrak{f}_{(n)}} \end{array} \right) .$$

More in detail, d_{t^*} is defined as follows.

We write σt^* for the degree +1 derivation on $\bigwedge^{\bullet}(s\mathfrak{g}_{(n)}^* \oplus ss\mathfrak{f}_{(n)}^*)$ which acts on $\mathfrak{sg}_{(n)}^*$ as t^* followed by a shift in degree and which acts on $\mathfrak{sf}_{(n)}^{(n)}$ as 0. Then, for any $a \in \mathfrak{sg}_{(n)}^*$ we have

$$d_{t^*}a := d_{\mathfrak{g}_{(n)}}a + \sigma t^*(a).$$

and

$$d_{t^*}\sigma t^*(a) := -\sigma t^*(d_{\mathfrak{g}_{(n)}}a) = -d_{t^*}d_{\mathfrak{g}_{(n)}}a.$$

Proposition 1 The differential d_{t^*} defined this way indeed satisfies $(d_{t^*})^2 = 0$.

Proof. For $a \in s\mathfrak{g}_{(n)}^*$ we have

$$d_{t^*}d_{t^*}a = d_{t^*}(d_{\mathfrak{g}_{(n)}} + \sigma t^*(a)) = \sigma t^*(d_{\mathfrak{g}_{(n)}}a) - \sigma t^*(d_{\mathfrak{g}_{(n)}}a) = 0.$$

Hence $(d_{t^*})^2$ vanishes on $\bigwedge^{\bullet}(s\mathfrak{g}_{(n)}^*)$. Since

$$d_{t^*} d_{t^*} \sigma t^*(a) = -d_{t^*} d_{t^*} d_{\mathfrak{g}_{(n)}} a$$

and since $d_{\mathfrak{g}_{(n)}}a \in \bigwedge^{\bullet}(s\mathfrak{g}_{(n)}^*)$ this implies $(d_{t^*})^2 = 0$.

We write

$$(\mathfrak{h}_{(n)}^* \overset{t^*}{\longleftarrow} \mathfrak{g}_{(n)}^*) := \left(\bigwedge^{\bullet} (s\mathfrak{g}_{(n)}^* \oplus ss\mathfrak{f}_{(n)}^*), d_{t^*} \right)$$

for the resulting qDGCA and

$$\big(\mathfrak{h}_{(n)} \stackrel{t}{\to} \mathfrak{g}_{(n)}\big)$$

for the corresponding Lie (n+1)-algebra.

Proposition 2 There is a canonical morphism

$$\mathfrak{g}_{(n)}^* \longleftarrow (\mathfrak{h}_{(n)}^* \stackrel{t^*}{\leftarrow} \mathfrak{g}_{(n)}^*)$$

with the property that

$$\mathfrak{h}_{(n)}^* \overset{t^*}{\overset{t^*}{\longleftarrow}} \mathfrak{g}_{(n)}^* \overset{\mathfrak{f}^*}{\overset{\tau}{\longleftarrow}} \mathfrak{g}_{(n)}^*) .$$

Proof. On components, this morphisms is the identity on $s\mathfrak{g}_{(n)}^*$ and 0 on $ss\mathfrak{h}_{(n)}^*$. One checks that this respects the differentials. The homotopy to the 0-morphism sends

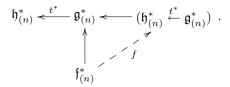
$$\tau : \sigma t^*(a) \mapsto t^*(a)$$
.

Proposition 3 Let

$$\mathfrak{h}_{(n)}^* \overset{t^*}{\longleftarrow} \mathfrak{g}_{(n)}^* \overset{}{\longleftarrow} \mathfrak{f}_{(n)}^*$$

be a sequence of qDGCAs with t^* as above and with the property that $\mathfrak{g}^*_{(n)} \longleftarrow \mathfrak{f}^*_{(n)}$ restricts, on the underlying vector spaces of generators, to the kernel of the linear map underlying t^* .

Then there is a unique morphism $f:\mathfrak{f}_{(n)}^*\to(\mathfrak{h}_{(n)}^*\stackrel{t^*}\leftarrow\mathfrak{g}_{(n)}^*)$ such that



Proof. The morphism f has to be in components the same as $\mathfrak{g}_{(n)}^* \leftarrow \mathfrak{f}_{(n)}^*$. By the assumption that this is in the kernel of t^* , the differentials are respected. \square

Remark. It must be possible to relax the assumptions on $\mathfrak{g}_{(n)}^* \leftarrow \mathfrak{f}_{(n)}^*$ while retaining a unique $\mathfrak{f}_{(n)}^* \rightarrow (\mathfrak{h}_{(n)}^* \stackrel{t^*}{\leftarrow} \mathfrak{g}_{(n)}^*)$ up to isomorphism. This would then show that

$$(\mathfrak{h}_{(n)}^* \stackrel{t^*}{\leftarrow} \mathfrak{g}_{(n)}^*) = \operatorname{coker}(t)^*$$

is the weak kernel of t^* .

Proposition 4 With the assumptions as before, the morphism $\mathfrak{f}_{(n)}^* \to (\mathfrak{h}_{(n)}^* \stackrel{t^*}{\leftarrow} \mathfrak{g}_{(n)}^*)$ has a – noncanonical – weak inverse

$$f^{-1}:(\mathfrak{h}_{(n)}^* \stackrel{t^*}{\leftarrow} \mathfrak{g}_{(n)}^*) \to \mathfrak{f}_{(n)}^*$$

Proof. We first construct a morphism f^{-1} and then show that it is weakly inverse to f. Choose a splitting of the vector space V underlying $\mathfrak{g}_{(n)}^*$ as

$$V = \ker(t^*) \oplus V_1$$
.

Take the component map of f^{-1} to be the identity on $\ker(t^*)$ and 0 on V_1 . Moreover, for $a \in V_1$ set

$$f^{-1}: \sigma t^*(a) \mapsto -d_{\mathfrak{g}_{(n)}} a |_{\bigwedge^{\bullet} \ker(t^*)}.$$

For $a \in \ker(t^*)$ we have

$$\begin{array}{ccc}
a & \longrightarrow & d_{\mathfrak{g}_{(n)}} a \\
\downarrow & & \downarrow & \downarrow \\
a & \longmapsto & d_{\mathfrak{g}_{(n)}} a
\end{array}$$

For $a \in V_1$ we have

and

$$\begin{array}{ccc}
\sigma t^{*}(a) & \longrightarrow & -\sigma t^{*}(d_{\mathfrak{g}_{(n)}}a) \\
\downarrow^{f^{-1}} & & \downarrow^{f^{-1}} \\
d_{\mathfrak{g}_{(n)}}a|_{\bigwedge^{\bullet} \ker(t^{*})} & \longrightarrow & d_{\mathfrak{g}_{(n)}}(d_{\mathfrak{g}_{(n)}}a|_{\bigwedge^{\bullet} \ker(t^{*})})
\end{array}.$$

Hence this is indeed a morphism of qDGCAs.

Next we check that f^{-1} is a weak inverse of f. Clearly

$$\mathfrak{f}_{(n)}^{*} \longleftarrow (\mathfrak{h}_{(n)}^{*} \stackrel{t^{*}}{\leftarrow} \mathfrak{g}_{(n)}^{*}) \longleftarrow \mathfrak{f}_{(n)}^{*}$$

is the identity on $\mathfrak{f}_{(n)}^*$. What remains is to construct a homotopy

$$(\mathfrak{h}_{(n)}^* \stackrel{t^*}{\leftarrow} \mathfrak{g}_{(n)}^*) \longleftarrow \mathfrak{f}_{(n)}^* \longleftarrow (\mathfrak{h}_{(n)}^* \stackrel{t^*}{\leftarrow} \mathfrak{g}_{(n)}^*) .$$

One checks that this is accomplished by taking τ to act on σV_1 as $\tau: \sigma V_1 \stackrel{\simeq}{\to} V_1$ and extend suitably.

Example. For μ an (n+1)-cocycle on \mathfrak{g} and \mathfrak{g}_{μ} the corresponding Baez-Crans type Lie n-algebra, the qDGCA of $(\Sigma^{n-1}\mathfrak{u}(1)^* \leftarrow \mathfrak{g}_{\mu}^*)$ is based on the vector space $(\bigwedge^{\bullet}(s\mathfrak{g}^* \oplus s^n\mathbb{R}^* \oplus s^{n+1}\mathbb{R}^*))$ with the differential on $s\mathfrak{g}^*$ being the Chevalley-Eilenberg differential. For $\{b\}$ the canonical basis of $s^n\mathbb{R}^*$ and $\{c\}$ the canonical basis of $s^{n+1}\mathbb{R}^*$ the differential on these generators is

$$db = -\mu + c$$

and

$$dc = 0$$
.

Then the morphism

$$f^{-1}: (\Sigma^{n-1}\mathfrak{u}(1)^* \leftarrow \mathfrak{g}_{\mu}^*) \to \mathfrak{g}^*$$

is the identity on $s\mathfrak{g}^*$, vanishes on b and sends

$$f^{-1}: c \mapsto \mu$$
.

Mapping cones and weak cokernels of 2-groups 3

Mapping cone of the identity

In [RobertsSchreiber:2007] the mapping cone of the identity morphism on a strict 2-group was studied.

Definition 2 The Gray groupoid which we denote either

$$T\Sigma G_{(2)}$$

and address it as the tangent 2-groupoid of $\Sigma G_{(2)}$, or

$$INN_0(G_{(2)})$$

and address it as the inner automorphism 2-groupoid of $\Sigma G_{(2)}$ or simply

$$(G_{(2)} \xrightarrow{\mathrm{Id}} G_{(2)})$$

and address it as the mapping cone of $\mathrm{Id}_{G_{(2)}}$ or as the 2-crossed module induced by $\mathrm{Id}_{G_{(2)}}$. This 2-groupoid $T\Sigma G_{(2)}$ is defined to be the strict pullback

$$T\Sigma G_{(2)} \longrightarrow (\Sigma G_{(2)})^{2}$$

$$\downarrow \qquad \qquad \downarrow^{\text{dom}}$$

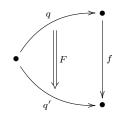
$$\{\bullet\} \longrightarrow \Sigma G_{(2)}$$

This means the following. An object of $T\Sigma G_{(2)}$ is a morphism

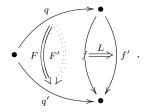
$$\bullet \xrightarrow{q} \bullet$$

in $\Sigma G_{(2)}$, hence an object of $G_{(2)}$.

A 1-morphism in $T\Sigma G_{(2)}$ is a filled triangle



in $\Sigma G_{(2)}$. Finally, a 2-morphism in $T^t\Sigma G_{(2)}$ looks like



The monoidal structure on $T\Sigma G_{(2)}$ is that induced from the embedding

$$T\Sigma G_{(2)} := \mathrm{INN}_0(\Sigma G_{(2)}) \hookrightarrow \mathrm{AUT}(G_{(2)})$$

discussion in [RobertsSchreiber:2007].

Recall for later use that this canonically sits in the sequence

$$G_{(2)} \hookrightarrow T\Sigma G_{(2)} \longrightarrow \Sigma G_{(2)}$$
.

3.2 Mapping cone of a faithful morphism

This has an obvious generalization to non-identity but faithful morphisms:

Let $G_{(2)}$ and $H_{(2)}$ be strict 2-groups and write $\Sigma G_{(2)}$ and $\Sigma H_{(2)}$ be the corresponding strict one object 2-groupoids.

Let

$$t: H_{(2)} \hookrightarrow G_{(2)}$$

be a morphism of strict 2-groups, faithful as a functor of the underlying 1-groupoids. This means we have a strict 2-functor

$$\Sigma t : \Sigma H_{(2)} \hookrightarrow \Sigma G_{(2)}$$
.

Definition 3 The morphism t defines a strict 2-groupoid with a weak monoidal structure that makes it a Gray groupoid, which we denote either

$$T^t \Sigma G_{(2)}$$

and address it as the tangent 2-groupoid of $\Sigma G_{(2)}$ relative to t, or

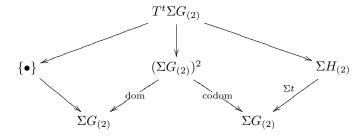
$$\operatorname{INN}_0^t(G_{(2)})$$

and address it as the inner automorphism 2-groupoid of $\Sigma G_{(2)}$ relative to t or simply

$$(\ H_{(2)} \xrightarrow{\quad t \quad} G_{(2)}\)$$

and address it as the mapping cone of t or as the 2-crossed module induced by t.

This 2-groupoid $T^t\Sigma G_{(2)}$ is defined to be the strict pullback



where

$$2 := \{ \bullet \xrightarrow{\simeq} \circ \}$$

is the fat point.

Equivalently this means that $T^t\Sigma G_{(2)}$ is the strict pullback

$$T^{t}\Sigma G_{(2)} \longrightarrow \Sigma G_{(2)}$$

$$\downarrow \qquad \qquad \downarrow =$$

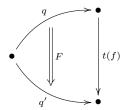
$$\Sigma H_{(2)} \xrightarrow{\Sigma t} \Sigma G_{(2)}$$

An object of $T^t\Sigma G_{(2)}$ is a morphism

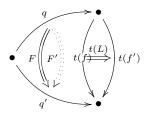
$$\bullet \xrightarrow{q} \bullet$$

in $\Sigma G_{(2)}$, hence an object of $G_{(2)}$.

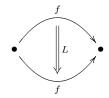
A 1-morphism in $T^t\Sigma G_{(2)}$ is a filled triangle



in $\Sigma G_{(2)}$, with f a morphism in $\Sigma H_{(2)}$, hence an object of $H_{(2)}$. Finally, a 2-morphism in $T^t\Sigma G_{(2)}$ looks like



with



a 2-morphism in $\Sigma H_{(2)}$, hence a morphism in $H_{(2)}$.

The monoidal structure on $T^t\Sigma G_{(2)}$ is that induced from the embedding

$$T^t \Sigma G_{(2)} \hookrightarrow T \Sigma G_{(2)}$$
.

Proposition 5 The 2-groupoid $T^t\Sigma G_{(2)}$ is codiscrete at top level. Therefore it is equivalent to its quotient by its 2-morphisms

$$T^t \Sigma G_{(2)} \simeq \pi_1(T^t \Sigma G_{(2)}).$$

This quotient is isomorphic to what in [CarrascoGarzónVitale:2006] is called (p. 595) the quotient pointed groupoid: $G_{(2)}/\langle H_{(2)}, t \rangle$:

$$\pi_1(T^t \Sigma G_{(2)}) \simeq G_{(2)}/\langle H_{(2)}, t \rangle$$
.

Proof. This is a matter of matching the items of the componentwise definition on the top of p. 595 in [CarrascoGarzónVitale:2006] to the above definition. \Box

[CarrascoGarzónVitale:2006] prove that $G_{(2)}/\langle H_{(2)},t\rangle$ is indeed the cokernel of t. See the last paragraph on p. 595 and item 2 on p. 596.