descent of the universal transition

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October 10, 2006

Abstract

To a given morphism $\mathcal{P}_2(Y) \to \mathcal{P}_2(X)$ of domains of 2-transport is associated the universal transition $\mathcal{P}_2(Y^\bullet)$. For sufficiently nice Y we have

$$\mathcal{P}_2(Y^\bullet) \simeq \mathcal{P}_2(X) \; .$$

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Let

$$p : \mathcal{P}_2(Y) \to \mathcal{P}_2(X)$$

be a morphism of domains of 2-transport.

Definition 1 *We say that* $p : \mathcal{P}_2(Y) \to \mathcal{P}_2(X)$ **contains all disks** *if each 2-morphism in* $\mathcal{P}_2(X)$ *is the image of one in* $\mathcal{P}_2(Y)$*.*

For instance consider the union of a good covering of X by open sets with a collection consisting of one contractible open neighbourhood for every image of the standard disk in X.

Proposition 1 *If* Y *contains all disks, then the universal transition* $\mathcal{P}_2(Y^\bullet)$ *associated with* Y *is equivalent, as a 2-category, to* $\mathcal{P}_2(X)$*,*

$$\mathcal{P}_2(Y^\bullet) \simeq \mathcal{P}_2(X)\ .$$

This equivalence is established using a 2-functor

$$s : \mathcal{P}_2(X) \to \mathcal{P}_2(Y^\bullet)$$

with only weak respect for composition. Hence the proposition does not hold within the 3-category of strict 2-categories with strict 2-functors between them.

We shall first construct a 2-functor

$$\tilde{s} : \mathcal{P}_2(X) \to \mathcal{P}_2(p)$$

and then show that it takes values only in the sub-2-category $\mathcal{P}_2(Y^\bullet) \subset \mathcal{P}_2(X)$.

$\tilde{s}$ is obtained from vertical composition with 2-morphisms of the kind t and $\bar{t}$ in $\mathcal{P}_2(p)$. In fact, we only need the combinations

$$\begin{tikzcd}
p(y_1) \arrow[r, "p(\gamma)"] \arrow[d, "t(y_1)"'] \arrow[dr, phantom, "t(\gamma)\ \Swarrow"] & p(y_2) \\
y_1 \arrow[r, "\gamma"'] & y_2 \arrow[u, "\bar{t}(y_2)"']
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
p(y_1) \arrow[r, "p(\gamma)"] \arrow[d, "t(y_1)"'] \arrow[dr, phantom, "t(\gamma)\ \Swarrow"] & p(y_2) \arrow[r, "\mathrm{Id}"] \arrow[d, "t(y_2)"] \arrow[dr, phantom, "\Swarrow"] & p(y_2) \\
y_1 \arrow[r, "\gamma"'] & y_2 \arrow[ur, "\bar{t}(y_2)"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
y_1 \arrow[r, "\gamma"] \arrow[dr, phantom, "\bar{t}(\gamma)\ \Swarrow"] & y_2 \arrow[d, "t(y_2)"] \\
p(y_1) \arrow[u, "t(y_1)"] \arrow[r, "p(\gamma)"'] & p(y_2)
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
& y_1 \arrow[r, "\gamma"] \arrow[d, "\bar{t}(y_1)"] \arrow[dr, phantom, "\bar{t}(\gamma)\ \Swarrow"] & y_2 \arrow[d, "t(y_2)"] \\
p(y_1) \arrow[ur, "t(y_1)"] \arrow[r, "\mathrm{Id}"] \arrow[ur, phantom, "\Swarrow", near end] & p(y_1) \arrow[r, "p(\gamma)"'] & p(y_2)
\end{tikzcd}\ ,$$

which, by slight abuse of notation, we still call t and $\bar{t}$.

It is crucial that these are one-sided inverses of each other:

Proposition 2

$$
\begin{tikzcd}
p(y_1) \arrow[rrrr, bend left, "p(\gamma)"] \arrow[rrrr, bend right, "p(\gamma)"'] \arrow[r, "t(y_1)"] & y_1 \arrow[r, "\gamma"] \arrow[r, phantom, shift left=4ex, "\Downarrow t(\gamma)"] \arrow[r, phantom, shift right=4ex, "\Downarrow \bar{t}(\gamma)"] & y_2 \arrow[r, "\bar{t}(y_2)"] & p(y_2)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
p(y_1) \arrow[rr, "p(\gamma)"] & & p(y_2)
\end{tikzcd}
$$

Proof. Use the fact that t and $\bar{t}$ fit into a special ambidextrous adjunction. $\square$

Definition 2 *Let p be such that it contains all disks, and choose a lift for each morphism in $\mathcal{P}_2(X)$. The 2-functor*

$$\tilde{s} : \mathcal{P}_2(X) \to \mathcal{P}_2(p)$$

is defined by the assignment

$$
\begin{tikzcd}
x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow\Sigma"] & y
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
 & (x,j) \arrow[r, "{(\gamma,j)}"] \arrow[dr, phantom, "\Swarrow \bar{t}_j(\gamma)"] & (y,j) \arrow[d, "\bar{t}_j(y)"] & \\
(x,i) \arrow[r, "\bar{t}_i(x)"] & x \arrow[u, "t_j(x)"] \arrow[d, "t_k(x)"'] \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow\Sigma"] \arrow[dr, phantom, "\Swarrow t_k(\gamma')"] & y \arrow[r, "t_l(y)"] & (y,l) \\
 & (x,k) \arrow[r, "{(\gamma',k)}"'] & (y,k) \arrow[u, "\bar{t}_k(y)"'] &
\end{tikzcd},
$$

where (x,i) is the chosen lift of x, (γ,j) is the lift of γ, and so on.

Proposition 3 *$\tilde{s}$ is indeed 2-functorial.*

Proof. Respect for vertical composition is an immediate consequence of prop.

2. Horizontal composition is respected up to the compositor

$$
\begin{tikzcd}
 & (x,r) \arrow[rrrr, "{(\gamma_2\circ\gamma_1,r)}"] \arrow[rrrr, phantom, shift right=8ex, "\Swarrow \bar{t}_r(\gamma_2\circ\gamma_1)"] & & & & (z,r) \arrow[d, "\bar{t}_r(x)"] & \\
(x,i) \arrow[r, "\bar{t}_i(x)"] & x \arrow[u, "t_r(x)"] \arrow[r, "\gamma_1"] \arrow[d, "t_j(x)"'] \arrow[dr, phantom, "\Swarrow t_j(\gamma_1)"] & y \arrow[rr, "\mathrm{Id}"] \arrow[dr, "t_l(y)"] \arrow[rr, phantom, shift right=3ex, "\Downarrow"] & & y \arrow[r, "\gamma_2"] \arrow[dd, "t_n(y)"] \arrow[ddr, phantom, "\Swarrow t_n(\gamma_2)"] & z \arrow[r, "t_o(z)"] & (z,o) \\
 & & & (y,l) \arrow[ur, "\bar{t}_l(y)"] & & & \\
 & (x,j) \arrow[r, "{(\gamma_1',j)}"'] & (y,j) \arrow[uu, "\bar{t}_j(y)"] & & (y,n) \arrow[r, "{(\gamma_2',n)}"'] & (z,n) \arrow[uu, "\bar{t}_n(y)"'] &
\end{tikzcd}
.
$$

Its coherence (associativity) is again a consequence of t being inverse to $\bar{t}$. $\square$

Proposition 4 *$\tilde{s}$ factors through $\mathcal{P}_2(Y^\bullet)$.*

$$\tilde{s}:\ \mathcal{P}_2(X) \xrightarrow{s} \mathcal{P}_2(Y^\bullet) \dashrightarrow \mathcal{P}_2(p)\ .$$

Proof. This is essentially trivial, since we know that $\mathcal{P}_2(X)$ is the sub-2-category of $\mathcal{P}_2(p)$ of all those 2-morphisms whose source and target object do not come from $\mathcal{P}_2(X)$.

But more explicitly, we can write

$$
\begin{tikzcd}
 & (x,j) \arrow[r, "(\gamma,j)"] \arrow[dr, phantom, "\Swarrow \bar{t}_j(\gamma)"] & (y,j) \arrow[d, "\bar{t}_j(y)"] & \\
(x,i) \arrow[r, "\bar{t}_i(x)"] & x \arrow[u, "t_j(x)"] \arrow[d, "t_k(x)"'] \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow \Sigma"] \arrow[dr, phantom, "\Swarrow t_k(\gamma')"] & y \arrow[r, "t_l(y)"] & (y,l) \\
 & (x,k) \arrow[r, "(\gamma',k)"'] & (y,k) \arrow[u, "\bar{t}_k(y)"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & (x,j) \arrow[r, "(\gamma,j)"] \arrow[dr, phantom, "\Swarrow \bar{t}_j(\gamma)"] & (y,j) \arrow[d, "\bar{t}_j(y)"] & \\
(x,i) \arrow[r, "\bar{t}_i(x)"] & x \arrow[u, "t_j(x)"] \arrow[r, "\gamma"] \arrow[d, "t_n(x)"] \arrow[dd, bend right=60, "\mathrm{Id}"'] \arrow[dr, phantom, "\Swarrow t_n(\gamma)"] & y \arrow[d, "t_n(y)"] \arrow[dd, bend left=60, "\mathrm{Id}"] & \\
 & (x,n) \arrow[r, bend left, "(\gamma,n)"] \arrow[r, bend right, "(\gamma',n)"'] \arrow[r, phantom, "\Downarrow (\Sigma,n)"] \arrow[d, "\bar{t}_n(x)"] \arrow[dr, phantom, "\Swarrow \bar{t}_n(\gamma')"] & (y,n) \arrow[d, "\bar{t}_n(x)"] & \\
 & x \arrow[r, "\gamma'"] \arrow[d, "t_k(x)"] \arrow[dr, phantom, "\Swarrow t_k(\gamma')"] & y \arrow[r, "t_l(y)"] & (y,l) \\
 & (x,k) \arrow[r, "(\gamma',k)"'] & (y,k) \arrow[u, "\bar{t}_k(y)"'] &
\end{tikzcd}
$$

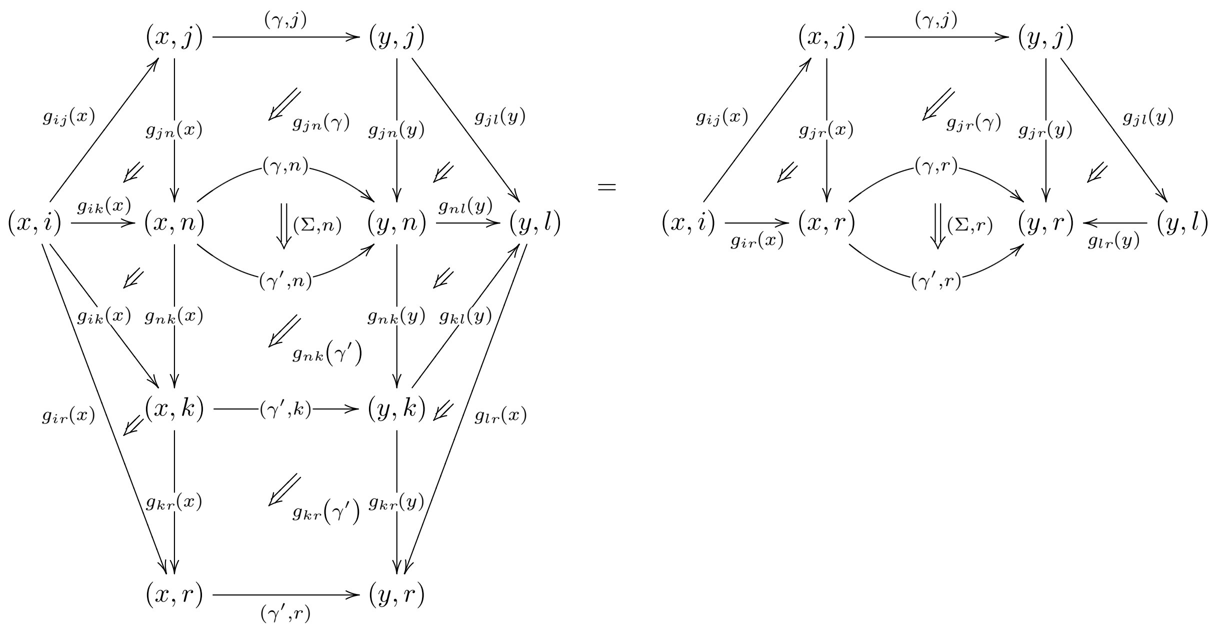
$$
=
\begin{tikzcd}
 & (x,j) \arrow[r, "(\gamma,j)"] \arrow[d, "g_{jn}(x)"] \arrow[dr, phantom, "\Swarrow g_{jn}(\gamma)"] & (y,j) \arrow[d, "g_{jn}(y)"] \arrow[dr, "g_{jl}(y)"] & \\
(x,i) \arrow[ur, "g_{ij}(x)"] \arrow[r, "g_{ik}(x)"] \arrow[dr, "g_{ik}(x)"'] & (x,n) \arrow[r, bend left, "(\gamma,n)"] \arrow[r, bend right, "(\gamma',n)"'] \arrow[r, phantom, "\Downarrow (\Sigma,n)"] \arrow[d, "g_{nk}(x)"] \arrow[dr, phantom, "\Swarrow g_{nk}(\gamma')"] & (y,n) \arrow[r, "g_{nl}(y)"] \arrow[d, "g_{nk}(y)"] & (y,l) \\
 & (x,k) \arrow[r, "(\gamma',k)"'] & (y,k) \arrow[ur, "g_{kl}(y)"'] &
\end{tikzcd}
,
$$

making the 2-morphism in $\mathcal{P}_2(Y^\bullet)$ manifest. $\square$

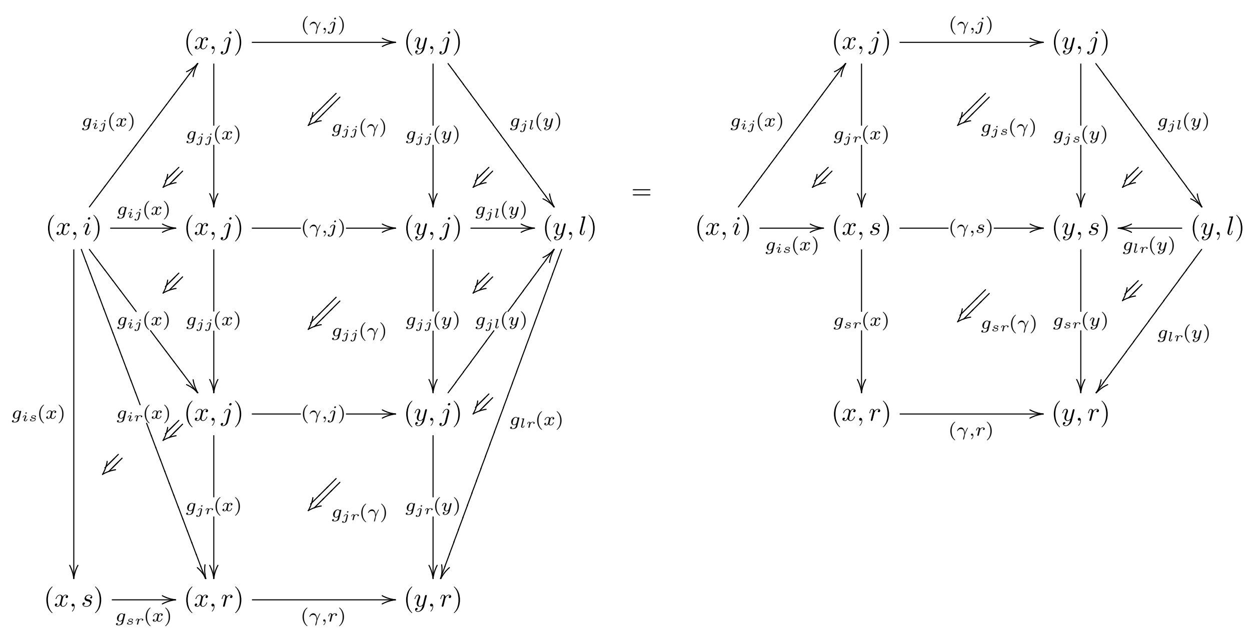
It remains to check that composing s with the canonical morphism

$$\mathcal{P}_2(Y^\bullet) \to \mathcal{P}_2(X)$$

is equivalent to the identity morphism. This is straightforward, but slightly tedious. There is an obvious choice for the pseudonatural isomorphisms one needs and all tin can equations involved, like



and



are seen to hold by repeatedly using the triangle and tetrahedron relation in $\mathcal{P}_2(Y^\bullet)$.