universal transition of transport

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Abstract

To any morphism $p : \mathcal{P}_2(Y) \to \mathcal{P}_2(X)$ of domains of 2-transport, we may associate a universal p-local transition. The associated object turns out to be the category of 2-paths in the 2-groupoid $Y^\bullet$. The associated factorization morphism turns out to be the descending trivialized transport.

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Universal local trivialization. Fix a morphism of transport domains

$$\mathcal{P}_2(Y) \xrightarrow{p} \mathcal{P}(X)\ .$$

We would like to understand in which sense there can be a *universal p-local trivialization*, such that every p-local trivialization factors through this universal one.

Definition 1 *For a given morphism of transport domains*

$$\mathcal{P}_2(Y) \xrightarrow{p} \mathcal{P}(X)$$

we denote the corresponding weak 2-colimit by

$$\mathcal{P}_2(p) \equiv \mathcal{P}_2(Y) \oplus_p \mathcal{P}_2(X)\ .$$

More precisely, $\mathcal{P}_2(p)$ comes with an adjoint equivalence

$$\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\overset{t}{\Swarrow}\sim"] & \mathcal{P}_2(X) \arrow[d] \\
\mathcal{P}_2(Y) \arrow[r] & \mathcal{P}_2(p)
\end{tikzcd}$$

such that for any other

$$\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\overset{f}{\Swarrow}\sim"] & \mathcal{P}_2(X) \arrow[d] \\
\mathcal{P}_2(Y) \arrow[r] & Q
\end{tikzcd}$$

we have

$$\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\overset{f}{\Swarrow}\sim"] & \mathcal{P}_2(X) \arrow[ddr] & \\
\mathcal{P}_2(Y) \arrow[drr] & & \\
& & Q
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\overset{t}{\Swarrow}\sim"] & \mathcal{P}_2(X) \arrow[d] \arrow[ddr] & \\
\mathcal{P}_2(Y) \arrow[r] \arrow[drr] & \mathcal{P}_2(p) \arrow[dr, dashed] \arrow[r, phantom, "="] \arrow[d, phantom, "="] & {} \\
& {} & Q
\end{tikzcd}
\ .$$

Proposition 1 *$\mathcal{P}_2(p)$ is given in terms of generators and relations as follows. The generators are the 2-morphisms of $\mathcal{P}_2(Y)$ and those of $\mathcal{P}_2(X)$, together with further morphism*

$$
\begin{tikzcd}
p(y_1) \arrow[r, "p(\gamma)"] \arrow[d, "t(y_1)"'] \arrow[dr, phantom, "\swarrow t(\gamma)"] & p(y_2) \arrow[d, "t(y_2)"] \\
y_1 \arrow[r, "\gamma"'] & y_2
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
y_1 \arrow[r, "\gamma"] \arrow[d, "\bar{t}(y_1)"'] \arrow[dr, phantom, "\swarrow t(\gamma)"] & y_2 \arrow[d, "\bar{t}(y_2)"] \\
p(y_1) \arrow[r, "p(\gamma)"'] & p(y_2)
\end{tikzcd}
$$

as well as

$$
\begin{tikzcd}
p(y) \arrow[r, "t(y)"] \arrow[rr, bend right, "\mathrm{Id}"'] & y \arrow[r, "\bar{t}(y)"] \arrow[d, phantom, "\Downarrow"] & p(y) \\
 & {} &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
y \arrow[r, "\bar{t}(y)"] \arrow[rr, bend right, "\mathrm{Id}"'] & p(y) \arrow[r, "t(y)"] \arrow[d, phantom, "\Downarrow"] & y \\
 & {} &
\end{tikzcd} ,
$$

subject to the relations which make t an adjoint equivalence.

For instance for every 2-morphism

$$
\begin{tikzcd}
y_1 \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow S"] & y_2
\end{tikzcd}
$$

in $\mathcal{P}_2(Y)$ we have a relation

$$\begin{tikzcd}
p(y_1) \arrow[r, "p(\gamma')"] \arrow[r, bend left=50, "p(\gamma)"] \arrow[r, phantom, bend left=25, "\Downarrow p(S)"] \arrow[d, "t(y_1)"'] \arrow[dr, phantom, "\swarrow t(\gamma')"] & p(y_2) \arrow[d, "t(y_y)"] \\
y_1 \arrow[r, "\gamma'"'] & y_2
\end{tikzcd}
=
\begin{tikzcd}
p(y_1) \arrow[r, "p(\gamma)"] \arrow[d, "t(y_1)"'] \arrow[dr, phantom, "\swarrow t(\gamma)"] & p(y_2) \arrow[d, "t(y_y)"] \\
y_1 \arrow[r, "\gamma"] \arrow[r, bend right=50, "\gamma'"'] \arrow[r, phantom, bend right=25, "\Downarrow S"] & y_2
\end{tikzcd}.$$

Proof. Given

$$\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\overset{f}{\swarrow}\sim"] & \mathcal{P}_2(X) \arrow[ddr, "a"] & \\
\mathcal{P}_2(Y) \arrow[drr, "b"'] & & \\
& & Q
\end{tikzcd}$$

we define

$$\begin{tikzcd}
\mathcal{P}_2 \arrow[dr, dashed, "{(a,b,f)}"] & \\
& Q
\end{tikzcd}$$

by using a on the generators $\mathcal{P}_2(X)$, b on the generators $\mathcal{P}_2(Y)$, and f on the generators involving t.

The adjoint equivalence obtained by the horizontal composition

$$\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\overset{t}{\swarrow}\sim"] & \mathcal{P}_2(X) \arrow[d] & \\
\mathcal{P}_2(Y) \arrow[r] & \mathcal{P}_2 \arrow[dr, dashed, "{(a,b,f)}"] & \\
& & Q
\end{tikzcd}$$

is obtained by applying $\mathcal{P}_2 \overset{(a,b,f)}{\dashrightarrow} Q$ to all the diagrams defining t. By construction, the result if f. For instance

$$\left(y_1 \xrightarrow{\gamma} y_2 \right) \overset{t}{\mapsto} \begin{tikzcd} p(y_1) \arrow[r, "p(\gamma)"] \arrow[d, "t(y_1)"'] \arrow[dr, phantom, "t(\gamma)\swArrow"] & p(y_2) \arrow[d, "t(y_2)"] \\ y_1 \arrow[r, "\gamma"'] & y_2 \end{tikzcd} \overset{(a,b,f)}{\mapsto} \begin{tikzcd} a(p(y_1)) \arrow[r, "a(p(\gamma))"] \arrow[d, "f(y_1)"'] \arrow[dr, phantom, "f(\gamma)\swArrow"] & a(p(y_2)) \arrow[d, "f(y_2)"] \\ b(y_1) \arrow[r, "b(\gamma)"'] & b(y_2) \end{tikzcd} .$$

□

Example 1

Using the universal p-local trivialization, we may now factor any p-local i-trivialization

$$\begin{tikzcd} \mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{tra}_Y"'] \arrow[dr, phantom, "t \swArrow \sim"] & \mathcal{P}_2(X) \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd}$$

of a transport $\mathrm{tra} : \mathcal{P}_2(X) \to T$ as

$$\begin{tikzcd} \mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "t \swArrow \sim"] & \mathcal{P}_2(X) \arrow[ddr, "\mathrm{tra}"] & \\ \mathcal{P}_2(Y) \arrow[drr, "i_*\mathrm{tra}_Y"'] & & \\ & & T \end{tikzcd} = \begin{tikzcd} \mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\swArrow \sim"] & \mathcal{P}_2(X) \arrow[d] & \\ \mathcal{P}_2(Y) \arrow[r] & \mathcal{P}_2(p) \arrow[dr, dashed, "(\mathrm{tra}, i_*\mathrm{tra}_Y, t)"] & \\ & & T \end{tikzcd} .$$

Notice that the functor

$$(\mathrm{tra}, i_*\mathrm{tra}_Y, t) : \ \mathcal{P}_2(p) \dashrightarrow T$$

is built precisely from the data of an object in the category $\mathrm{Triv}_{i,p}$ of p-local i-trivializations.

Universal local transition. By similar reasoning, we may find the *univsersal* p*-local transition.*

Notice that to the universal p-local trivialization

$$
\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[r, "p"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\overset{t}{\swarrow}\sim"] & \mathcal{P}_2(X) \arrow[d] \\
\mathcal{P}_2(Y) \arrow[r] & \mathcal{P}_2(p)
\end{tikzcd}
,
$$

like to any other local trivialization, we may associate the corresponding transition:

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{P}_2(Y) \arrow[dr] & \\
\mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{P}_2(Y) \arrow[ur, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] & \Downarrow g & \mathcal{P}_2(p) \\
 & & \mathcal{P}_2(Y) \arrow[ur] &
\end{tikzcd}
\overset{\phi}{\equiv}
\begin{tikzcd}[column sep=small]
 & & \mathcal{P}_2(Y) \arrow[drr] \arrow[d, phantom, "\Downarrow \bar{t}"] & & \\
\mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{P}_2(Y) \arrow[ur, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] \arrow[r, "p"] & \mathcal{P}_2(X) \arrow[rr] \arrow[d, phantom, "\Downarrow t"] & & \mathcal{P}_2(p) \\
 & & \mathcal{P}_2(Y) \arrow[urr] & &
\end{tikzcd}
,
$$

giving rise to a transition triangle and a tetrahedron law, as usual. But $\mathcal{P}_2(p)$ will not be the universal object for these transitions. This motivates

Definition 2 *Denote by* $\mathcal{P}_2(Y^\bullet)$ *the object sitting in a diagram*

$$
\begin{tikzcd}
\mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{v}{\swarrow}\sim"] & \mathcal{P}_2(Y) \arrow[d] \\
\mathcal{P}_2(Y) \arrow[r] & \mathcal{P}_2(Y^\bullet)
\end{tikzcd}
$$

together with a morphism

$$
\begin{tikzcd}
 & p_2 \arrow[dr, "p_{23}^* v"] \arrow[d, phantom, "w\Downarrow"] & \\
p_1 \arrow[ur, "p_{12}^* v"] \arrow[rr, "p_{13}^* v"'] & & p_3
\end{tikzcd}
$$

satisfying a tetrahedron law, such that $\mathcal{P}_2\left(Y^\bullet\right)$ is universal in the sense that for any other

$$\begin{tikzcd}
\mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "f \swArrow \sim"] & \mathcal{P}_2\left(Y\right) \arrow[d] \\
\mathcal{P}_2\left(Y\right) \arrow[r] & Q
\end{tikzcd}$$

and a corresponding triangle making a tetrahedron 2-commute, we have

$$\begin{tikzcd}
\mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "f \swArrow \sim"] & \mathcal{P}_2\left(Y\right) \arrow[dd] \\
\mathcal{P}_2\left(Y\right) \arrow[dr] & \\
& Q
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "v \swArrow \sim"] & \mathcal{P}_2\left(Y\right) \arrow[d] \arrow[ddr] & \\
\mathcal{P}_2\left(Y\right) \arrow[r] \arrow[ddrr, phantom, ""] \arrow[drr] & \mathcal{P}_2\left(Y^\bullet\right) \arrow[dr, dashed] & \\
& & Q
\end{tikzcd}
.$$

Proposition 2 *$\mathcal{P}_2\left(Y^\bullet\right)$ is given in terms of generators and relations as follows. The generators are the 2-morphisms of $\mathcal{P}_2\left(Y\right)$, together with further morphisms*

$$\begin{tikzcd}
p_1^*x \arrow[r, "p_1^*\gamma"] \arrow[d] \arrow[dr, phantom, "\swArrow"] & p_2^*y \arrow[d] \\
p_2^*x \arrow[r, "p_2^*\gamma"'] & p_2^*y
\end{tikzcd}$$

and

$$\begin{tikzcd}
& p_2^*x \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\
p_1^*x \arrow[ur] \arrow[rr] & {} & p_3^*x
\end{tikzcd}$$

subject to the relations familiar from 2-transitions.

Proof. The construction is entirely analogous to that in the proof of prop. 1. $\square$

Remark. $\mathcal{P}_2(Y^\bullet)$ is nothing but the 2-category of 2-paths within the 2-groupoid

$$Y^\bullet = \begin{tikzcd} Y^{[3]} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Y^{[2]} \arrow[r, shift left] \arrow[r, shift right] & Y \end{tikzcd} .$$

Morphism here are free combinations of pieces of path in Y combined with jumps between parts of Y with the same projection p to X.

Example 2 (factorization of transition)

Given any p-local i-transition of 2-transport

$$\begin{tikzcd} & & T' \arrow[dr, "i"] & \\ \mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{P}_2(Y) \arrow[ur, "\mathrm{tra}_Y"] \arrow[dr, "\mathrm{tra}_Y"'] \arrow[rr, phantom, "\Downarrow g"] & & T \\ & & T' \arrow[ur, "i"'] & \end{tikzcd}$$

with corresponding transition triangle

$$\begin{tikzcd} & p_2^*\mathrm{tra}_Y \arrow[dr, "p_{23}^*"] \arrow[d, phantom, "\Downarrow f"] & \\ p_1^*\mathrm{tra}_Y \arrow[ur, "p_{12}^* g"] \arrow[rr, "p_{13}^*"'] & {} & p_3^*\mathrm{tra}_Y \end{tikzcd}$$

we may factor it through the universal transition as

$$\begin{tikzcd} \mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathcal{P}_2(Y) \arrow[ddr, "i_*\mathrm{tra}_Y"] \arrow[dl, phantom, "\swarrow^{g}\sim"] & \\ \mathcal{P}_2(Y) \arrow[drr, "i_*\mathrm{tra}_Y"'] & & \\ & & T \end{tikzcd} = \begin{tikzcd} \mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\swarrow^{v}\sim"] & \mathcal{P}_2(Y) \arrow[d] & \\ \mathcal{P}_2(Y) \arrow[r] & \mathcal{P}_2(Y^\bullet) \arrow[dr, dashed, "(\mathrm{tra}_Y, g, f)"] & \\ & & T \end{tikzcd} .$$

Notice that the functor

$$(\mathrm{tra}_Y, g, f) : \mathcal{P}_2(Y^\bullet) \dashrightarrow T$$

is built precisely from the data of an object of the category $\mathrm{Trans}_{i,p}$ of p-local i-transitions.

Example 3

We may factor the transition obtained from the universal local trivialization through the universal transition:

$$
\begin{tikzcd}
 & & \mathcal{P}_2(Y) \arrow[dr] \arrow[d, phantom, "\Downarrow\bar{t}"] & \\
\mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{P}_2(Y) \arrow[ur, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] \arrow[r, "p"] & \mathcal{P}_2(X) \arrow[r] \arrow[d, phantom, "\Downarrow t"] & \mathcal{P}_2(p) \\
 & & \mathcal{P}_2(Y) \arrow[ur] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{P}_2\left(Y^{[2]}\right) \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\swarrow v\,\sim"] & \mathcal{P}_2(Y) \arrow[d] & \\
\mathcal{P}_2(Y) \arrow[r] & \mathcal{P}_2(Y^\bullet) \arrow[dr, dashed] & \\
 & & \mathcal{P}_2(p)
\end{tikzcd}
\; .
$$

Now the morphism

$$
\mathcal{P}_2(Y^\bullet) \dashrightarrow \mathcal{P}_2(p)
$$

embeds $\mathcal{P}_2(Y^\bullet)$ as a sub-2-category into $\mathcal{P}_2(X)$, namely that which is obtained from $\mathcal{P}_2(p)$ by removing all generators coming from $\mathcal{P}_2(X)$.

Iterated trivializations. In example 2 we associate to any p-local i-trivialization of 2-transport

$$
\mathrm{tra} : \mathcal{P}_2(X) \to T
$$

the 2-functor

$$
(\mathrm{tra}_Y, g, f) : \mathcal{P}_2(Y^\bullet) \to T\,.
$$

The latter has, by construction, the special property that it becomes i-trivial when pulled back along

$$
\mathcal{P}_2(Y) \to \mathcal{P}_2(Y^\bullet)\;.
$$

It need not, however, in general be i-trivial itself. But it may be i-trivializable

$$
\begin{tikzcd}
\mathcal{P}_2(Y^\bullet) \arrow[r, "\mathrm{Id}"] \arrow[d, "(\mathrm{tra}_Y, g, f)_Y"'] \arrow[dr, phantom, "\swarrow\sim"] & \mathcal{P}_2(Y^\bullet) \arrow[d, "(\mathrm{tra}_Y, g, f)"] \\
T' \arrow[r, "i"'] & t
\end{tikzcd}
\; .
$$

In this case, we might want to address $(\mathcal{P}_2(Y), g, f)_Y$ as a **full p-local i-trivialization**.

Example 4 (from line bundle gerbes to Deligne cocycles)

For i being the obvious injection

$$i : \Sigma(U(1) \to) \to \Sigma(1D\mathrm{Vect}_{\mathbb{C}}) \; ,$$

a p-local i-trivialization of some

$$\mathrm{tra} : \mathcal{P}_2(X) \to \Sigma(1D\mathrm{Vect}_{\mathbb{C}})$$

is a line bundle gerbe with connection. The fact that the corresponding (tra_Y, g, f) is not itself i-trivial reflects the fact that the line bundle gerbe consists of transition bundles instead of transition functions.

For sufficiently well-behaved Y (for instance for any good covering of X) the second trivialization step, leading to the full i-trivilization, amounts to passing to Deligne cocycles $(\mathrm{tra}_Y, g, f)_Y$ for the line bundle gerbe (tra_Y, g, f).