

universal transition of transport

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October 9, 2006

Abstract

To any morphism $p : \mathcal{P}_2(Y) \rightarrow \mathcal{P}_2(X)$ of domains of 2-transport, we may associate a universal p -local transition. The associated object turns out to be the category of 2-paths in the 2-groupoid Y^\bullet . The associated factorization morphism turns out to be the descending trivialized transport.

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Universal local trivialization. Fix a morphism of transport domains

$$\mathcal{P}_2(Y) \xrightarrow{p} \mathcal{P}(X) .$$

We would like to understand in which sense there can be a *universal p-local trivialization*, such that every p-local trivialization factors through this universal one.

Definition 1 For a given morphism of transport domains

$$\mathcal{P}_2(Y) \xrightarrow{p} \mathcal{P}(X)$$

we denote the corresponding weak 2-colimit by

$$\mathcal{P}_2(p) \equiv \mathcal{P}_2(Y) \oplus_p \mathcal{P}_2(X) .$$

More precisely, $\mathcal{P}_2(p)$ comes with an adjoint equivalence

$$\begin{array}{ccc} \mathcal{P}_2(Y) & \xrightarrow{p} & \mathcal{P}_2(X) \\ \text{Id} \downarrow & \swarrow t \sim & \downarrow \\ \mathcal{P}_2(Y) & \longrightarrow & \mathcal{P}_2(p) \end{array}$$

such that for any other

$$\begin{array}{ccc} \mathcal{P}_2(Y) & \xrightarrow{p} & \mathcal{P}_2(X) \\ \text{Id} \downarrow & \swarrow f \sim & \downarrow \\ \mathcal{P}_2(Y) & \longrightarrow & Q \end{array}$$

we have

$$\begin{array}{ccc} \mathcal{P}_2(Y) & \xrightarrow{p} & \mathcal{P}_2(X) \\ \text{Id} \downarrow & \swarrow f \sim & \searrow \\ \mathcal{P}_2(Y) & & Q \end{array} = \begin{array}{ccc} \mathcal{P}_2(Y) & \xrightarrow{p} & \mathcal{P}_2(X) \\ \text{Id} \downarrow & \swarrow t \sim & \downarrow \\ \mathcal{P}_2(Y) & \longrightarrow & \mathcal{P}_2(p) \\ & \searrow & \swarrow \\ & & Q \end{array} .$$

Proposition 1 $\mathcal{P}_2(p)$ is given in terms of generators and relations as follows. The generators are the 2-morphisms of $\mathcal{P}_2(Y)$ and those of $\mathcal{P}_2(X)$, together with further morphism

$$\begin{array}{ccc}
 p(y_1) & \xrightarrow{p(\gamma)} & p(y_2) \\
 \downarrow t(y_1) & \searrow t(\gamma) & \downarrow t(y_2) \\
 y_1 & \xrightarrow{\gamma} & y_2
 \end{array}$$

and

$$\begin{array}{ccc}
 y_1 & \xrightarrow{\gamma} & y_2 \\
 \downarrow \bar{t}(y_1) & \searrow t(\gamma) & \downarrow \bar{t}(y_2) \\
 p(y_1) & \xrightarrow{p(\gamma)} & p(y_2)
 \end{array}$$

as well as

$$\begin{array}{ccccc}
 p(y) & \xrightarrow{t(y)} & y & \xrightarrow{\bar{t}(y)} & p(y) \\
 & \searrow & \Downarrow & \nearrow & \\
 & & \text{Id} & &
 \end{array}$$

and

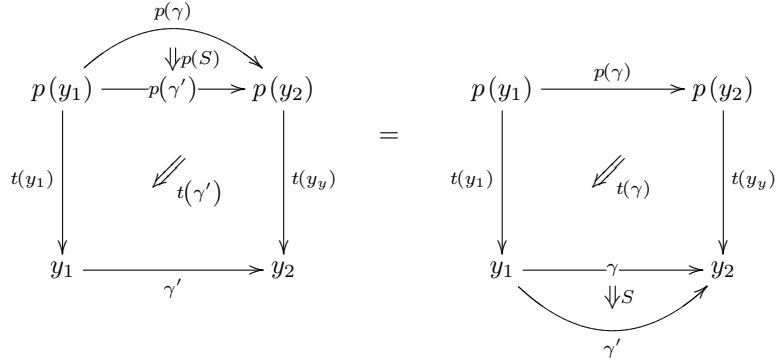
$$\begin{array}{ccccc}
 y & \xrightarrow{\bar{t}(y)} & p(y) & \xrightarrow{t(y)} & y \\
 & \searrow & \Downarrow & \nearrow & \\
 & & \text{Id} & &
 \end{array}$$

subject to the relations which make t an adjoint equivalence.

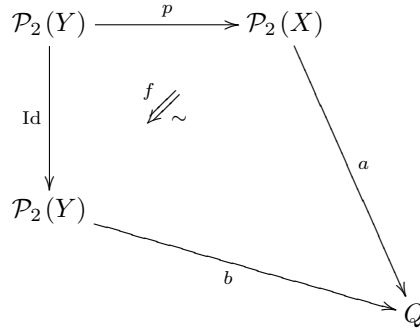
For instance for every 2-morphism

$$\begin{array}{ccc}
 & \gamma & \\
 y_1 & \curvearrowright & y_2 \\
 & \Downarrow S & \\
 & \gamma' &
 \end{array}$$

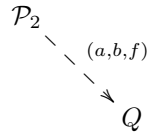
in $\mathcal{P}_2(Y)$ we have a relation



Proof. Given

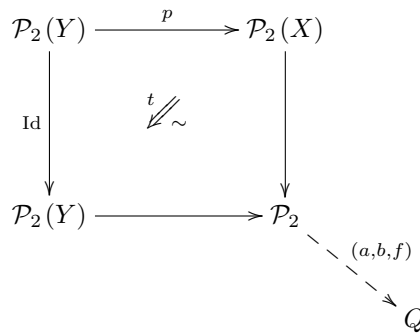


we define



by using a on the generators $\mathcal{P}_2(X)$, b on the generators $\mathcal{P}_2(Y)$, and f on the generators involving t .

The adjoint equivalence obtained by the horizontal composition



is obtained by applying $\mathcal{P}_2 \xrightarrow{(a,b,f)} Q$ to all the diagrams defining t . By construction, the result is f . For instance

$$\begin{array}{ccc}
 (y_1 \xrightarrow{\gamma} y_2) \xrightarrow{t} & \begin{array}{ccc} p(y_1) & \xrightarrow{p(\gamma)} & p(y_2) \\ \downarrow t(y_1) & \searrow t(\gamma) & \downarrow t(y_2) \\ y_1 & \xrightarrow{\gamma} & y_2 \end{array} & \xrightarrow{(a,b,f)} \begin{array}{ccc} a(p(y_1)) & \xrightarrow{a(p(\gamma))} & a(p(y_2)) \\ \downarrow f(y_1) & \searrow f(\gamma) & \downarrow f(y_2) \\ b(y_1) & \xrightarrow{b(\gamma)} & b(y_2) \end{array} .
 \end{array}$$

□

Example 1

Using the universal p -local trivialization, we may now factor any p -local i -trivialization

$$\begin{array}{ccc}
 \mathcal{P}_2(Y) & \xrightarrow{p} & \mathcal{P}_2(X) \\
 \text{tra}_Y \downarrow & \searrow t \sim & \downarrow \text{tra} \\
 T' & \xrightarrow{i} & T
 \end{array}$$

of a transport $\text{tra} : \mathcal{P}_2(X) \rightarrow T$ as

$$\begin{array}{ccc}
 \mathcal{P}_2(Y) & \xrightarrow{p} & \mathcal{P}_2(X) \\
 \text{Id} \downarrow & \searrow t \sim & \downarrow \text{tra} \\
 \mathcal{P}_2(Y) & \xrightarrow{i_* \text{tra}_Y} & T
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{P}_2(Y) & \xrightarrow{p} & \mathcal{P}_2(X) \\
 \text{Id} \downarrow & \searrow \sim & \downarrow \\
 \mathcal{P}_2(Y) & \xrightarrow{\quad} & \mathcal{P}_2(p) \\
 & & \searrow (\text{tra}, i_* \text{tra}_Y, t) \\
 & & T
 \end{array}
 .$$

Notice that the functor

$$(\text{tra}, i_* \text{tra}_Y, t) : \mathcal{P}_2(p) \dashrightarrow T$$

is built precisely from the data of an object in the category $\text{Triv}_{i,p}$ of p -local i -trivializations.

Universal local transition. By similar reasoning, we may find the *universal p-local transition*.

Notice that to the universal p -local trivialization

$$\begin{array}{ccc} \mathcal{P}_2(Y) & \xrightarrow{p} & \mathcal{P}_2(X) \\ \text{Id} \downarrow & \swarrow \sim & \downarrow \\ \mathcal{P}_2(Y) & \longrightarrow & \mathcal{P}_2(p) \end{array},$$

like to any other local trivialization, we may associate the corresponding transition:

$$\begin{array}{ccc} & \mathcal{P}_2(Y) & \\ \nearrow \text{Id} & & \searrow \\ \mathcal{P}_2(Y^{[2]}) \xrightarrow[p_2]{p_1} \mathcal{P}_2(Y) & \Downarrow g & \mathcal{P}_2(p) \\ \searrow \text{Id} & & \nearrow \end{array} \stackrel{\phi}{\cong} \begin{array}{ccccc} & \mathcal{P}_2(Y) & & \mathcal{P}_2(Y) & \\ \nearrow \text{Id} & & \Downarrow \bar{t} & & \searrow \\ \mathcal{P}_2(Y^{[2]}) \xrightarrow[p_2]{p_1} \mathcal{P}_2(Y) & \xrightarrow{-p} & \mathcal{P}_2(X) & \longrightarrow & \mathcal{P}_2(p) \\ \searrow \text{Id} & & \Downarrow t & & \nearrow \\ & \mathcal{P}_2(Y) & & & \end{array},$$

giving rise to a transition triangle and a tetrahedron law, as usual. But $\mathcal{P}_2(p)$ will not be the universal object for these transitions. This motivates

Definition 2 Denote by $\mathcal{P}_2(Y^\bullet)$ the object sitting in a diagram

$$\begin{array}{ccc} \mathcal{P}_2(Y^{[2]}) & \xrightarrow{p_1} & \mathcal{P}_2(Y) \\ p_2 \downarrow & \swarrow \sim & \downarrow \\ \mathcal{P}_2(Y) & \longrightarrow & \mathcal{P}_2(Y^\bullet) \end{array}$$

together with a morphism

$$\begin{array}{ccc} & p_2 & \\ p_{12}^* v \nearrow & & \searrow p_{23}^* v \\ p_1 & \xrightarrow[p_{13}^* v]{} & p_3 \end{array}$$

satisfying a tetrahedron law, such that $\mathcal{P}_2(Y^\bullet)$ is universal in the sense that for any other

$$\begin{array}{ccc}
 \mathcal{P}_2(Y^{[2]}) & \xrightarrow{p_1} & \mathcal{P}_2(Y) \\
 \downarrow p_2 & \searrow f \sim & \downarrow \\
 \mathcal{P}_2(Y) & \longrightarrow & Q
 \end{array}$$

and a corresponding triangle making a tetrahedron 2-commute, we have

$$\begin{array}{ccc}
 \mathcal{P}_2(Y^{[2]}) & \xrightarrow{p_1} & \mathcal{P}_2(Y) \\
 \downarrow p_2 & \searrow f \sim & \searrow \\
 \mathcal{P}_2(Y) & & Q
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{P}_2(Y^{[2]}) & \xrightarrow{p_1} & \mathcal{P}_2(Y) \\
 \downarrow p_2 & \searrow v \sim & \downarrow \\
 \mathcal{P}_2(Y) & \longrightarrow & \mathcal{P}_2(Y^\bullet) \\
 & \searrow & \searrow \\
 & & Q
 \end{array}$$

Proposition 2 $\mathcal{P}_2(Y^\bullet)$ is given in terms of generators and relations as follows. The generators are the 2-morphisms of $\mathcal{P}_2(Y)$, together with further morphisms

$$\begin{array}{ccc}
 p_1^*x & \xrightarrow{p_1^*\gamma} & p_2^*y \\
 \downarrow & \searrow & \downarrow \\
 p_2^*x & \xrightarrow{p_2^*\gamma} & p_2^*y
 \end{array}$$

and

$$\begin{array}{ccc}
 & p_2^*x & \\
 & \uparrow & \downarrow \\
 p_1^*x & \longrightarrow & p_3^*x
 \end{array}$$

subject to the relations familiar from 2-transitions.

Proof. The construction is entirely analogous to that in the proof of prop. 1. \square

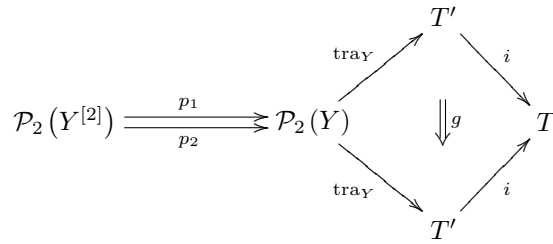
Remark. $\mathcal{P}_2(Y^\bullet)$ is nothing but the 2-category of 2-paths within the 2-groupoid

$$Y^\bullet = Y^{[3]} \rightrightarrows Y^{[2]} \rightrightarrows Y .$$

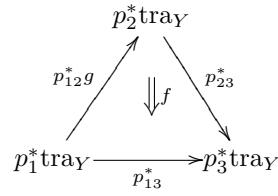
Morphism here are free combinations of pieces of path in Y combined with jumps between parts of Y with the same projection p to X .

Example 2 (factorization of transition)

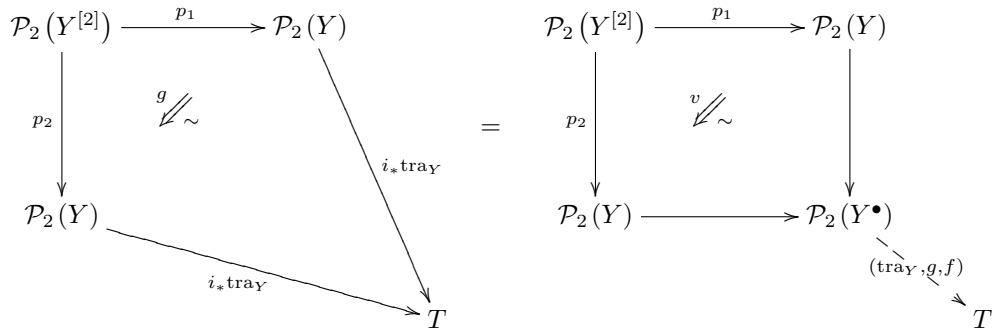
Given any p -local i -transition of 2-transport



with corresponding transition triangle



we may factor it through the universal transition as



Notice that the functor

$$(\text{tray}, g, f) : \mathcal{P}_2(Y^\bullet) \dashrightarrow T$$

is built precisely from the data of an object of the category $\text{Trans}_{i,p}$ of p -local i -transitions.

Example 3

We may factor the transition obtained from the universal local trivialization through the universal transition:

$$\begin{array}{ccc}
 & & \mathcal{P}_2(Y) \\
 & \text{Id} \nearrow & \downarrow \bar{t} \\
 \mathcal{P}_2(Y^{[2]}) \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} \mathcal{P}_2(Y) & \xrightarrow{p} & \mathcal{P}_2(X) \longrightarrow \mathcal{P}_2(p) \\
 & \text{Id} \searrow & \downarrow t \\
 & & \mathcal{P}_2(Y)
 \end{array} = \begin{array}{ccc}
 \mathcal{P}_2(Y^{[2]}) & \xrightarrow{p_1} & \mathcal{P}_2(Y) \\
 \downarrow p_2 & \swarrow v \sim & \downarrow \\
 \mathcal{P}_2(Y) & \longrightarrow & \mathcal{P}_2(Y^\bullet) \\
 & & \dashrightarrow \mathcal{P}_2(p)
 \end{array} .$$

Now the morphism

$$\mathcal{P}_2(Y^\bullet) \dashrightarrow \mathcal{P}_2(p)$$

embeds $\mathcal{P}_2(Y^\bullet)$ as a sub-2-category into $\mathcal{P}_2(X)$, namely that which is obtained from $\mathcal{P}_2(p)$ by removing all generators coming from $\mathcal{P}_2(X)$.

Iterated trivializations. In example 2 we associate to any p -local i -trivialization of 2-transport

$$\text{tra} : \mathcal{P}_2(X) \rightarrow T$$

the 2-functor

$$(\text{tra}_Y, g, f) : \mathcal{P}_2(Y^\bullet) \rightarrow T .$$

The latter has, by construction, the special property that it becomes i -trivial when pulled back along

$$\mathcal{P}_2(Y) \rightarrow \mathcal{P}_2(Y^\bullet) .$$

It need not, however, in general be i -trivial itself. But it may be i -trivializable

$$\begin{array}{ccc}
 \mathcal{P}_2(Y^\bullet) & \xrightarrow{\text{Id}} & \mathcal{P}_2(Y^\bullet) \\
 \downarrow (\text{tra}_Y, g, f)_Y & \swarrow \sim & \downarrow (\text{tra}_Y, g, f) \\
 T' & \xrightarrow{i} & t
 \end{array} .$$

In this case, we might want to address $(\mathcal{P}_2(Y), g, f)_Y$ as a **full p -local i -trivialization**.

Example 4 (from line bundle gerbes to Deligne cocycles)

For i being the obvious injection

$$i : \Sigma(U(1) \rightarrow) \rightarrow \Sigma(1D\text{Vect}_{\mathbb{C}}) ,$$

a p -local i -trivialization of some

$$\text{tra} : \mathcal{P}_2(X) \rightarrow \Sigma(1D\text{Vect}_{\mathbb{C}})$$

is a line bundle gerbe with connection. The fact that the corresponding (tra_Y, g, f) is not itself i -trivial reflects the fact that the line bundle gerbe consists of transition bundles instead of transition functions.

For sufficiently well-behaved Y (for instance for any good covering of X) the second trivialization step, leading to the full i -trivialization, amounts to passing to Deligne cocycles $(\text{tra}_Y, g, f)_Y$ for the line bundle gerbe (tra_Y, g, f) .