Parallel Transport and Quantization

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Abstract

What is the relation between parallel transport over n-dimensional worldvolumes and the associated n-dimensional quantum field theory?

Contents

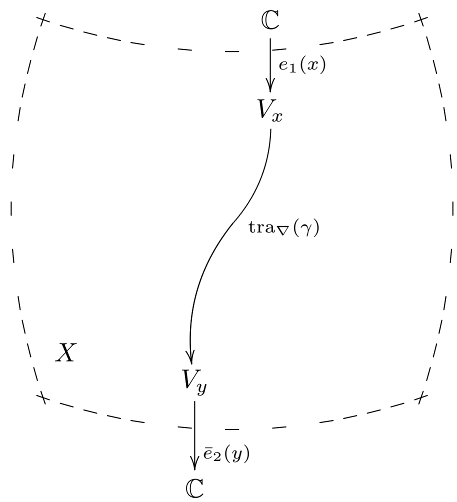
Models of the physics of charged particles are usually formulated in terms of vector bundles

$$V \to X$$

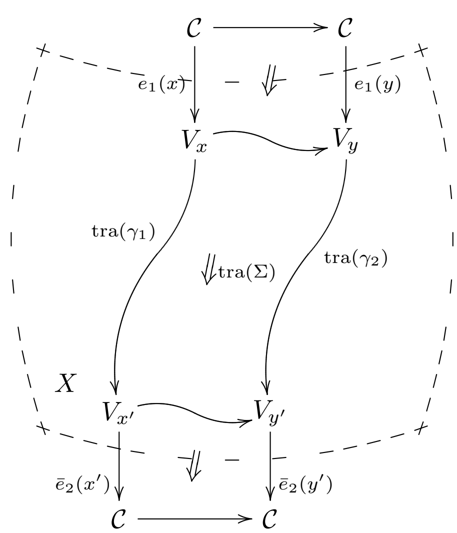
with connection

$$\nabla\,.$$

The part of this formalism most directly connected to what we actually observe in nature is the **parallel transport.**



The concept of parallel transport can be generalized from particles to 2-particles (strings).



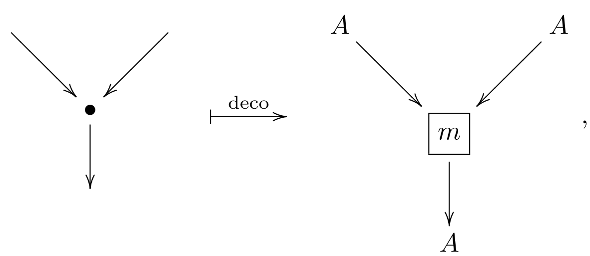
There is a mystery that demands to be understood:

Mystery 1 *The theory of gerbes with connection in terms of local data exhibits a lot of structural resemblance to state sum models of 2-dimensional quantum field theory.*

Why is that?

Does this point to a deeper pattern that we might want to understand?

— two pictures go here, both showing a 2-dimensional cobordism with a dual triangulation drawn on it. In the first case the triangulation is labeled by certain p-form data and describes the surface holonomy of a gerbe with connection. In the second case the triangulation is decorated by a Frobenius algebra, and encodes a correlator in a 2-dimensional quantum field theory. —



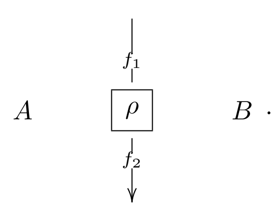
Our **claim** is that all these formulas are special cases of those describing a **locally trivialized 2-transport**.

$$
\begin{tikzcd}
 & \cdot \arrow[dr] & \\
\cdot \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3ex, "\Downarrow\bullet"] & & \cdot
\end{tikzcd}
\quad \overset{\text{deco}}{\longmapsto} \quad
\begin{tikzcd}
 & \mathbb{1} \arrow[dr, "A_{jk}"] & \\
\mathbb{1} \arrow[ur, "A_{ij}"] \arrow[rr, "A_{ik}"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow m_{ijk}"] & & \mathbb{1}
\end{tikzcd}
$$

We simply have to replace globular diagrams

$$
\begin{tikzcd}
A \arrow[r, bend left=50, "f_1"{name=U}] \arrow[r, bend right=50, "f_2"'{name=D}] & B
\arrow[from=U, to=D, Rightarrow, "\rho"]
\end{tikzcd}
$$

by string diagrams



Plan.

- First understand parallel n-transport.
- Then understand its quantization.

1 $n = 1$: The Charged Particle

Consider a particle on X, charged under a vector bunde $V \to X$ with connection ∇ associated to a G-principal bundle $P \to X$.

1.1 The Classical Structure

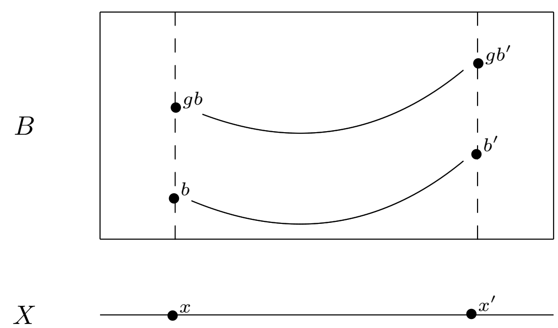
For each path

$$x \xrightarrow{\gamma} x'$$

in base space the connection allows us to find a path

$$b \xrightarrow{\tilde{\gamma}} b'$$

in the total space of the bundle, which is everywhere parallel to γ.



We say that b' is obtained from **parallel transporting** b along γ from the fiber B_x to the fiber $B_{x'}$.

This way a connection assigns, by parallel transport, to each path γ in base space a map

$$\mathrm{tra}(\gamma) : B_x \to B_y$$

between the fibers over the endpoints

This assignment of maps between fibers to paths in base space has some special properties:

- The G-invariance of the choice of horizontal subspaces implies that these maps between the fibers commute with the G-action on the fibers.
- In particular, this implies that these maps are invertible, since G acts freely and transitively on each fiber.
- The map $\mathrm{tra}(\gamma)$ is independent of the parameterization of γ.
- If $\bar{\gamma}$ is obtained from γ by reversing the direction, then $\mathrm{tra}(\bar{\gamma})$ is the inverse of $\mathrm{tra}(\gamma)$.

- If γ is the composition of two paths γ_1 and γ_2, then

$$\mathrm{tra}(\gamma) = \mathrm{tra}(\gamma_2) \circ \mathrm{tra}(\gamma_1)\,.$$

Clearly, all this is trying to tell us that **parallel transport is a functor**

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$$

that sends paths in base space to morphisms of G-torsors.

1.2 The Quantum Structure

kinematics	**dynamics**
vector bundle $V \to X$	connection ∇
space of states	evolution operator
H	$U(t) : H \to H$
objects	morphisms
space of sections	path integral
straightforward	subtle

Table 1: **Quantization** involves a kinematical and a dynamical aspect.

Kinematics. The **space of states** of the charged particle is obtained by summing the parallel transport over all points.

$$\Gamma(V) \simeq \mathrm{colim}_X \mathrm{tra} := \int_X \mathrm{tra}$$

We *push the functor forward to a point.*

Freed emphasized how this is similar to doing the path integral itself. Accordingly, Freed and Hopkins, in their work on Chern-Simons theory, like to write the above as

$$\Gamma(V) = \int_X e^{iS_{\mathrm{objects}}} d\mu\,.$$

Dynamics. The dynamics also comes from a sum, the path integral

$$\psi(y) = \int_{PX} \phi(\gamma(0))\, e^{-iS(\gamma)} D\gamma\,,$$

which, infinitesimally, comes from the Hamiltonian

$$\Delta = -\frac{1}{2m}\nabla^2\,.$$

So quantum propagation is a functor

$$U : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Hilb}$$

which sends

$$U : (\, \bullet \xrightarrow{\ t\ } \bullet \,) \mapsto (\, \Gamma(V) \xrightarrow{\ e^{it\Delta}\ } \Gamma(V) \,)\,.$$

Summary. So quantization has mapped a functor assigning classical phases to embedded paths

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$$

to a functor which sends abstract paths to quantum phases

$$U : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Hilb}\,.$$

2 $n = 2$, The Charged 2-Particle (String)

Raising the dimension: Categorification. Paths form a category:

$$\begin{tikzcd}
x_0 \arrow[r] & x_1 \arrow[dd] & & \\
& & x_4 \arrow[r] & x_5 \arrow[dd] \\
& x_2 \arrow[r] & x_3 \arrow[u] & \\
& & & x_6
\end{tikzcd}$$

2-paths form a 2-category

$$\begin{tikzcd}
x_0 \arrow[r] \arrow[dddd] & x_1 \arrow[dd] & & \\
& & x_4 \arrow[r] & x_5 \arrow[ddd] \\
& x_2 \arrow[r] \arrow[dr, phantom, "\Swarrow"] & x_3 \arrow[u] & \\
& & & \\
y_1 \arrow[rrr] & & & x_6
\end{tikzcd}$$

$$\begin{tikzcd}
x_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & x_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & x_2 \arrow[d] \\
y_0 \arrow[r] & y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & y_2 \arrow[d] \\
& z_1 \arrow[r] & z_2
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x_0 \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Swarrow"] & & x_2 \arrow[dd] \\
y_0 \arrow[r] & y_1 \arrow[d] & \\
& z_1 \arrow[r] & z_2
\end{tikzcd}$$

a group is a category with a single object and all morphisms being invertible

$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \;=\; \bullet \xrightarrow{g_1 g_2} \bullet$$

a **2-group** is a 2-category with a single object, and everything invertible in a suitabel sense:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_1'"'] \arrow[r, phantom, "\Downarrow h_1"] & \bullet \arrow[r, bend left=50, "g_2"] \arrow[r, bend right=50, "g_2'"'] \arrow[r, phantom, "\Downarrow h_2"] & \bullet \end{tikzcd} \;=\; \begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1 \cdot g_2"] \arrow[r, bend right=50, "g_1' \cdot g_2'"'] \arrow[r, phantom, "\Downarrow h_1 \cdot \alpha(g_1)(h_2)"] & \bullet \end{tikzcd}$$

and

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, "g_2" description] \arrow[r, bend right=50, "g_3"'] \arrow[r, phantom, shift left=3, "\Downarrow h_1"] \arrow[r, phantom, shift right=3, "\Downarrow h_2"] & \bullet \end{tikzcd} \;=\; \begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_3"'] \arrow[r, phantom, "\Downarrow h_2 \cdot h_1"] & \bullet \end{tikzcd} \,,$$

2.1 The Classical Structure

Smooth Transport. **Fact.** A smooth functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to \Sigma G$$

comes from the path ordered exponential of a 1-form:

$$\mathrm{tra} : (\; x \xrightarrow{\gamma} y \;) \mapsto (\; \bullet \xrightarrow{\mathrm{P}\exp \int_\gamma A} \bullet \;)\,.$$

Fact. A smooth 2-functor

$$\mathrm{tra} : \mathcal{P}_2(X) \to \Sigma G_2$$

comes similarly from a 1- and a 2-form.

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow_{\Sigma}"] & x_2 \arrow[d] \\ z_0 \arrow[r] & z_2 \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} x_0 \arrow[r, "1+A_{x_0,x_1}"] \arrow[d] \arrow[dr, phantom, "\swarrow 1+B(x_0)"] & x_1 \arrow[r, "1+A_{x_0,x_1}"] \arrow[d] \arrow[dr, phantom, "\swarrow 1+B(x_1)"] & x_2 \arrow[d] \\ y_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow 1+B(y_0)"] & y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow 1+B(y_1)"] & y_2 \arrow[d] \\ z_0 \arrow[r] & z_1 \arrow[r] & z_2 \end{tikzcd}$$

Locally Trivializable smooth Transport We say a 2-functor $\mathrm{tra} : \mathcal{P}_2(X) \to T$ is p-locally i-trivializable if there is a cover

$$p : U \to X$$

and an equivalence

$$\begin{tikzcd} \mathcal{P}_2(U) \arrow[r, "p"] \arrow[d, "\mathrm{tra}_U"'] \arrow[dr, phantom, "\overset{\sim}{\swArrow}_{t}"] & \mathcal{P}_2(X) \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd} .$$

This says that tra locally looks like something that takes values just in T'.

For instance for

$$i : \Sigma(U(n)) \xrightarrow{\rho} \mathrm{Vect}$$

the canonical rep, this describes a vector bundle with connection.

For

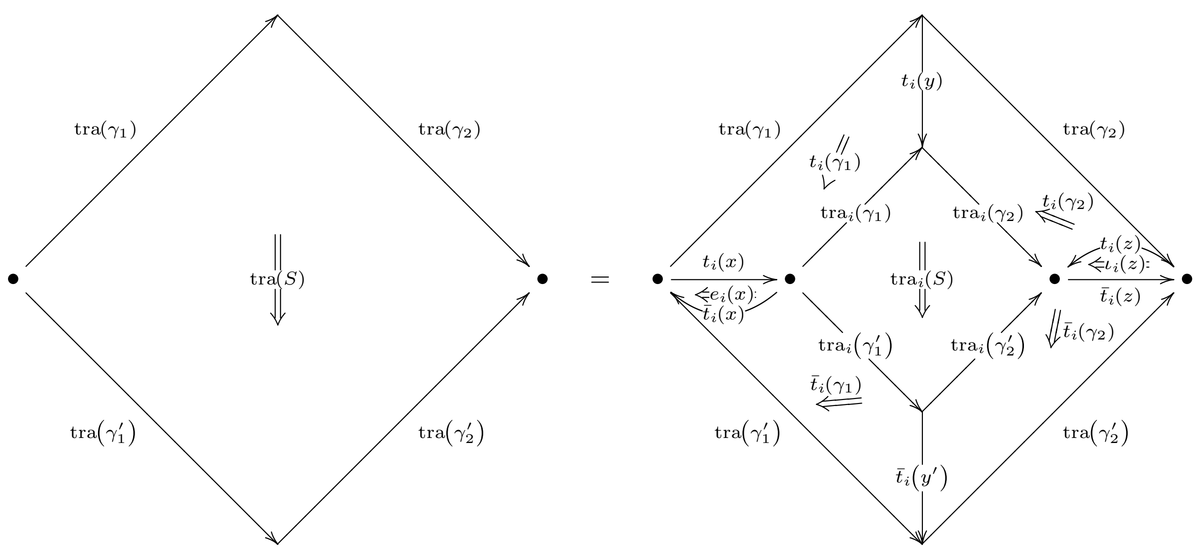
$$i : \Sigma\Sigma U(1) \xrightarrow{\rho} 2\mathrm{Vect}$$

the canonical rep, this describe a line bundle gerbe with connection (and curving).

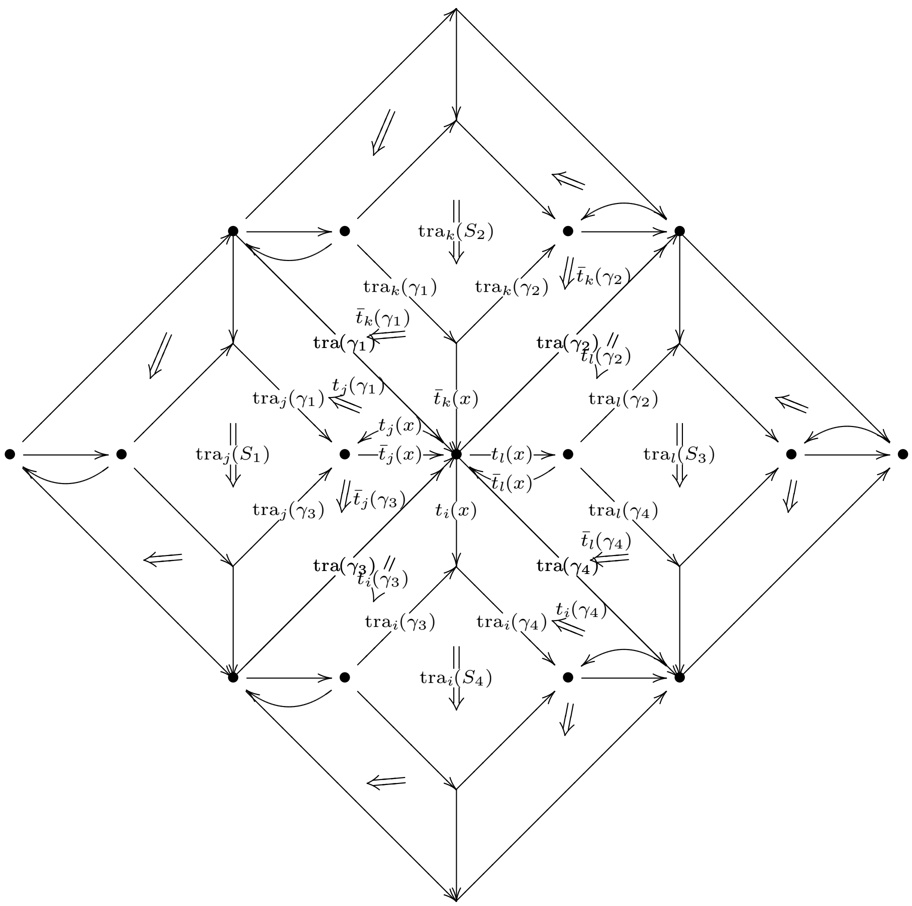
	1-anafunctors		**2-anafunctors**
	$\begin{tikzcd} \mathcal{P}_1(U) \arrow[r, "p"] \arrow[d, "\mathrm{tra}_U"'] \arrow[dr, phantom, "\overset{\sim}{\swarrow}_t"] & \mathcal{P}_1(X) \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd}$	as p-local i-trivialization	$\begin{tikzcd} \mathcal{P}_2(U) \arrow[r, "p"] \arrow[d, "\mathrm{tra}_U"'] \arrow[dr, phantom, "\overset{\sim}{\swarrow}_t"] & \mathcal{P}_2(X) \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd}$
	$\begin{tikzcd} & p_2^*\mathrm{tra}_U \arrow[dr, "p_{23}^*g"] & \\ p_1^*\mathrm{tra}_U \arrow[ur, "p_{12}^*g"] \arrow[rr, "p_{13}^*g"'] & & p_3^*\mathrm{tra}_U \end{tikzcd}$	as descent data/ transition data	$\begin{tikzcd} p_2^*\mathrm{tra}_U \arrow[r, "p_{23}^*g"] & p_3^*\mathrm{tra}_U \arrow[d, "p_{34}^*g"] & & p_2^*\mathrm{tra}_U \arrow[r, "p_{23}^*g"] \arrow[dr, "p_{24}^*g"] & p_3^*\mathrm{tra}_U \arrow[d, "p_{34}^*g"] \arrow[dl, phantom, "\swArrow p_{234}^*f"] \\ p_1^*\mathrm{tra}_U \arrow[u, "p_{12}^*g"] \arrow[ur, "p_{13}^*g"] \arrow[r, "p_{14}^*g"'] \arrow[ur, phantom, "p_{123}^*f\, \seArrow" near start] \arrow[ur, phantom, "\Downarrow p_{134}^*f" near end] & p_4^*\mathrm{tra}_U & = & p_1^*\mathrm{tra}_U \arrow[u, "p_{12}^*g"] \arrow[r, "p_{14}^*g"'] \arrow[ur, phantom, "p_{124}^*f \Downarrow"] & p_4^*\mathrm{tra}_U \end{tikzcd}$
	$\begin{tikzcd} \mathcal{P}_1(U^\bullet) \arrow[r, "(\mathrm{tra}_U, g)"] \arrow[d, "p"'] & T' \\ \mathcal{P}_1(X) & \end{tikzcd}$	as spans	$\begin{tikzcd} \mathcal{P}_2(U^\bullet) \arrow[r, "(\mathrm{tra}_U, g, f)"] \arrow[d, "p"'] & T' \\ \mathcal{P}_2(X) & \end{tikzcd}$

Table 2: We generalize 1-anafunctors to **2-anafunctors** by regarding an anafunctor as an instance of **descent data** or **transition data**.

Local Data on Triangulations This local trivialization is what gives rise to decorated dual triangulations.



The global transport is locally expressed in terms of the i-trivial transport, surrounded by trivialization morphisms.



Along regions where two different local trivializations meet, these trivializations morphisms combine into the transition morphisms that decorate a dual triangulation.

2.2 The Quantum Structure

We formulate the proceudre of **quantization by push-forward** in a way that generalizes to the n-categorical setup.

A charged n-particle...

$$\left(\mathrm{par} \xrightarrow{\gamma \in \mathrm{conf}} \mathrm{tar} \xrightarrow{\mathrm{tra}} \mathrm{phas} \right)$$

... comes with
a configuration space of maps
from its parameter space
into its target space...

$$\begin{tikzcd}
 & & \mathrm{conf} \\
\mathrm{tar} & \mathrm{conf} \times \mathrm{par} \arrow[l, "\mathrm{ev}"'] \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\
 & & \mathrm{par}
\end{tikzcd}$$

... and a coupling to
a transport functor
on target space...

$$\begin{tikzcd}
 & & {[\mathrm{conf}, \mathrm{phas}]} \arrow[dl, "p_1^*"'] \\
{[\mathrm{tar}, \mathrm{phas}]} \arrow[r, "\mathrm{ev}^*"] & {[\mathrm{conf} \times \mathrm{par}, \mathrm{phas}]} & \\
 & & {[\mathrm{par}, \mathrm{phas}]} \arrow[ul, "p_2^*"]
\end{tikzcd}$$

...which induces transport functors
on configuration space
and on parameter space...

$$\begin{tikzcd}
 & & {[\mathrm{conf}, \mathrm{phas}]} \\
{[\mathrm{tar}, \mathrm{phas}]} \arrow[r, "\mathrm{ev}^*"] \arrow[urr, bend left, "\mathrm{t}"] \arrow[drr, bend right, "\mathrm{q}"'] & {[\mathrm{conf} \times \mathrm{par}, \mathrm{phas}]} \arrow[ur, "\bar{p}_1^*"] \arrow[dr, "\bar{p}_2^*"'] & \\
 & & {[\mathrm{par}, \mathrm{phas}]}
\end{tikzcd}$$

...that are known as the
transgression
and the quantization
of the n-particle.

$$\begin{tikzcd}
 & t(\mathrm{tra}) \\
\mathrm{tra} \arrow[ur, mapsto, bend left, "\text{transgression}"] \arrow[dr, mapsto, bend right, "\text{quantization}"'] & \\
 & q(\mathrm{tra})
\end{tikzcd}$$

Table 3: **The story of the charged n-particle.** A drama in three acts.

Example. For instance: take a string coupled to a gerbe and feed it into this machinery. The quantization step by push-forward produces the existence of gerbe modules coupled to the endpoints of the string.

What is such a morphism like? Being a morphism of 2-functors, it is a pseudonatural transformation. This means e is determined by an assignment

$$e : (\,(x,i) \longrightarrow (x,j)\,) \quad \mapsto \quad \begin{tikzcd} \mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[d, "e_i(x)"'] \arrow[dr, phantom, "\swarrow e_{ij}(x)"] & \mathbb{C} \arrow[d, "e_j(x)"] \\ \mathbb{C} \arrow[r, "\mathbb{C}"'] & \mathbb{C} \end{tikzcd} ,$$

for each point x in a double intersection of the cover, where $e_i(x)$ and $e_j(x)$ are $\mathbb{C}$-bimodules, hence vector spaces, and where $e_{ij}(x) : e_j(x) \to e_i(x)$ is a morphism of $\mathbb{C}$-bimoudles, hence a linear map.

The consistency condition this assignment has to satisfy is

$$\begin{tikzcd} & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[d, phantom, "\Downarrow f_{ijk}(x)"] & \\ \mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"] \arrow[d, "e_i(x)"'] \arrow[drr, phantom, "\swarrow e_{ik}(x)"] & {} & \mathbb{C} \arrow[d, "e_k(x)"] \\ \mathbb{C} \arrow[rr, "\mathbb{C}"'] & & \mathbb{C} \end{tikzcd} \quad = \quad \begin{tikzcd} & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[dd, "e_j(x)"] & \\ \mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[dd, "e_i(x)"'] \arrow[r, phantom, "\swarrow e_{ij}(x)"] & {} \arrow[r, phantom, "\swarrow e_{jk}(x)"] & \mathbb{C} \arrow[dd, "e_k(x)"] \\ & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & \\ \mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"'] & {} & \mathbb{C} \end{tikzcd}$$

for all x in triple overlaps of the cover.

If you like formulas better, think of this equivalently as saying that

$$e_{ij} \circ e_{jk} = f_{ijk} e_{ik} \,.$$

It follows that the section e of our line-2-bundle is, over the endpoints of the open string, much like an ordinary vector bundle, but one whose transition cocycle involves a certain "twist" which is measured by the cocycle data of the line-2-bundle.

Such structures are equivalently known as

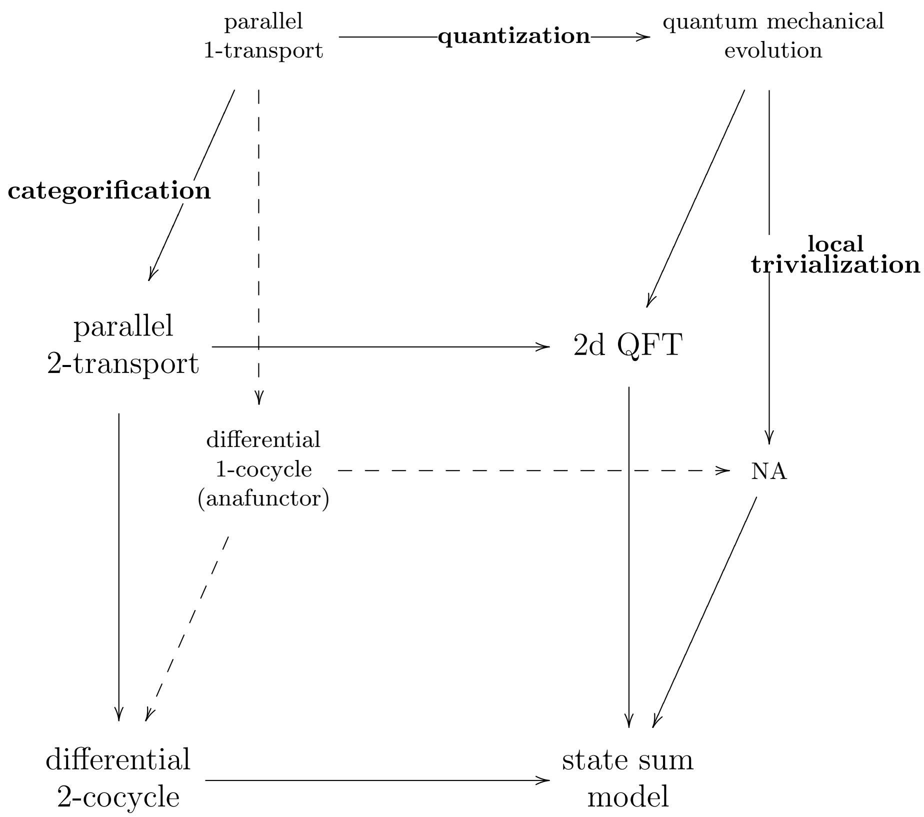
- twisted vector bundles
- gerbe modules
- twisted representations of $U^{[2]}$

- D-branes with Chan-Paton bundles .

In conclusion, we find that

Proposition 1 *A section of a line-2-bundle ($\simeq$ line bundle gerbe) with respect to the open string $\{a \to b\}$ is a D-brane over a, another D-brane over b together with a morphism of D-branes over $a \to b$.*

3 What did we find?



	classical data		quantum theory
	background field	**n-particle**	
name of **n-functor**	parallel transport	action	quantum propagation
image of **n-functor**	monodromy	classical phases	quantum amplitudes
	with values in $\mathrm{phas} = n\mathrm{Vect}$		
domain	on target space tar	on configuration space $\mathrm{conf} \subset [\mathrm{par}, \mathrm{tar}]$	on parameter space par
in **symbols**	$\mathrm{tra} : \mathrm{tar} \to \mathrm{phas}$	$\mathrm{tra}_* : \mathrm{conf} \to [\mathrm{par}, \mathrm{phas}]$	$q(\mathrm{tra}) : \mathrm{par} \to \mathrm{phas}$
operation **in physics terms**	coupling	quantization	
correspondence	$\mathrm{tar} \xleftarrow{\mathrm{ev}} \mathrm{conf} \times \mathrm{par} \xrightarrow{p} \mathrm{par}$		
operation **in symbols**	$[\mathrm{tar}, \mathrm{phas}] \xrightarrow{\mathrm{ev}^*} [\mathrm{conf} \times \mathrm{par}, \mathrm{phas}] \xrightarrow{\bar{p}^*} [\mathrm{par}, \mathrm{phas}]$ coupling, quantization $\mathrm{tar} \longmapsto \mathrm{ev}^*\mathrm{tar} \longmapsto \bar{p}^*\mathrm{ev}^*\mathrm{tar}$		
elements	flat sections $e : 1 \to \mathrm{tra}$ in $\Gamma(\mathrm{tra}) = \mathrm{Hom}(1, \mathrm{tra})$	states $\psi : 1_\bullet \to q(\mathrm{tra})$ in $\mathrm{Hom}(1_*, \mathrm{tra}_*) \simeq \mathrm{Hom}(1_\bullet, q(\mathrm{tra}))$	
pairing of **elements**	holonomy		correlator

Table 4: **The charged n-particle and its quantization**. The process begins with a parallel transport n-functor tra for an n-bundle with connection, modelling a physical background field. It continues by specifying certain maps into the domain of the parallel transport and transgressing tra to the configuration space of all these maps. This models the coupling of the background field to a charged n-particle (a point particle, a string, a membrane, etc.). Finally, the transgressed n-functor may be pushed forward to a point. This yields the quantum theory of the charged n-particle coupled to the given background field.

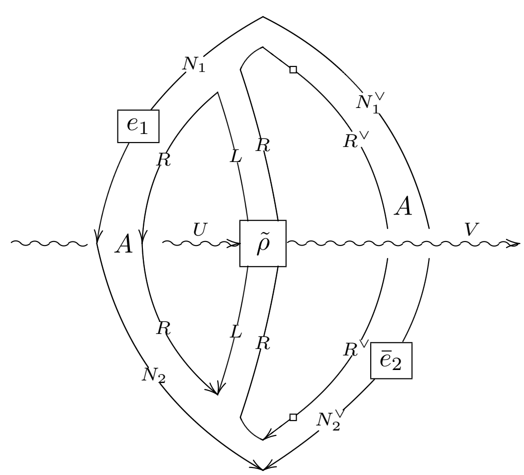
4 Application to 2d CFT

From locally trivializing 2-vector transport over the disk, we get decorations of the form appearing in the FFRS construction of RCFT.

$$
\begin{tikzcd}[row sep=large, column sep=huge]
\mathbb{1} \arrow[r, "U"] \arrow[d, "L"'] \arrow[dddddd, bend right=90, "\mathbb{1}"'] \arrow[dddddd, phantom, bend right=50, "\overset{e}{\Leftarrow}"] \arrow[dr, phantom, "\Swarrow +"] & \mathbb{1} \arrow[d, "L"] \arrow[dddddd, bend left=90, "\mathbb{1}"] \arrow[dddddd, phantom, bend left=50, "\overset{\tilde{i}}{\Leftarrow}"] \\
A \arrow[r, "A\otimes^+ U"] \arrow[d, "R"] \arrow[dd, bend right=60, "A"'] \arrow[dd, phantom, bend right=30, "\overset{\tilde{e}}{\Leftarrow}"] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & A \arrow[d, "R"] \arrow[dd, bend left=60, "A"] \arrow[dd, phantom, bend left=30, "\overset{i}{\Leftarrow}"] \\
\mathbb{1} \arrow[r, "U"] \arrow[d, "L"] \arrow[dr, phantom, "\Swarrow +"] & \mathbb{1} \arrow[d, "L"] \\
A \arrow[r, bend left, "A\otimes^+ U"] \arrow[r, bend right, "A\otimes^- V"'] \arrow[r, phantom, "\Downarrow\rho"] \arrow[d, "R"] \arrow[dd, bend right=60, "A"'] \arrow[dd, phantom, bend right=30, "\overset{\tilde{e}}{\Leftarrow}"] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & A \arrow[d, "R"] \arrow[dd, bend left=60, "A"] \arrow[dd, phantom, bend left=30, "\overset{i}{\Leftarrow}"] \\
\mathbb{1} \arrow[r, "V"] \arrow[d, "L"] \arrow[dr, phantom, "\Swarrow -"] & \mathbb{1} \arrow[d, "L"] \\
A \arrow[r, "A\otimes^- V"] \arrow[d, "R"'] \arrow[dr, phantom, "\Swarrow \mathrm{Id}"] & A \arrow[d, "R"] \\
\mathbb{1} \arrow[r, "V"'] & \mathbb{1}
\end{tikzcd}
\in \mathrm{Mor}_2(\mathcal{C}) \subset \mathrm{Mor}_2(\mathbf{BiMod}(\mathcal{C}))
$$

$\leftrightarrow$

$$\langle \bar{e}_2 | I_\rho e_1 \rangle \quad = $$



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