the 1-dimensional 3-vector space

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November 23, 2006

Abstract

We explain how, for any braided abelian monoidal category $\mathcal{C}$, the 3-category $\Sigma(\mathrm{Bim}(\mathcal{C}))$ plays the role of the 3-category of canonical 1-dimensional 3-vector spaces. We make some comments on the resulting concept of line-3-bundles with connection and show how the 3-category of twisted bimodules arises from morphisms of almost-trivial line-3-bundles with connection.

Let $\mathcal{C}$ be a braided abelian monoidal category.

You may want to think of the examples $\mathcal{C} = \mathrm{Vect}_k$ for some field k, or $\mathcal{C} = \mathrm{Mod}_R$, for some commutative ring R. But for the applications we have in mind, we will have a nontrivial braiding. In particular, $\mathcal{C}$ might be a modular tensor category.

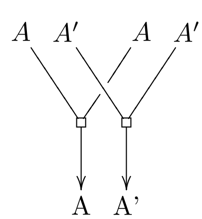
I denote the 2-category whose objects are algebras internal to $\mathcal{C}$, whose morphisms are bimodules and whose 2-morphisms are bimodule homomorphisms by $\mathrm{Bim}(\mathcal{C})$.

We can think of this as a 2-category of 2-vector spaces, due to the canonical inclusion

$$\mathrm{Bim}(\mathcal{C}) \stackrel{\subseteq}{\to} \mathrm{Mod}_{\mathcal{C}} \,.$$

Remarkably, since $\mathcal{C}$ is assumed to be braided, we get that $\mathrm{Bim}(\mathcal{C})$ is a monoidal 2-category.

For A and A' two algebras, their tensor product $A \otimes A'$ is the algebra which is $A \otimes A'$ as an object in $\mathcal{C}$ and equipped with the product obtained by using the braiding to exchange A with A':



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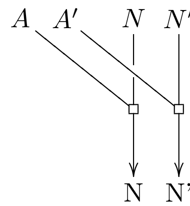
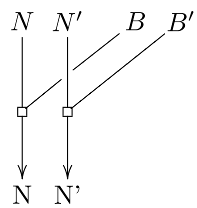
Accordingly, the left A-module N and the left A'-module N' are tensored to form the $A \otimes A'$-module $N \otimes N'$ with the action given by using the braiding:



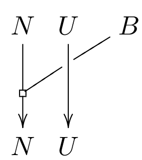
Similarly, if N is a right B-module and N' is a right A'-module, the right action of $B \otimes B'$ on $N \otimes N'$ is



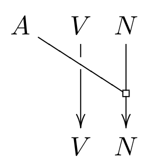
A simple special case of this turns out to be interesting in applications. The tensor unit $\mathbb{1}$ of $\mathcal{C}$ with the trivial algebra structure on it is always an algebra internal to $\mathcal{C}$. Any object of $\mathcal{C}$ is a $\mathbb{1}$-$\mathbb{1}$ bimodule. This yields a canonical inclusion

$$\Sigma(\mathcal{C}) \xrightarrow{\subset} \mathrm{Bim}(\mathcal{C}) \; .$$

This means that for any A-B bimodule N, and any object U in $\mathcal{C}$, we may consider $N \otimes U$ as another A-B bimodule, with the obvious left action and with the right action given by



Similarly, for V any object of $\mathcal{C}$, we obtain the A-B bimodule $V \otimes N$ with the obvious right action and the left action given by



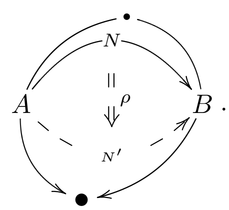
Quite literally, we can think of the tensor structure on $\mathrm{Bim}(\mathcal{C})$ as obtained from arranging bimodules in front of each other.

The formal expression of this geometric intuition is that from the monoidal 2-category $\mathrm{Bim}(\mathcal{C})$ we can form the suspension, $\Sigma(\mathrm{Bim}(\mathcal{C}))$, which is the 3-category with a single object $\bullet$, such that $\mathrm{End}(\bullet) = \mathrm{Bim}(\mathcal{C})$, and such that composition across that single object is the tensor product on $\mathrm{Bim}(\mathcal{C})$.

If

$$\begin{tikzcd} A \arrow[r, bend left=50, "N"] \arrow[r, bend right=50, "N'"'] \arrow[r, phantom, "\Downarrow \rho"] & B \end{tikzcd}$$

is a 2-morphism in $\mathrm{Bim}(\mathcal{C})$, we draw the corresponding 3-morphism in $\Sigma(\mathrm{Bim}(\mathcal{C}))$ as



Since $\mathcal{C}$ is braided, by assumption, it can itself be regarded as a 3-category with a single object and a single morphism. This is the double suspension $\Sigma(\Sigma(\mathcal{C}))$ of $\mathcal{C}$. As before, we have a canonical inclusion

$$\Sigma(\Sigma(\mathcal{C})) \xrightarrow{\subset} \Sigma(\mathrm{Bim}(\mathcal{C}))\,.$$

This inclusion should be thought of as analogous to the canonical inclusion

$$\Sigma(\mathbb{C}) \xrightarrow{\subset} \mathrm{Vect}_{\mathbb{C}} \ .$$

Notice that we may think of $\Sigma(\mathrm{Bim}(\mathcal{C}))$ as the 3-category obtained by acting with $\mathrm{Bim}(\mathcal{C})$ on itself. The single object then corresponds to $\mathrm{Bim}(\mathcal{C})$ itself, a morphism colored by an algebra A then corresponds to the 2-functor

$$A \otimes \cdot : \mathrm{Bim}(\mathcal{C}) \to \mathrm{Bim}(\mathcal{C})\,,$$

and so on.

Therefore we have a canonical embedding

$$\Sigma(\mathrm{Bim}(\mathcal{C})) \xrightarrow{\subset} \mathrm{Mod}_{\mathrm{Bim}(\mathcal{C})} \ .$$

I suspect that under suitable conditions the similar inclusion $\mathrm{Bim}(\mathcal{C}) \overset{\subseteq}{\to} \mathrm{Mod}_{\mathcal{C}}$ is in fact an equivalence. It seems that Ostrik has at least shown that for well behaved $\mathcal{C}$ this inclusion is at least essentially surjective on objects.

We might even be tempted to *define* the well-behaved part of $\mathrm{Mod}_{\mathcal{C}}$ to be that in the image of this inclusion.

Just suppose for the moment this were so. Then

$$\mathrm{Mod}_{\mathrm{Bim}(\mathcal{C})} \simeq \mathrm{Mod}_{\mathrm{Mod}_{\mathcal{C}}}$$

and

$$\Sigma(\mathrm{Bim}(\mathcal{C})) \xrightarrow{\subset} \mathrm{Mod}_{\mathrm{Mod}_{\mathcal{C}}} \;.$$

But here the right hand side is rightly addressed as the 3-category of 3-vector spaces.

For that reason, just like we may address $\mathcal{C}$ itself as the canonical 1-dimensional $\mathcal{C}$-module category, it seems right to address $\mathrm{Bim}(\mathcal{C})$ as the canonical 1-dimensional $\mathrm{Mod}_{\mathcal{C}}$-module 2-category. Or, more suggestively, as the canonical 1-dimensional 3-vector space.

Adopting this point of view, we make the following definitions, all with respect to a fixed choice of braided abelian monoidal category $\mathcal{C}$.

Definition 1 *A* **3-vector-bundle** *with connection is a transport 3-functor*

$$\mathcal{P} \to \mathrm{Mod}_{\mathrm{Mod}_{\mathcal{C}}} \,.$$

Recall that we have talked about this chain of inclusions:

$$\Sigma(\Sigma(\mathcal{C})) \xrightarrow{j} \Sigma(\mathrm{Bim}(\mathcal{C})) \xrightarrow{i} \mathrm{Mod}_{\mathrm{Mod}_{\mathcal{C}}} \;.$$

If $\mathcal{C}$ is itself already a category of modules, for instance if $\mathcal{C} = \mathrm{Vect}_{\mathbb{C}} = \mathrm{Mod}_{\mathbb{C}}$, we get yet another inclusion:

$$\begin{tikzcd}
\Sigma(\Sigma(\Sigma(\mathbb{C}))) \arrow[r, "k"] & \Sigma(\Sigma(\mathbf{Vect}_{\mathbb{C}})) \arrow[r, "j"] \arrow[d, equal] & \Sigma(\mathrm{Bim}(\mathbf{Vect}_{\mathbb{C}})) \arrow[r, "i"] \arrow[d, equal] & \mathrm{Mod}_{\mathrm{Mod}_{\mathrm{Vect}_{\mathbb{C}}}} \arrow[d, equal] \\
& \Sigma(\Sigma(\mathcal{C})) \arrow[r, "j"] & \Sigma(\mathrm{Bim}(\mathcal{C})) \arrow[r, "i"] & \mathrm{Mod}_{\mathrm{Mod}_{\mathcal{C}}}
\end{tikzcd} \;.$$

For each such inclusion, we get a notion of trivial, or locally trivial, 3-vector bundle.

Definition 2 *An i-trivial 3-vector bundle with connection, called a* **line-3-bundle with connection***, is a transport 3-functor*

$$\mathcal{P} \to \Sigma(\mathrm{Bim}(\mathcal{C}))\,.$$

The $i \circ j \circ k$-trivial n-vector bundle shall be denoted by 1. It plays a role for defining the spaces of (flat) sections of a 3-vector bundle. In general, we say

Definition 3 *The 3-functor*

$$1 : \mathcal{P} \to \Sigma(\mathrm{Bim})$$

is that which sends everything to the identity.

Proposition 1 *Let the domain $\mathcal{P}$ be a 2-category, i.e. a 3-category with only identity 3-morphisms. Endomorphisms of the trivial 3-vector bundle* 1 *on $\mathcal{P}$ are the same as 2-functors to* $\mathrm{Bim}(\mathcal{C})$*.*

$$\mathrm{End}(1) \simeq [\mathcal{P}, \mathrm{Bim}(\mathcal{C})]\,.$$

Proof. This will become clear, shortly. $\square$

A degenerate but interesting example in between general line 3- bundles and the completely trivial bundle 1 are those that are $i \circ j$-trivial.

We shall be interested in those especially for the case where the domain $\mathcal{P}$ is what we call the (open, disklike) 2-particle.

Definition 4 *The* **3-particle** *is, for the present purpose, the 2-category*

$$\mathrm{par} = \left\{ \begin{tikzcd} k \arrow[rr, "S"] \arrow[dr] \arrow[rr, bend left=60, no head, "a"] & & k' \arrow[dl] \\ & b & \end{tikzcd} \right\}$$

that consists of two objects, two nontrivial 1-morphisms and one nontrivial 2-morphism, as shown.

Example 1 (morphisms of $(i \circ j)$-trivial line 3-bundles over the open 3-particle)

A general line-3-bundle on par is nothing but any bimodule.

An $(i \circ j)$-trivial line-3-bundle with connection on par is nothing but any $\mathbb{1}$-$\mathbb{1}$-bimodule, hence nothing but any object of $\mathcal{C}$.

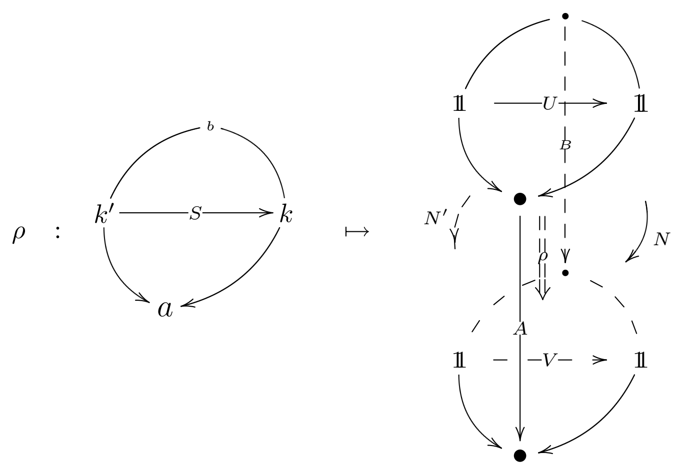
Let's write

$$1_U : \mathrm{par} \to \Sigma(\mathrm{Bim}(\mathcal{C}))$$

for the $(i \circ j)$-trivial 3-bundle with connection that assigns $U \in \mathrm{Obj}(\mathcal{C})$ to S:

$$1_U \;:\; \left(\begin{tikzcd} k \arrow[rr, "S"] \arrow[dr] \arrow[rr, bend left=60, no head, "a"] & & k' \arrow[dl] \\ & b & \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} \mathbb{1} \arrow[rr, "U"] \arrow[dr] \arrow[rr, bend left=60, no head, "\bullet"] & & \mathbb{1} \arrow[dl] \\ & \bullet & \end{tikzcd} \right) .$$

A morphism $\rho 1_U \to 1_V$ is a filled tin can 3-morphism



in $\Sigma(\mathrm{Bim}(\mathcal{C}))$.

Cutting this open, this is a 3-morphism ρ from

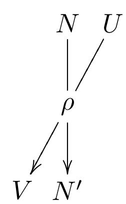
$$\begin{tikzcd}
\bullet \arrow[r, "\mathbb{1}"] \arrow[r, bend left=60, "\mathbb{1}"] \arrow[r, phantom, bend left=30, "\Downarrow U"] \arrow[d, "B"'] \arrow[dr, phantom, "\Swarrow N"] & \bullet \arrow[d, "A"] \\
\bullet \arrow[r, "\mathbb{1}"'] & \bullet
\end{tikzcd},$$

to

$$\begin{tikzcd}
\bullet \arrow[r, "\mathbb{1}"] \arrow[d, "B"'] \arrow[dr, phantom, "\Swarrow N'"] & \bullet \arrow[d, "A"] \\
\bullet \arrow[r, "\mathbb{1}"] \arrow[r, bend right=60, "\mathbb{1}"'] \arrow[r, phantom, bend right=30, "\Downarrow V"] & \bullet
\end{tikzcd}.$$

In other words, ρ is a morphism from the $A \otimes \mathbb{1}$-$B \otimes \mathbb{1}$-bimodule $N \otimes U$ to the

$\mathbb{1} \otimes A$-$\mathbb{1} \otimes B$-bimodule $V \otimes N'$:

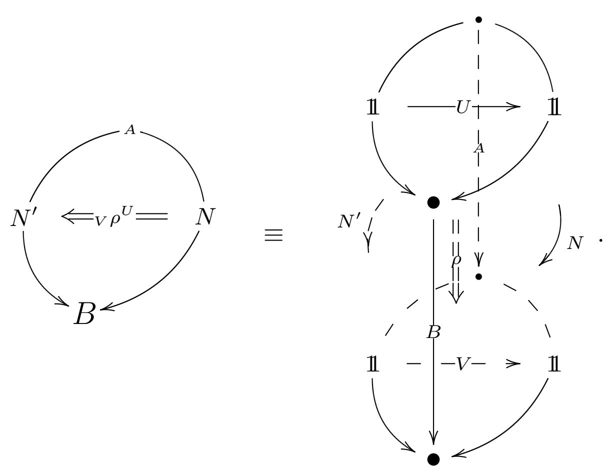


All tin cans ρ in $\Sigma(\mathrm{Bim}(\mathcal{C}))$ of this kind, with top and bottom a $\mathbb{1}$-$\mathbb{1}$ bimodule, form a 2-category in the obvious way. We will address this as

Definition 5 *The 2-category* $\mathrm{TwBim}(\mathcal{C})$ *of* **twisted bimodules** *is the 2-category of tin cans in* $\Sigma(\mathrm{Bim}(\mathcal{C}))$ *whose top and bottom are* $\mathbb{1}$-$\mathbb{1}$*-bimodules,*

$$
\mathrm{TwBim}(\mathcal{C}) \equiv \left\{ \begin{tikzcd} A \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow {}_V\rho^U"] & B \end{tikzcd} \right\} .
$$

Here



Sometimes it is useful to think of TwBim as a 3-category, too. The 3-morphisms then come from composing 3-morphisms in $\Sigma(\mathrm{Bim}(\mathcal{C}))$ at the top and bottom of those tin cans.