Differentiating Lie Groupoids

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March 15, 2007

1 Introduction

The concept of a *groupoid* is a rather natural one. As is that of a *Lie groupoid*.

Every Lie groupoid may be differentiated to yield a *Lie algebroid*. However, maybe somewhat surprisingly, the standard definition of a Lie algebroid has an appearence which is nowhere close to the simple elegance of the definition of a Lie groupoid.

While one may tend to accept this as a sad fact of life, it becomes increasingly annoying as one tries to categorify these concepts: passing from (Lie) groupoids to (Lie) 2-groupoids is, again, the most natural thing in the world. But the analogous step on the Lie algebroid side – which surely ought to exist – is, when using the standard definition of a Lie algebroid, quite non-obvious.

In fact, to the best of my knowledge, no direct definition of Lie 2-algebroid has ever appeared.

(What does exists is an "indirect" definition, using a detour through Baez-Crans Lie-2-algebras, their relation to L_∞-algebras, the relation of those to quasi-free differential algebras and finally their known relation to Lie 1-algebroids.)

Here I would like to try to improve on this situation by re-formulating the definition of the Lie-algebroid

$$\mathrm{Lie}(\mathrm{Gr})$$

associated to any Lie groupoid

$$\mathrm{Gr}$$

using only canonical and natural ingredients.

In order to accomplish this, I invoke the point of view that

- every Lie groupoid, Gr, is canonically a Gr-equivariant principal Gr-bundle over its space of objects.

While possibly still sounding a little intricate, this is a very natural point of view, since it is, as I shall make explicit, nothing but the "integrated Yoneda embedding" of the Lie groupoid, which gives rise to the functor

$$\mathrm{tra}_{\mathrm{Gr}} : \mathrm{Gr} \to C^\infty$$

that sends objects to the target fibers over them and morphisms to the post-composition with these:

$$\mathrm{tra}_{\mathrm{Gr}} : (\ x \xrightarrow{f} y\ ) \mapsto (\ t^{-1}(x) \xrightarrow{f\circ\cdot} t^{-1}(y)\ )\,.$$

2 Canonical Ingredients

In this section I simply list a couple of standard facts and constructions. These will then be used in the next section to swiftly say how a Lie algebroid arises from a Lie groupoid.

Fact 1 *Every Lie groupoid, when regarded as a span*

$$\begin{tikzcd} & \mathrm{Mor}(C) \arrow[dl, "t"'] \arrow[dr, "s"] & \\ \mathrm{Obj}(C) & & \mathrm{Obj}(C) \end{tikzcd}$$

internal to smooth manifolds, canonically becomes a Gr*-principal bundle*

$$\begin{tikzcd} \mathrm{Mor}(\mathrm{Gr}) \arrow[d, "p"'] \arrow[dr, "\lambda"] & \\ \mathrm{Obj}(\mathrm{Gr}) & \mathrm{Obj}(\mathrm{Gr}) \end{tikzcd}$$

(also known as a Gr*-torsor) over its own space of objects, with the target map playing the role of the bundle projection and the source map that of the "momentum map" (or "anchor map").*

This bundle is equivariant *with respect to the canonical* Gr*-action on its own space of objects.*

In the language of parallel transport functors, the same fact has the following, maybe more immediate, formulation (where GrTor denotes the category of Gr-torsors *over a point*).

Fact 2 *We have a smoothly locally trivializable* Gr*-principal parallel transport*

$$R : \mathrm{Gr} \to \mathrm{GrTor}$$

acting by "right translation"

$$R : (\ x \xrightarrow{f} y\ ) \mapsto (\ t^{-1}(x) \xrightarrow{f\circ\cdot} t^{-1}(y)\ )\,.$$

In components this reads

$$R(\ x \xrightarrow{g} y\ ) : (\ w \xrightarrow{f} x\ ) \mapsto (\ w \xrightarrow{f} x \xrightarrow{g} y\ )\,.$$

(Notice that, while the Gr-bundle $\mathrm{Mor}(\mathrm{Gr}) \to \mathrm{Obj}(\mathrm{Gr})$ does have a global section, it has no *equivariant* global section.)

This functor encodes the target map and the composition in the groupoid, by way of an "integrated Yoneda embedding". The source map in Gr appears, from this point of view, as a natural transformation on this functor:

Fact 3 *Write*

$$S : \mathrm{Gr} \to C^\infty$$

for the functor that sends everything to $\mathrm{Id}_{\mathrm{Obj(Gr)}}$. *Then the source map, s, of* Gr *is a natural transformation*

$$s : R \to S\,.$$

(Here the application of the faithful forgetful functor $\mathrm{GrTor} \to C^\infty$, which just forgets the groupoid action on a smooth manifold, is to be understood implicitly.)

Fact 4 *We have the following three functors.*

- *The* **tangent bundle functor**

  $$T : C^\infty \to \mathrm{VectBun}$$

  sends smooth spaces to their tangent bundle and sends smooth maps to their differential:

  $$T : (\ X \xrightarrow{f} Y\ ) \mapsto (\ TX \xrightarrow{df} TY\ )\,.$$

  That this assignmnet respects composition is nothing but the chain rule of calculus.

- *The* **section functor**

  $$\Gamma : \mathrm{VectBun}(M) \to \mathrm{Vect}$$

  sends a vector bundle over M to its space of sections and sends a morphism of vector bundles to the induced map on their sections

  $$\Gamma : (\ V \xrightarrow{f} W\ ) \mapsto (\ \Gamma(V) \xrightarrow{\Gamma(f)} \Gamma(W)\ )\,.$$

- *The* **composition of both**, *defined on each isomorphism class,*

  $$\Gamma \circ T : C^\infty|_{\sim M} \to \mathrm{Vect}$$

  in fact factors through the forgetful functor

  $$\mathrm{LieAlg} \to \mathrm{Vect}\,,$$

  since the space of section of a tangent *vector bundle TX canonically carries the structure of the Lie algebra of vector fields on X.*

To combine these facts neatly, consider the following definition.

Write

$$I : \mathcal{P}_1(X) \to \mathrm{Vect}$$

for the tensor unit in the category $[\mathcal{P}_1(X), \mathrm{Vect}]$ of functors into vector spaces, inherited from the standard monoidal structure on Vect.

Definition 1 *Let* $\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$ *be a smoothly locally trivializable vector bundle with connection. A* **flat section** *or* **covariantly constant section** *of* tra *is a morphism*

$$e : I \to \mathrm{tra}\,.$$

We write

$$\Gamma_{\mathrm{fl}}(\mathrm{tra}) := [I, \mathrm{tra}]$$

for the **vector space of flat sections** *of* tra*.*

It follows that to any parallel transport with values in smooth spaces we may canonically associate the Lie algebra of flat sections of the associated vector bundle of vector fields on the fibers.

Definition 2 *Given a parallel transport with values in smooth spaces*

$$\mathrm{tra} : \mathcal{P}_1(X) \to C^\infty$$

we write

$$\mathrm{Lie}(\mathrm{tra}) := \Gamma_{\mathrm{fl}}(\Gamma \circ T \circ \mathrm{tra})$$

for the associated **Lie algebra of flat sections of the associated vector bundle of vector fields on the fibers***.*

3 Lie Algebroids

We have seen that, essentially by the Yoneda embedding, any Lie groupoid Gr is encoded in a functor

$$R : \mathrm{Gr} \to C^\infty\,,$$

giving the right action of the groupoid on itself (encoding target and composition maps), together with a transformation

$$s : R \to S\,,$$

(encoding the source map).

Applying definition 2 to this transformation yields a morphism of Lie algebras

$$\rho := ds : \mathrm{Lie}(R) \to \mathrm{Lie}(S)\,.$$

This is the Lie algebroid obtained from differentiating the Lie groupoid Gr.

To see more clearly how this reproduces the standard way in which the defintion of a Lie algebroid is formulated, notice that

$$\mathrm{Lie}(S) \simeq \Gamma(T\mathrm{Obj}(\mathrm{Gr}))$$

and

$$\mathrm{Lie}(R) \simeq \Gamma(\bigcup_{x \in \mathrm{Obj}(\mathrm{Gr})} T_{\mathrm{Id_x}} t^{-1}(x))\,.$$