# Tangent Categories

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**Abstract**

An arrow-theoretic formulation of tangency is proposed. This gives rise to a notion of tangent $n$-bundle for any $n$-groupoid. Properties and examples are discussed.

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### 0.1 Introduction

The present comments are rooted in two seemingly unrelated motivations:

- the desire to better understand the true nature of inner automorphism $n$-categories
- the desire to realize the concepts of tangency and supergeometry in a purely and genuinely arrow-theoretic and $n$-categorical way.

Maybe surprisingly, this is in fact very closely related, apparently.

The notion of tangent category and tangent bundle given in the following is just a simple variation of the familiar concept of comma categories, albeit generalized to $n$-categories. While very simple, it still seems to me that there is something interesting going on here. I present the concept in a slightly redundant fashion which is supposed to suggest to the inclined reader the more general picture which seems to be at work in the background.

### 0.2 Tangent $n$-categories

**Definition 1** (the point)**.** *The point is the (n-)category*

$$\mathrm{pt} := \{\bullet\}$$

*We shall carefully distinguish this from the (n-category)*

$$\mathbf{pt} := \{ \bullet \xrightarrow{\sim} \circ \} ,$$

*consisting of two objects connected by an equivalence.*

The category **pt** might be called the "fat point" or even the "superpoint". It is of course equivalent to the point – but not isomorphic. We fix one injection

$$i : \ \mathrm{pt} \hookrightarrow \mathbf{pt}$$

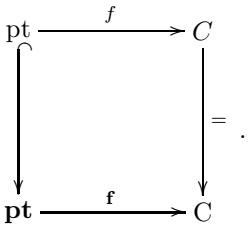
$$i : \bullet \mapsto \bullet$$

once and for all.

It is useful to think of morphisms

$$\mathbf{f} : \mathbf{pt} \to C$$

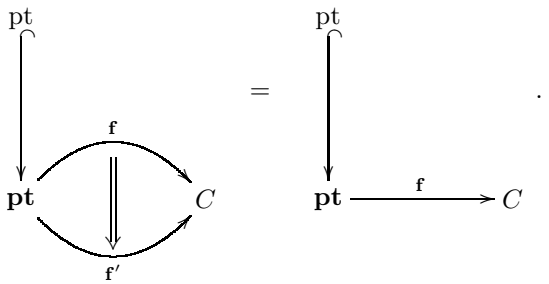
from the fat point to some codomain $C$ as labeled by the corresponding image of the ordinary point




**Definition 2** (tangent $n$-bundle)**.** *Given any $n$-category $C$, we define its tangent (n-)bundle*

$$TC \subset \mathrm{Hom}_{n\mathrm{Cat}}(\mathbf{pt}, C)$$

*to be that sub $n$-category of morphisms from the fat point into $C$ which collapses to a 0-category when pulled back along the fixed inclusion $i : \ \mathrm{pt} \hookrightarrow \mathbf{pt}$ :*




*The tangent $n$-bundle is a disjoint union*

$$TC = \bigoplus_{x \in \mathrm{Obj}(C)} T_x C$$

*of tangent $n$-categories at each object $x$ of $C$. In this way it is an $n$-bundle*

$$p : TC \longrightarrow \mathrm{Disc}(C)$$

*over the space of objects of $C$.*

As befits a tangent bundle, the tangent $n$-bundle has a canonical section

$$e_{\mathrm{Id}} : \mathrm{Disc}(C) \to TC$$

which sends every object of $C$ to the Identity morphism on it.

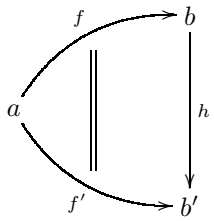
**Example (slice categories).** For $C$ any 1-groupoid, its tangent 1-category is the comma category

$$TC = ((\mathrm{Disc}(C) \hookrightarrow C) \downarrow \mathrm{Id}_C) \,.$$

This is the disjoint union of all co-over categories on all objects of $C$

$$TC = \bigoplus_{a \in \mathrm{Obj}(C)} (a \downarrow C)$$

Objects of $TC$ are morphisms $f : a \to b$ in $C$, and morphisms $f \xrightarrow{h} f'$ in $TC$ are commuting triangles



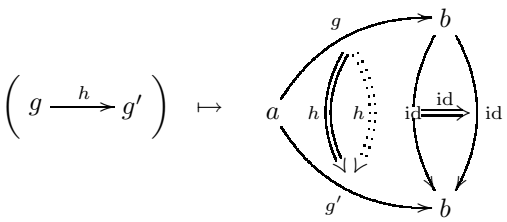

in $C$.

**Proposition 1.** *For any $n$-category $C$, its tangent $n$-bundle $TC$ fits into an exact sequence*

$$\mathrm{Mor}(C) \hookrightarrow TC \twoheadrightarrow C \ .$$

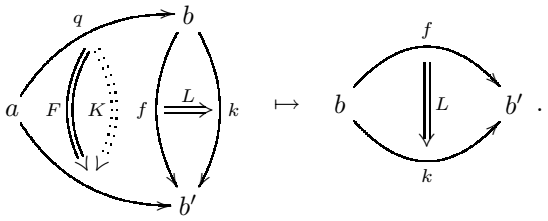
Proof. For definiteness we assume that $n = 2$ and that we are working in the world of strict 2-categories, strict 2-functors between them, pseudonatural transformation between these and modifications between those.

The strict inclusion 2-functor on the left is



for $g, g'$ any two morphisms in $C$ and $h$ any 2-morphism between them.

The strict surjection 2-functor on the right is




The image of the injection is precisely the preimage under the surjection of the identity 2-morphism on the identity 1-morphism of the single object. This means the sequence is exact. □

## 0.3 Inner automorphisms

Usually, for $G$ any group, inner and outer automorphisms are regarded as sitting in a short exact sequence

$$\mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \ .$$

But we shall find shortly that we ought to be distinguishing inner automorphisms that differ by elements in the center. Then one has an exact sequence of the form

$$\mathrm{Z}(G) \longrightarrow \mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \ .$$

With this definition of inner automorphisms of ordinary groups, we have of course $\mathrm{Inn}(G) \simeq G$. But this degenration of concepts disappears as we move to the proper $n$-categorical context.

First recall the standard definitions of center and automorphism of groupoids:

**Definition 3.** *Given any $n$-groupoid $C$*

- *the automorphism $(n+1)$-group*

$$\mathrm{AUT}(C) \subset \mathrm{Hom}_{n\mathrm{Cat}}(C, C)$$

  *is the core of the endomorphism $(n+1)$-category of $C$*

- *the center of $C$ is*

$$Z(C) := \Sigma \mathrm{AUT}(\mathrm{Id}_C) \, .$$

To these two standard definitions, we add the following one, which is supposed to be the proper $n$-categorical generalization of the concept of inner automorphisms.

**Definition 4** (inner automorphisms)**.** *Given any $n$-category $C$, the tangent $n$-groupoid*

$$\mathrm{INN}(C) := T_{\mathrm{Id}_C}(\mathrm{Mor}(n\mathrm{Cat}))$$

*is the $n$-groupoid of inner automorphisms of $C$.*

**Proposition 2.** *For any $C$, we have an exact sequence*

$$Z(C) \longrightarrow \mathrm{INN}(C) \longrightarrow \mathrm{AUT}(C) \longrightarrow \mathrm{OUT}(C) \ .$$

*Moreover, for $\mathbf{C} := \mathrm{Mor}(n\mathrm{Cat})$, this sits inside the exact sequence from proposition 1 as*

$$\begin{array}{ccccccc} Z(C) & \longrightarrow & \mathrm{INN}(C) & \longrightarrow & \mathrm{AUT}(C) & \longrightarrow & \mathrm{OUT}(C) \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Mor}(\mathbf{C}) & \longrightarrow & T\mathbf{C} & \longrightarrow & \mathbf{C} & & \end{array} \ .$$

**Example.** For $G$ an ordinary group we write

$$\mathrm{INN}(G) := \mathrm{INN}(\Sigma G) \, .$$

Then

$$\mathrm{INN}(G) := T_{\mathrm{Id}_{\Sigma G}}(n\mathrm{Cat}) \simeq T_\bullet(\Sigma G)$$

is the codiscrete groupoid over the elements of $G$. Its nature as a groupoid is manifest from its realization as

$$\mathrm{INN}(G) = T_\bullet(\Sigma G) \, .$$

But it is also a 2-group. The monoidal structure is that coming from its realization as $\mathrm{INN}(G) := T_{\mathrm{Id}_{\Sigma G}}(n\mathrm{Cat})$.

**Proposition 3.** *For $G_{(2)}$ any strict 2-group, we have an inclusion*

$$T_\bullet \Sigma G_{(2)} \subset T_{\mathrm{Id}_{\Sigma G_{(2)}}}(2\mathrm{Cat}) \, .$$

*This realizes $T_\bullet \Sigma G_{(2)}$ as a subcategory of $T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$.*

Proof. This is described in the work with David Roberts. □

**Remark.** I have a slight problem with the notation here. With David what we called $\mathrm{INN}(G_{(2)})$ is $T_\bullet \Sigma G_{(2)}$. But above I am using $\mathrm{INN}(G_{(2)})$ for $T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$.