Tangent Categories

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Abstract

An arrow-theoretic formulation of tangency is proposed. This gives rise to a notion of tangent n-bundle for any n-groupoid. Properties and examples are discussed.

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0.1 Introduction

The present comments are rooted in two seemingly unrelated motivations:

- the desire to better understand the true nature of inner automorphism n-categories
- the desire to realize the concepts of tangency and supergeometry in a purely and genuinely arrow-theoretic and n-categorical way.

Maybe surprisingly, this is in fact very closely related, apparently.

The notion of tangent category and tangent bundle given in the following is just a simple variation of the familiar concept of comma categories, albeit generalized to n-categories. While very simple, it still seems to me that there is something interesting going on here. I present the concept in a slightly redundant fashion which is supposed to suggest to the inclined reader the more general picture which seems to be at work in the background.

0.2 Tangent n-categories

Definition 1 (the point)**.** *The point is the (n-)category*

$$\mathrm{pt} := \{\bullet\}$$

We shall carefully distinguish this from the (n-category)

$$\mathbf{pt} := \{ \bullet \xrightarrow{\sim} \circ \} ,$$

consisting of two objects connected by an equivalence.

The category **pt** might be called the "fat point" or even the "superpoint". It is of course equivalent to the point – but not isomorphic. We fix one injection

$$i : \ \mathrm{pt} \hookrightarrow \mathbf{pt}$$

$$i : \bullet \mapsto \bullet$$

once and for all.

It is useful to think of morphisms

$$\mathbf{f} : \mathbf{pt} \to C$$

from the fat point to some codomain C as labeled by the corresponding image of the ordinary point

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "f"] \arrow[d, hook] & C \arrow[d, "="] \\ \mathbf{pt} \arrow[r, "\mathbf{f}"] & C \end{tikzcd} .$$

Definition 2 (tangent n-bundle)**.** *Given any n-category C, we define its tangent $(n$-$)$bundle*

$$TC \subset \mathrm{Hom}_{n\mathrm{Cat}}(\mathbf{pt}, C)$$

to be that sub n-category of morphisms from the fat point into C which collapses to a 0-category when pulled back along the fixed inclusion $i : \ \mathrm{pt} \hookrightarrow \mathbf{pt}$:

$$\begin{tikzcd} \mathrm{pt} \arrow[d, hook] & \\ \mathbf{pt} \arrow[r, bend left, "\mathbf{f}"] \arrow[r, bend right, "\mathbf{f}'"'] \arrow[r, phantom, "\Downarrow"] & C \end{tikzcd} \quad = \quad \begin{tikzcd} \mathrm{pt} \arrow[d, hook] & \\ \mathbf{pt} \arrow[r, "\mathbf{f}"] & C \end{tikzcd} .$$

The tangent n-bundle is a disjoint union

$$TC = \bigoplus_{x \in \mathrm{Obj}(C)} T_x C$$

of tangent n-categories at each object x of C. In this way it is an n-bundle

$$p : TC \longrightarrow \mathrm{Disc}(C)$$

over the space of objects of C.

As befits a tangent bundle, the tangent n-bundle has a canonical section

$$e_{\mathrm{Id}} : \mathrm{Disc}(C) \to TC$$

which sends every object of C to the Identity morphism on it.

Example (slice categories). For C any 1-groupoid, its tangent 1-category is the comma category

$$TC = ((\mathrm{Disc}(C) \hookrightarrow C) \downarrow \mathrm{Id}_C)\,.$$

This is the disjoint union of all co-over categories on all objects of C

$$TC = \bigoplus_{a \in \mathrm{Obj}(C)} (a \downarrow C)$$

Objects of TC are morphisms $f : a \to b$ in C, and morphisms $f \xrightarrow{h} f'$ in TC are commuting triangles

$$\begin{tikzcd} & b \arrow[dd, "h"] \\ a \arrow[ur, bend left, "f"] \arrow[dr, bend right, "f'"'] \arrow[r, phantom, "\Vert"] & \\ & b' \end{tikzcd}$$

in C.

Proposition 1. *For any n-category C, its tangent n-bundle TC fits into an exact sequence*

$$\mathrm{Mor}(C) \hookrightarrow TC \twoheadrightarrow C\ .$$

Proof. For definiteness we assume that $n = 2$ and that we are working in the world of strict 2-categories, strict 2-functors between them, pseudonatural transformation between these and modifications between those.

The strict inclusion 2-functor on the left is

$$\left(g \xrightarrow{h} g' \right) \mapsto \begin{tikzcd} & b \arrow[dd, bend right, "\mathrm{id}"'] \arrow[dd, bend left, "\mathrm{id}"] \arrow[dd, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] \\ a \arrow[ur, bend left, "g"] \arrow[dr, bend right, "g'"'] \arrow[r, phantom, "h \Downarrow\ \ h \Downarrow"] & \\ & b \end{tikzcd}$$

for g, g' any two morphisms in C and h any 2-morphism between them.

The strict surjection 2-functor on the right is

$$\begin{tikzcd} & b \arrow[dd, bend right, "f"'] \arrow[dd, bend left, "k"] \arrow[dd, phantom, "\overset{L}{\Rightarrow}"] \\ a \arrow[ur, bend left, "q"] \arrow[dr, bend right] \arrow[r, phantom, "F \Downarrow \; K \Downarrow"] & \\ & b' \end{tikzcd} \mapsto \begin{tikzcd} b \arrow[r, bend left, "f"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow L"] & b' \end{tikzcd} .$$

The image of the injection is precisely the preimage under the surjection of the identity 2-morphism on the identity 1-morphism of the single object. This means the sequence is exact. □

0.3 Inner automorphisms

Usually, for G any group, inner and outer automorphisms are regarded as sitting in a short exact sequence

$$\mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \ .$$

But we shall find shortly that we ought to be distinguishing inner automorphisms that differ by elements in the center. Then one has an exact sequence of the form

$$\mathrm{Z}(G) \longrightarrow \mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \ .$$

With this definition of inner automorphisms of ordinary groups, we have of course $\mathrm{Inn}(G) \simeq G$. But this degenration of concepts disappears as we move to the proper n-categorical context.

First recall the standard definitions of center and automorphism of groupoids:

Definition 3. *Given any n-groupoid C*

- *the automorphism $(n+1)$-group*

$$\mathrm{AUT}(C) \subset \mathrm{Hom}_{n\mathrm{Cat}}(C, C)$$

  is the core of the endomorphism $(n+1)$-category of C

- *the center of C is*

$$Z(C) := \Sigma \mathrm{AUT}(\mathrm{Id}_C) \,.$$

To these two standard definitions, we add the following one, which is supposed to be the proper n-categorical generalization of the concept of inner automorphisms.

Definition 4 (inner automorphisms)**.** *Given any n-category C, the tangent n-groupoid*

$$\mathrm{INN}(C) := T_{\mathrm{Id}_C}(\mathrm{Mor}(n\mathrm{Cat}))$$

is the n-groupoid of inner automorphisms of C.

Proposition 2. *For any C, we have an exact sequence*

$$Z(C) \longrightarrow \mathrm{INN}(C) \longrightarrow \mathrm{AUT}(C) \longrightarrow \mathrm{OUT}(C)\ .$$

Moreover, for $\mathbf{C} := \mathrm{Mor}(n\mathrm{Cat})$, this sits inside the exact sequence from proposition 1 as

$$\begin{tikzcd} Z(C) \arrow[r] \arrow[d, hook] & \mathrm{INN}(C) \arrow[r] \arrow[d, hook] & \mathrm{AUT}(C) \arrow[r] \arrow[d, hook] & \mathrm{OUT}(C) \\ \mathrm{Mor}(\mathbf{C}) \arrow[r] & T\mathbf{C} \arrow[r] & \mathbf{C} & \end{tikzcd} .$$

Example. For G an ordinary group we write

$$\mathrm{INN}(G) := \mathrm{INN}(\Sigma G)\,.$$

Then

$$\mathrm{INN}(G) := T_{\mathrm{Id}_{\Sigma G}}(n\mathrm{Cat}) \simeq T_\bullet(\Sigma G)$$

is the codiscrete groupoid over the elements of G. Its nature as a groupoid is manifest from its realization as

$$\mathrm{INN}(G) = T_\bullet(\Sigma G)\,.$$

But it is also a 2-group. The monoidal structure is that coming from its realization as $\mathrm{INN}(G) := T_{\mathrm{Id}_{\Sigma G}}(n\mathrm{Cat})$.

Proposition 3. *For $G_{(2)}$ any strict 2-group, we have an inclusion*

$$T_\bullet \Sigma G_{(2)} \subset T_{\mathrm{Id}_{\Sigma G_{(2)}}}(2\mathrm{Cat})\,.$$

This realizes $T_\bullet \Sigma G_{(2)}$ as a subcategory of $T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$.

Proof. This is described in the work with David Roberts. □

Remark. I have a slight problem with the notation here. With David what we called $\mathrm{INN}(G_{(2)})$ is $T_\bullet \Sigma G_{(2)}$. But above I am using $\mathrm{INN}(G_{(2)})$ for $T_{\mathrm{Id}_{\Sigma G_{(2)}}}(\mathrm{Mor}(2\mathrm{Cat}))$.