

Tangent Categories

Urs Schreiber

July 21, 2007

Abstract

An arrow-theoretic formulation of tangency is proposed. This gives rise to a notion of tangent n -bundle for any n -groupoid. Properties and examples are discussed.

Contents

0.1	Introduction	1
0.2	Tangent n -categories	1
0.3	Inner automorphisms	4

0.1 Introduction

The present comments are rooted in two seemingly unrelated motivations:

- the desire to better understand the true nature of inner automorphism n -categories
- the desire to realize the concepts of tangency and supergeometry in a purely and genuinely arrow-theoretic and n -categorical way.

Maybe surprisingly, this is in fact very closely related, apparently.

The notion of tangent category and tangent bundle given in the following is just a simple variation of the familiar concept of comma categories, albeit generalized to n -categories. While very simple, it still seems to me that there is something interesting going on here. I present the concept in a slightly redundant fashion which is supposed to suggest to the inclined reader the more general picture which seems to be at work in the background.

0.2 Tangent n -categories

Definition 1 (the point). *The point is the (n -)category*

$$\text{pt} := \{\bullet\}$$

We shall carefully distinguish this from the (n -category)

$$\mathbf{pt} := \{ \bullet \xrightarrow{\sim} \circ \},$$

consisting of two objects connected by an equivalence.

The category \mathbf{pt} might be called the “fat point” or even the “superpoint”. It is of course equivalent to the point – but not isomorphic. We fix one injection

$$i : \mathbf{pt}^{\subset} \longrightarrow \mathbf{pt}$$

$$i : \bullet \mapsto \bullet$$

once and for all.

It is useful to think of morphisms

$$\mathbf{f} : \mathbf{pt} \rightarrow C$$

from the fat point to some codomain C as labeled by the corresponding image of the ordinary point

$$\begin{array}{ccc} \mathbf{pt} & \xrightarrow{f} & C \\ \downarrow & & \downarrow = \\ \mathbf{pt} & \xrightarrow{\mathbf{f}} & C \end{array} .$$

Definition 2 (tangent n -bundle). Given any n -category C , we define its tangent (n -)bundle

$$TC \subset \text{Hom}_{n\text{Cat}}(\mathbf{pt}, C)$$

to be that sub n -category of morphisms from the fat point into C which collapses to a 0-category when pulled back along the fixed inclusion $i : \mathbf{pt}^{\subset} \longrightarrow \mathbf{pt}$:

$$\begin{array}{ccc} \mathbf{pt} & & \mathbf{pt} \\ \downarrow & & \downarrow \\ \mathbf{pt} & \xrightarrow{\mathbf{f}} & C \\ \uparrow & & \uparrow \\ \mathbf{pt} & & \mathbf{pt} \end{array} \quad = \quad \begin{array}{ccc} \mathbf{pt} & & \\ \downarrow & & \\ \mathbf{pt} & \xrightarrow{\mathbf{f}} & C \end{array} .$$

The tangent n -bundle is a disjoint union

$$TC = \bigoplus_{x \in \text{Obj}(C)} T_x C$$

of tangent n -categories at each object x of C . In this way it is an n -bundle

$$p : TC \longrightarrow \text{Disc}(C)$$

over the space of objects of C .

As befits a tangent bundle, the tangent n -bundle has a canonical section

$$e_{\text{Id}} : \text{Disc}(C) \rightarrow TC$$

which sends every object of C to the Identity morphism on it.

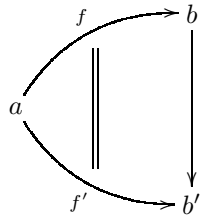
Example (slice categories). For C any 1-groupoid, its tangent 1-category is the comma category

$$TC = ((\text{Disc}(C) \hookrightarrow C) \downarrow \text{Id}_C).$$

This is the disjoint union of all co-over categories on all objects of C

$$TC = \bigoplus_{a \in \text{Obj}(C)} (a \downarrow C)$$

Objects of TC are morphisms $f : a \rightarrow b$ in C , and morphisms $f \xrightarrow{h} f'$ in TC are commuting triangles



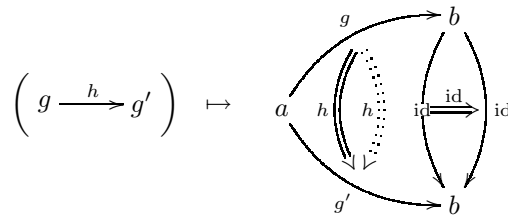
in C .

Proposition 1. For any n -category C , its tangent n -bundle TC fits into an exact sequence

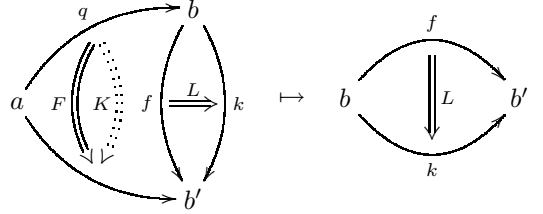
$$\text{Mor}(C) \hookrightarrow TC \twoheadrightarrow C.$$

Proof. For definiteness we assume that $n = 2$ and that we are working in the world of strict 2-categories, strict 2-functors between them, pseudonatural transformation between these and modifications between those.

The strict inclusion 2-functor on the left is



for g, g' any two morphisms in C and h any 2-morphism between them.
 The strict surjection 2-functor on the right is



The image of the injection is precisely the preimage under the surjection of the identity 2-morphism on the identity 1-morphism of the single object. This means the sequence is exact. \square

0.3 Inner automorphisms

Usually, for G any group, inner and outer automorphisms are regarded as sitting in a short exact sequence

$$\text{Inn}(G) \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) .$$

But we shall find shortly that we ought to be distinguishing inner automorphisms that differ by elements in the center. Then one has an exact sequence of the form

$$Z(G) \longrightarrow \text{Inn}(G) \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) .$$

With this definition of inner automorphisms of ordinary groups, we have of course $\text{Inn}(G) \simeq G$. But this degeneration of concepts disappears as we move to the proper n -categorical context.

First recall the standard definitions of center and automorphism of groupoids:

Definition 3. *Given any n -groupoid C*

- *the automorphism $(n + 1)$ -group*

$$\text{AUT}(C) \subset \text{Hom}_{n\text{Cat}}(C, C)$$

is the core of the endomorphism $(n + 1)$ -category of C

- *the center of C is*

$$Z(C) := \Sigma \text{AUT}(\text{Id}_C) .$$

To these two standard definitions, we add the following one, which is supposed to be the proper n -categorical generalization of the concept of inner automorphisms.

Definition 4 (inner automorphisms). *Given any n -category C , the tangent n -groupoid*

$$\text{INN}(C) := T_{\text{Id}_C}(\text{Mor}(n\text{Cat}))$$

is the n -groupoid of inner automorphisms of C .

Proposition 2. *For any C , we have an exact sequence*

$$Z(C) \longrightarrow \text{INN}(C) \longrightarrow \text{AUT}(C) \longrightarrow \text{OUT}(C) .$$

Moreover, for $\mathbf{C} := \text{Mor}(n\text{Cat})$, this sits inside the exact sequence from proposition 1 as

$$\begin{array}{ccccccc} Z(C) & \longrightarrow & \text{INN}(C) & \longrightarrow & \text{AUT}(C) & \longrightarrow & \text{OUT}(C) \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Mor}(\mathbf{C}) & \longrightarrow & T\mathbf{C} & \longrightarrow & \mathbf{C} & & \end{array} .$$

Example. For G an ordinary group we write

$$\text{INN}(G) := \text{INN}(\Sigma G) .$$

Then

$$\text{INN}(G) := T_{\text{Id}_{\Sigma G}}(n\text{Cat}) \simeq T_{\bullet}(\Sigma G)$$

is the codiscrete groupoid over the elements of G . Its nature as a groupoid is manifest from its realization as

$$\text{INN}(G) = T_{\bullet}(\Sigma G) .$$

But it is also a 2-group. The monoidal structure is that coming from its realization as $\text{INN}(G) := T_{\text{Id}_{\Sigma G}}(n\text{Cat})$.

Proposition 3. *For $G_{(2)}$ any strict 2-group, we have an inclusion*

$$T_{\bullet}\Sigma G_{(2)} \subset T_{\text{Id}_{\Sigma G_{(2)}}}(2\text{Cat}) .$$

This realizes $T_{\bullet}\Sigma G_{(2)}$ as a subcategory of $T_{\text{Id}_{\Sigma G_{(2)}}}(\text{Mor}(2\text{Cat}))$.

Proof. This is described in the work with David Roberts. □

Remark. I have a slight problem with the notation here. With David what we called $\text{INN}(G_{(2)})$ is $T_{\bullet}\Sigma G_{(2)}$. But above I am using $\text{INN}(G_{(2)})$ for $T_{\text{Id}_{\Sigma G_{(2)}}}(\text{Mor}(2\text{Cat}))$.