How to get a spinning string from here to there

Urs Schreiber

June 3, 2007

joint work, in parts, with

John Baez
Alissa Crans
Jim Stasheff
Danny Stevenson
Konrad Waldorf

inspired, in parts, by

Stephan Stolz *and* Peter Teichner

Contents

String(n) is some topological group, to be described in a moment. For various reasons it appears to be of interest to understand what a String(n)-bundle with connection would be. Here are some of these reasons.

1 The need for String(n)-Bundles

Killingback and others noticed [1, 2] that

1	super particles	couple to	Spin(n)-bundles	with connection

like

2	super strings	couple to	String(n)-bundles	with (?)

Using the Atiyah-Segal observation that

1	quantum (super) particles	are functors	$1\mathrm{Cob}_S \to \mathrm{Hilb}_S$

like

2	quantum (super) strings	are functors	$2\mathrm{Cob}_S \to \mathrm{Hilb}_S$

this should translate into a precise statement (about representations of cobordisms categories). Back then few people thought of categorification. But Stolz and Teichner later made two remarks.

First Remark. First, following Dan Freed, Segal's original viewpoint should be refined to

1	quantum (super) particles	are functors	$1\mathrm{Cob}_S \to \mathrm{Hilb}_S$

like

2	quantum (super) strings	are 2-functors	$\mathrm{Cob}_S^{\mathrm{ext}} \to 2\mathrm{Hilb}_S$

This is nowadays known as *extended quantum field theory*.

Second Remark. Moreover, it should be true that

1	Spin(n) bundles	with connection	are related to	K-cohomology

like

2	String(n)-bundles	with connection	are related to	elliptic cohomology

All in all, this is supposed to be considerable reason to be interested in String(n)-bundles with connection.

2 What is String(n), anyway?

There is the classical definition of String(n), and there is a "revisionist" one. The latter is maybe intuitively more accessible.

Revisionist definition: String(n) **as stringy** Spin(n)**.** Superstrings (in their RNS incarnation) are sometimes called *spinning strings*. Indeed, a superstring is much like a continuous line of spinors.

This suggests that the corresponding gauge group is the loop group

$$\Omega \mathrm{Spin}(n)$$

or maybe its Kac-Moody central extension

$$\hat{\Omega}_k \mathrm{Spin}(n)$$

or maybe the path group

$$P\mathrm{Spin}(n)\,.$$

Or maybe all of these. In fact, there are canonical group homomorphisms

$$\hat{\Omega}_k \mathrm{Spin}(n) \xrightarrow{t} P\mathrm{Spin}(n) \xrightarrow{\alpha} \mathrm{Aut}(\hat{\Omega}_k \mathrm{Spin}(n))\ .$$

These satisfy two compatibility conditions which say that the groups here conspire to form a (strict Fréchet-Lie) *2-group*

$$G_{(2)}\,.$$

A 2-group is a category which behaves like a group.

Every toplogical category like this may be turned into a big ordinary topological space by taking its nerve and forming its geometric realization.

	0-simplices	1-simplices	2-simplices	etc.
nerve of category: (simplicial set)	g	$g_1 \xrightarrow{h} g_2$	$\begin{tikzcd} & g_3 \arrow[dr, "h_2"] & \\ g_1 \arrow[ur, "h_1"] \arrow[rr, "h_1 \cdot h_2"] & & g_2 \end{tikzcd}$	
geometric realization:	fill with standard simplices in $\mathbb{R}^n$ – glue along common faces			

If the category is a 2-group, the realization of its nerve is a topological group.

For $G_{(2)}$, this nerve is [8, 9]

$$|G_{(2)}| \simeq \mathrm{String}(n)\,.$$

ON FIRST READING, SKIP TO 3, NOW

Classical definition. This is all that is needed about String(n) in the following. But for completeness, here is the classical definition.

Definition 1 *The* **string group** String_G *of a simple, simply connected, compact topological group G is (a model for) the 3-connected topological group with the same homotopy groups as G, except*

$$\pi_3(\mathrm{String}_G) = 0\,,$$

which, furthermore, fits into the exact sequence

$$1 \longrightarrow (BU(1) \simeq K(\mathbb{Z},2)) \longrightarrow \mathrm{String}_G \longrightarrow G \longrightarrow 1$$

of topological groups.

The string group proper is obtained by setting $G = \mathrm{Spin}(n)$.

$$\mathrm{String}(n) := \mathrm{String}_{\mathrm{Spin}(n)}\,.$$

The way to see that such a group is a plausible candidate for something generalizing the Spin-group, which, recall, fits into the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 1\,,$$

is to note that the first few homotopy groups π_k of $O(n)$ are

$k =$	0	1	2	3	4	5	6	7
$\pi_k(O(n)) =$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}$

Starting with $O(n)$, we can successively "kill" the lowest nonvanishing homotopy groups, thus obtaining first $SO(n)$ (the connected component), then $\mathrm{Spin}(n)$ (the universal cover) and finally $\mathrm{String}(n)$ (the 3-connected cover). Notice that with π_3 vanishing, $\mathrm{String}(n)$ cannot be a Lie group – but it can be a Lie 2-group.

Usually (see [5]), the definition of String_G includes also a condition on the boundary map $\pi_3(G) \xrightarrow{\partial} \pi_2(K(\mathbb{Z},2))$. Our definition above is really geared towards the application where $G = \mathrm{Spin}(n)$, for which we find it more natural.

Namely, recall that every short exact sequence of topological groups

$$0 \to A \to B \to C \to 0\,,$$

which happens to be a fibration, gives rise to a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(B) \longrightarrow \pi_n(C) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots\,.$$

In our case this becomes

$$\cdots \longrightarrow \pi_n(K(\mathbb{Z},2)) \longrightarrow \pi_n(\mathrm{String}_G) \longrightarrow \pi_n(G) \xrightarrow{\partial} \pi_{n-1}(K(\mathbb{Z},2)) \longrightarrow \cdots\,.$$

Demanding that $\pi_3(\mathrm{String}_G) = 0$ and assuming that also $\pi_2(\mathrm{String}_G) = 0$ (which we noticed above is the case for $G = \mathrm{Spin}(n)$) implies that we find inside this long exact sequence the short exact sequence

$$0 \longrightarrow (\pi_3(G) \simeq \mathbb{Z}) \xrightarrow{\partial} \mathbb{Z} \longrightarrow 0\,.$$

But this implies that the boundary map ∂ here is an isomorphism, hence that it acts on $\mathbb{Z}$ either by multiplication with $k = 1$ or $k = -1$. (This number is really the "level" governing this construction. If I find the time I will explain this later.)

In [5] this logic is applied the other way around. Instead of demanding that $\pi_3(\mathrm{String}_G) = 0$ it is demanded that the boundary map

$$\pi_3(G) \xrightarrow{\partial} \mathbb{Z}$$

is given by multiplication with the level, namely a specified element in $H^4(BG)$.

3 Connections, functorially: from here to there

Our statement

1	quantum (super) particles	are functors	$1\mathrm{Cob}_S \to \mathrm{Hilb}_S$

like

2	quantum (super) strings	are 2-functors	$\mathrm{Cob}_S^{\mathrm{ext}} \to 2\mathrm{Hilb}_S$

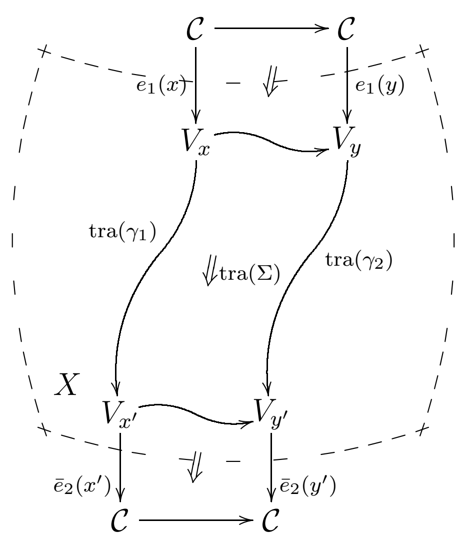
should be thought of as the quantization of the statement

1	parallel transport of	particles	is a functor	$\mathcal{P}_1(X) \to \mathrm{Vect}$

like

2	parallel transport of	strings	is a 2-functor	$\mathcal{P}_2(X) \to 2\mathrm{Vect}$

Here $\mathcal{P}_n(X)$ is the n-groupoid of n-paths in the space X.



In fact, a smooth spinor bundle with connection is enirely encoded [17] in a smooth functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$$

which sends paths to the parallel transport along them

$$\mathrm{tra} : (\, x \xrightarrow{\gamma} y \,) \mapsto \left(\begin{array}{ccc} V_x & \xrightarrow{\mathrm{tra}(\gamma)} & V_y \\ & & \\ (\text{fiber over } x) & \xrightarrow{\text{identitfy along path } \gamma} & (\text{fiber over } y) \end{array} \right)$$

and which factors locally through a functor with values in $\mathrm{Spin}(n)$:

$$\begin{tikzcd}
\mathcal{P}_1(Y) \arrow[r, "\pi"] \arrow[d, "\mathrm{triv}"', "(\text{local structure})"'] \arrow[dr, phantom, "\swarrow\sim"] & \mathcal{P}_1(X) \arrow[d, "\mathrm{tra}", "(\text{global structure})"] \\
\Sigma\mathrm{Spin}(n) \arrow[r, "\rho", "(\text{representation})"'] & \mathrm{Vect}
\end{tikzcd} .$$

4 2-Connections on $\mathrm{String}(n)$-2-bundles

The above has a straightforward categorification. But first let's look at what Stolz and Teichner did.

Stolz-Teichner's definition of a connection on a $\mathrm{String}(n)$-bundle. They observe the following important

Fact. *To every* $\mathrm{String}(n)$*-bundle is canonically associated a vonNeumann algebra bundle.*

But algebras, vonNeumann or not, naturally live not in a 1-category, but in the 2-category

$$\mathrm{Bim} .$$

So with the $\mathrm{String}(n)$-associated algebra bundle given, Stolz and Teichner consider parallel transport 2-functors

$$
\begin{tikzcd}
x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow\Sigma"] & y
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
A_x \arrow[rr, bend left, "N_\gamma"] \arrow[rr, bend right, "N_{\gamma'}"'] \arrow[rr, phantom, "\Downarrow"] & & A_y
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{(fiber over } x\text{)} \arrow[rr, bend left, "\text{bimodule for path } \gamma"] \arrow[rr, bend right, "\text{bimodule for path } \gamma'"'] \arrow[rr, phantom, "\Downarrow \text{ iso of bimodules}"] & & \text{(fiber over } y\text{)}
\end{tikzcd}
$$

Why? Remember: String(n) is the realization of the nerve of a 2-group $G_{(2)}$.

To get a $G_{(2)}$-associated 2-vector bundle,

$$
\begin{tikzcd}
\mathcal{P}_2(Y) \arrow[rr, "\pi"] \arrow[d, "\text{triv}"', "\text{(local structure)}"'] \arrow[drr, phantom, "\swArrow\sim"] & & \mathcal{P}_2(X) \arrow[d, "\text{tra}", "\text{(global structure)}"] \\
\Sigma G_{(2)} \arrow[rr, "\rho", "\text{(representation)}"'] & & 2\mathrm{Vect}
\end{tikzcd} ,
$$

we just need a 2-representation

$$\rho : \Sigma G_{(2)} \to 2\mathrm{Vect} \,.$$

5 The canonical 2-Representation

What is a 2-Vector space? Noticing that

$$\mathrm{Vect}_{\mathbb{C}} = {}_{\mathbb{C}}\mathrm{Mod}$$

and that Vect is again monoidal, we set

$$n\mathrm{Vect} = {}_{(n-1)\mathrm{Vect}}\mathrm{Mod}\,.$$

General 2Vect is large and unwieldy. But we have a chain of canonical inclusions

$$\mathrm{Intertwin} \hookrightarrow \mathrm{Bim} \hookrightarrow {}_{\mathrm{Vect}}\mathrm{Mod}$$

algebras	algebras
homomorphisms	bimodules
intertwiners	bimod. homom.

(and, by the way $\mathrm{KV2Vect} \hookrightarrow \mathrm{Bim} \hookrightarrow {}_{\mathrm{Vect}}\mathrm{Mod}$).

Canonical strict 2-Rep on Intertwin**.** Every strict 2-group $G_{(2)}$ comes from a crossed module of two ordinary groups

$$G_{(2)} = (\, H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)\,)\,.$$

For an ordinary representation

$$\rho_0 : \Sigma H \to \mathrm{Vect}$$

of H and with

$$A := \langle \mathrm{im}(\rho_0) \rangle$$

the algebra generated from that we get a 2-functor

$$\rho : \Sigma G_{(2)} \to \mathrm{Intertwin} \hookrightarrow \mathrm{Bim} \hookrightarrow 2\mathrm{Vect}$$

by

$$\rho \;:\; \begin{tikzcd} \bullet \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "g'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "h"] & \bullet \end{tikzcd} \;\mapsto\; \begin{tikzcd} A \arrow[r, bend left, "\alpha(g)"{name=U}] \arrow[r, bend right, "\alpha(g')"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\cdot\rho_0(h)"] & A \end{tikzcd}\,.$$

So:

For every strict 2-group $G_{(2)} = (t : H \to G)$ an ordinary representation ρ_0 of H induces a notion of $G_{(2)}$-associated 2-vector bundles.
Their typical fiber is (the module category of) the algebra generated from the image of ρ_0.

For the standard reps of $\hat{\Omega}_k\mathrm{Spin}(n)$ this algebra is a von Neumann type III factor.

6 Line 2-bundles versus $\mathrm{String}(n)$-2-bundles

A simple example to keep in mind are rank-1 2-vector bundles – "line 2-bundles". These are the 2-vector bundles canonically associated to the 2-group

$$G_{(2)} = \Sigma U(1) = (U(1) \to 1)$$

by the standard representation of $U(1)$ on $\mathbb{C}$:

- Let $G_{(2)} = \Sigma U(1)$,
- then $|G_{(2)}| \simeq PU(H)$;
- local semi trivialization of ρ-associated $\Sigma U(1)$-2-bundles are line bundle gerbes [18];
- indeed, these have the same classification as $PU(H)$-bundles, namely by classes in $H^3(X, \mathbb{Z})$.
- The typical fiber is (Morita!)-equivalent to $\mathbb{C}$, hence these are $K(H)$-bundles (where $K(H)$ is the algebra of finite rank operators.)

We can essentially send this example from $G_{(2)} = \Sigma U(1)$ to the string 2-group by using three results (two of which we already discussed):

Three results that clarify the situation for $\mathrm{String}(n)$:

- [9, 8]: There is a strict Fréchet Lie 2-group

  $$G_{(2)} = \mathrm{Spin}(n)_k = (t : \hat{\Omega}_k\mathrm{Spin}(n) \to P\mathrm{Spin}(n))$$

  such that $String(n)$ is the geometrical realization of its nerve

  $$\mathrm{String}(n) \simeq |\mathrm{Spin}(n)_k| \,.$$

- [6, 7]: $G_{(2)}$-2-bundles have the same classification as $|G_{(2)}|$-1-bundles.
- For every strict Lie 2-group there is a canonical 2-representation

$$G_{(2)} \xrightarrow{\rho} \mathrm{Intertwin} \hookrightarrow \mathrm{Bim} \hookrightarrow 2\mathrm{Vect} .$$

Caveat : Extending the canonical 2-rep from Lie to Fréchet is technically subtle. One has to pass from finite-dimensional vector spaces to Hilbert spaces and replace bimodule tensor products by Connes fusion.

	line 2-bundle	**String 2-bundle**
structure 2-group	$(U(1) \to 1)$	$(\hat{\Omega}\mathrm{Spin}(n) \to P\mathrm{Spin}(n))$
nerve of that	$PU(H)$	$\mathrm{String}(n)$
associated 2-vector bundle	finite-rank operators	von-Neumann algebras

Bottom line: 2-Vector bundles with 2-connection associated by the canonical 2-rep to $G_{(2)}$-principal 2-bundles give a good definition for $\mathrm{String}(n) \simeq |G_{(2)}|$ connections.

This definition is, in many respects, very similar to that by Stolz-Teichner (for instance: fake flatness!).

Is it equivalent? I don't know.

But it is useful, for instance for the following considerations.

7 Cartan connection with values in Lie n-algebras

When passing from the topological group $\mathrm{String}(n)$ to the 2-group $G_{(2)}$, we pass from the realm of pure topology to the realm of differential geometry. But it's still infinite-dimensional (Fréchet) differential geometry. We might want to go one step further and describe connections on $\mathrm{String}(n)$-bundles in terms of plain old differential forms.

⟵ topological	—Fréchet Lie—	Lie ⟶
2-functors on String(n)-associated bundles	associated $G_{(2)}$-2-vector transport	Lie($G_{(2)}$)-Cartan connection
Stolz-Teichner	BCSSW	Stasheff-S.

This can be done by passing from Lie n-groups to Lie n-algebras.

Differentiating parallel n-transport. For differentiating a parallel transport 2-functor

$$\mathcal{P}_2(X) \to \Sigma G_{(2)}$$

it is convenient to first pass to the corresponding curvature 3-transport [19, 18, 20].

	transport	**curvature**	**Bianchi**
	n	$(n+1)$	$(n+2)$
	$\mathrm{tra} : \mathcal{P}_n(X) \to \Sigma G_{(n)}$	$\mathrm{curv}_{\mathrm{tra}} : \Pi_{n+1}(X) \to \Sigma \mathrm{INN}(G_{(n)})$	$\mathrm{curv}_{\mathrm{curv}_{\mathrm{tra}}}$
	arbitrary	flat (as $(n+1)$-transport) $=$ Bianchi identity	trivial (as $(n+2)$-transport)
$n = 1$	$A \in \Omega^1(Y, \mathfrak{g})$	$F_A \in \Omega^2(Y, \mathfrak{g})$	$d_A F_A \in \Omega^3(Y, \mathfrak{g})$
$n = 2$	$\begin{pmatrix} A, \\ B \end{pmatrix} \in \begin{matrix} \Omega^1(Y,\mathfrak{g}) \\ \times\Omega^2(Y,\mathfrak{h}) \end{matrix}$	$\begin{pmatrix} \beta := F_A + t_* B, \\ H := d_A B \end{pmatrix} \in \begin{matrix} \Omega^2(Y,\mathfrak{g}) \\ \times\Omega^3(Y,\mathfrak{h}) \end{matrix}$	$\begin{pmatrix} d_A\beta - t_* H, \\ d_A H - \beta\wedge H \end{pmatrix} \in \begin{matrix} \Omega^3(Y,\mathfrak{g}) \\ \times\Omega^4(Y,\mathfrak{g}) \end{matrix}$

For one, smooth $(n+1)$-functors on the fundamental $(n+1)$-groupoid $\Pi_{n+1}(Y)$ are related, differentially, to *pseudo*-1-functors on the pair groupoid.

(illustration goes here)

So upon differentiation, our parallel transport n-functors turn into morphisms from the tangent algebroid

$$\mathrm{Vect}(Y)$$

to the Lie $(n+1)$-algebra

$$\mathrm{inn}(\mathfrak{g}_{(n)})$$

of inner derivations of the Lie n-algebra

$$\mathfrak{g}_{(n)} := \mathrm{Lie}(G_{(n)})\,.$$

	Lie n-groupoids	differentiation $\longmapsto$	**Lie n-algebras** ($\simeq$ n-term L_∞-algebras)	$\simeq$	**quasi free differential graded commutative algebras** (qfDGCAs)
morphism	$\begin{array}{c}\Sigma(\mathrm{INN}(G_{(n)}))\\ \uparrow F\\ \Pi_{n+1}(P)\end{array}$		$\begin{array}{c}\mathrm{inn}(\mathfrak{g}_{(n)})\\ \uparrow f\\ \mathrm{Vect}(P)\end{array}$		$\begin{array}{c}(\bigwedge^\bullet s\mathfrak{g}^*_{(n)}, d_{\mathrm{inn}(\mathfrak{g}_{(n)})})\\ \downarrow f^*\\ (\Omega^\bullet(P), d)\end{array}$
description	smooth pseudofunctor from pair groupoid of X to inner automorphisms of structure Lie n-group $G_{(n)}$		morphism of Lie n-algebroids $\simeq$ n-term L_∞-algebras from tangent algebroid of X to inner derivation Lie $(n+1)$-algebra $\mathfrak{g}_{(n)} := \mathrm{Lie}(G_{(n)})$		dual morphism of qfDGCAs

The n-connections which factor through $\mathfrak{g}_{(n)}$ itself, however, are the *flat* n-connections: all their curvature k-forms vanish:

n=1 n=2

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, hook] & \mathrm{inn}(\mathfrak{g}) \\
\mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0"'] & \mathrm{Vect}(X) \arrow[u, "(A)"]
\end{tikzcd}$$

n=2 n=3

$$\begin{tikzcd}
(\mathfrak{h} \to \mathfrak{g}) \arrow[r, hook] & \mathrm{inn}(\mathfrak{h} \to \mathfrak{g}) \\
\mathrm{Vect}(X) \arrow[u, "(A{,}B)", "{F_A+t_*B=0 \atop d_AB=0}"'] & \mathrm{Vect}(X) \arrow[u, "(A{,}B)"]
\end{tikzcd}$$

8 String and Chern-Simons connections

In fact, in terms of Lie 2-algebras, that big scary topological group $\mathrm{String}(n)$ becomes a small, handy Lie 2-algebra [8, 9]

$$\mathrm{string}(\mathfrak{g}) := \mathfrak{g}_\mu \,,$$

which is essentially just a simple Lie algebra $\mathfrak{g}$ together with a multiple of the canonical 3-cocycle μ.

This is just a first example of a general pattern [11, 13]:

- For every Lie $(n+1)$-cocycle μ on $\mathfrak{g}$ there is a Lie n-algebra
$$\mathfrak{g}_\mu \,.$$
- For every invariant degree $(n+1)$-polynomial k on $\mathfrak{g}$ there is a Lie $(2n+1)$-algebra
$$\mathrm{ch}_k(\mathfrak{g}) \,.$$

- For every transgressive element (Chern-Simons form) there is a short (weakly) exact sequence

$$0 \longrightarrow \mathfrak{g}_{\mu_k} \hookrightarrow \mathrm{cs}_k(\mathfrak{g}) \twoheadrightarrow \mathrm{ch}_k(\mathfrak{g}) \longrightarrow 0\,.$$

$$
\begin{tikzcd}
G \arrow[r] & EG \arrow[r] & BG & \text{top. spaces} \\
\mathrm{Disc}(G) \arrow[u, mapsto, "|\cdot|"'] \arrow[r, hook, "i"] \arrow[d, mapsto, "\mathrm{Lie}"'] & \mathrm{INN}(G) \arrow[u, mapsto, "|\cdot|"'] \arrow[r, two heads, "p"] \arrow[d, mapsto, "\mathrm{Lie}"'] & \Sigma G \arrow[u, mapsto, "|\cdot|"'] \arrow[d, mapsto] & \text{Lie groupoids} \\
(\bigwedge^\bullet(s\mathfrak{g}^*), d_{\mathfrak{g}}) \arrow[d, dash, "(\cdot)^*"', "\sim"] & (\bigwedge^\bullet(s\mathfrak{g}^* \oplus ss\mathfrak{g}^*), d_{\mathrm{inn}(\mathfrak{g})}) \arrow[l, two heads] \arrow[d, dash, "(\cdot)^*"', "\sim"] & (\bigwedge^\bullet(ss\mathfrak{g})^*) \arrow[l, hook'] & \text{free graded comm. algebras} \\
\mathfrak{g} \arrow[r, hook] & \mathrm{inn}(\mathfrak{g}) & & \text{Lie 2-algebras}
\end{tikzcd}
$$

cocycle | Chern-Simons element | inv. polynomial

$$
\begin{tikzcd}
 & 0 & & \\
0 & p^*k \arrow[u, mapsto, "d_{\mathrm{inn}(g)}"'] \arrow[d, bend right, "\tau"'] & k \arrow[l, mapsto, "p^*"'] & \text{elements in cohomology} \\
\mu \arrow[u, mapsto, "d_{\mathfrak{g}}"'] & cs \arrow[l, mapsto, "i^*"'] \arrow[u, mapsto, "d_{\mathrm{inn}(\mathfrak{g})}"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, hook] & \mathrm{inn}(\mathfrak{g}) \arrow[r, "="] & \mathrm{inn}(\mathfrak{g}) & \\
\mathfrak{g}_\mu \arrow[u, two heads] \arrow[r, hook] & \mathrm{cs}_k(\mathfrak{g}) \arrow[u, two heads] \arrow[r, two heads] \arrow[d, dash, "\sim"] & \mathrm{ch}_k(\mathfrak{g}) \arrow[u, two heads] & \text{Lie } (2n+1)\text{-algebras} \\
 & \mathrm{inn}(\mathfrak{g}_\mu) & &
\end{tikzcd}
$$

Baez-Crans | Chern-Simons | Chern

The Lie $(2n+1)$-algebra

$$\mathrm{cs}_k(\mathfrak{g})$$

is essentially defined by the fact that a connection form with values in it is the corresponding Chern-Simons form $\mathrm{CS}_k(A)$:

$$\begin{tikzcd}
 & \text{Baez-Crans} & \text{Chern-Simons} & \text{Chern} \\
1 & 2n & 2n+1 & 2n+1 \\
\mathfrak{g} & \mathfrak{g}_{\mu_k} \arrow[l, two heads] \arrow[r, hook] & \mathrm{cs}_k(\mathfrak{g}) \arrow[r, two heads] & \mathrm{ch}_k(\mathfrak{g}) \\
\mathrm{Vect}(X) \arrow[u, "{\substack{(A) \\ F_A=0}}"] & \mathrm{Vect}(X) \arrow[u, "{\substack{(A,B) \\ F_A=0 \\ dB+\mathrm{CS}_k(A)=0}}"] & \mathrm{Vect}(X) \arrow[u, "{\substack{(A,B,C) \\ C=dB+\mathrm{CS}_k(A)}}"] & \mathrm{Vect}(X) \arrow[u, "{\substack{(A,C) \\ dC=k((F_A)^{n+1})}}"]
\end{tikzcd} .$$

For the special case of interest here, $\mathrm{String}(n)$, where everything is controlled by the canonical 3-cocycle

$$\mu = \langle \cdot, [\cdot, \cdot] \rangle$$

and the corresponding characteristic class

$$k(F_A) = \langle F_A \wedge F_A \rangle$$

we have

$$
\begin{tikzcd}
n=1 & n=2 & n=3 & n=3 & \\
\mathfrak{g} \arrow[d, equal] & \mathrm{string}_k(\mathfrak{g}) \arrow[l, two heads] \arrow[r, hook] \arrow[d, equal, "\sim"] & \mathrm{inn}(\mathrm{string}_k(\mathfrak{g})) \arrow[d, equal, "\sim"] & & \text{as strict as possible} \\
\mathfrak{g} & \mathfrak{g}_k \arrow[l, two heads] \arrow[r, hook] & \mathrm{cs}_k(\mathfrak{g}) \arrow[r, two heads] & \mathrm{ch}_k(\mathfrak{g}) & \text{as small as possible} \\
\mathrm{Vect}(X) \arrow[u, "(A)", "F_A=0" description] & \mathrm{Vect}(X) \arrow[u, "(A{,}B)", "\substack{F_A=0 \\ dB+k\mathrm{CS}(A)=0}" description] & \mathrm{Vect}(X) \arrow[u, "(A{,}B{,}C)", "C=dB+k\mathrm{CS}(A)" description] & \mathrm{Vect}(X) \arrow[u, "(A{,}C)", "dC=\langle F_A\wedge F_A\rangle" description] &
\end{tikzcd}
,
$$

for

$$(A, B, C) \in \Omega^1(X, \mathfrak{g}) \times \Omega^2(X) \times \Omega^3(X)\,.$$

This should be one incarnation of the statement found in [5]:

String connections trivialize Chern-Simons theory.

References

[1] T. Killingback, *World Sheet Anomalies And Loop Geometry*, Nucl. Phys. B**288**, 578 (1987)

[2] M. Murray, D. Stevenson, *Higgs fields, bundle gerbes and string structures*, Commun. Math. Phys. **243** (2003), 541-555

[3] J. Mickelsson, *Kac-Moody groups, topology of the Dirac determinant bundle, and fermionization*, Commun. Math. Phys. **110** (1987), 173-183

[4] R. Brown and C. B. Spencer, *$\mathcal{G}$-groupoids, crossed modules, and the classifying space of a topological group*, Proc. Kon. Akad. v. Wet. **79** (1976), 296–302.

[5] S. Stolz, P. Teichner, *What is an elliptic object?*, in U. Tillmann (ed.) *Topology, Geometry and Quantum Field Theory* Proceedings of the 2002 Symposium in Honour of Graeme Segal, Lond. Math. Soc. Lect. Ser. 308, Cambridge (2004), available at `http://math.ucsd.edu/~teichner/Papers/Oxford.pdf`

[6] B. Jurco, *Crossed Module Bundle Gerbes; Classification, String Group and Differential Geometry*, `math.DG/0510078`

[7] J. Baez, D. Stevenson (in preparation)

[8] A. Henriques, *Integrating L_∞-Algebras*,

[9] J. Baez, A. Crans, D. Stevenson, U. S., *From Loop Groups to 2-Groups*,

[10] J. Baez, A. Lauda, *Higher-Dimensional Algebra V: Lie 2-Groups*, Theor. Appl. Cat. **12** (2004), 423-491.

[11] J. Baez, A. Crans, *Higher-Dimensional Algebra VI: Lie 2-Algebras*, Theor. Appl. Cat. **12** (2004), 492-528.

[12] T. Bartels, *Higher gauge theory: 2-Bundles*, PhD thesis, draft available at `http://math.ucr.edu/~toby/bundle.ps`

[13] U.S., J. Stasheff, *Structure of Lie n-Algebras* (in preparation)

[14] U.S., J. Stasheff, *Connections for Lie n-Algebras* (in preparation)

[15] U.S., J. Stasheff, *Zoo of Lie n-Algebras* (in preparation)

[16] D. Stevenson, *About the String Gerbe*, private notes, available at
`http://www.math.uni-hamburg.de/home/schreiber/string_gerbe_v1.pdf`

[17] U. S., K. Waldorf, *Parallel Transport and Functors*

[18] U. S., K. Waldorf, *Parallel 2-Transport and 2-Functors* (in preparation)

[19] U. S., *n-Curvature* (preprint)

[20] Anders Kock