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Quantum Symmetry and Braid Group Statistics in G-Spin Models

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Abstract. In two-dimensional lattice spin systems in which the spins take values in a finite group G we find a non-Abelian "parafermion" field of the form *order* $\times$ *disorder* that carries an action of the Hopf algebra $\mathscr{D}(G)$, the double of G. This field leads to a "quantization" of the Cuntz algebra and allows one to define amplifying homomorphisms on the $\mathscr{D}(G)$-invariant subalgebra that create the $\mathscr{D}(G)$-charges and generalize the endomorphisms in the Doplicher-Haag-Roberts program. The so-obtained category of representations of the observable algebra is shown to be equivalent to the representation category of $\mathscr{D}(G)$. The representation of the braid group generated by the statistics operator and the corresponding statistics parameter are calculated in each sector.

1. Introduction

Let G be a finite group. Consider G-valued spin configurations on the 2-dimensional square lattice, that is maps $\sigma:\mathbf{Z}^2 \to G$. The energy or Euclidean action fucntional of σ is

$$S(\sigma) = \sum_{\langle x,y\rangle} f(\sigma_x^{-1}\sigma_y)\,, \tag{1.1}$$

where the summation runs over nearest neighbour pairs of points in $\mathbf{Z}^2$ and $f:G \to \mathbf{R}$ is a function of the positive type. This kind of classical statistical systems or the corresponding quantum field theories will be called G-spin models.

Our first motivation for studying such models is that they provide the simplest examples of lattice field theories exhibiting quantum symmetry, that is a symmetry that cannot be described by a group. If $G = Z(N)$, G-spin models reduce to the well known Ising and $Z(N)$ spin models. $Z(N)$ models, or in general G-spin models with an Abelian group G, are known to have a symmetry group $G \times \hat{G}$, where $\hat{G}$ denotes the Pontryagin dual of G (the group of characters of G). The factor G is the symmetry related to the order parameters and is realized – if the temperature is not too low – on the Hilbert space by unitary operators $Q(g)$, $g \in G$ defined formally as follows. Let $\{|\sigma\rangle \mid \sigma : \mathbf{Z} \to G\}$ be the field diagonal basis then

$$Q(g)\,|\sigma\rangle = |\ldots, g\sigma_x, g\sigma_{x+1}, \ldots\rangle\,. \tag{1.2}$$

The factor $\hat{G}$ is the disorder symmetry and is related to the existence of solitons or kinks. If the temperature is not too high the solitons are stable therefore the Hilbert space decomposes into inequivalent sectors labelled by the (left) twist $\sigma_\infty\sigma_{-\infty}^{-1}$ in the boundary conditions. Let $P(h)$ denote the projection

$$P(h)\,|\sigma\rangle = \delta_{\sigma_\infty, h\sigma_{-\infty}}\,|\sigma\rangle\,. \tag{1.3}$$

Utilizing the Abelianness of G we can build up from the P-s the unitary operators

$$\hat{Q}(\hat{g}) = \sum_{h\in G} (\hat{g}; h)\,P(h)\,, \tag{1.4}$$

where $\hat{g} \in \hat{G}$ and $(\,;\,) : \hat{G} \times G \to U(1)$ is the canonical pairing. In this way $(g, \hat{g}) \mapsto Q(g)\hat{Q}(\hat{g})$ becomes a unitary representation of the order-disorder symmetry group $G \times \hat{G}$.

If the group G is non-Abelian the Pontryagin dual loses its meaning, so does the $\hat{Q}$, but the algebra generated by $Q(g)$ and $P(h)$ is still a symmetry algebra of the model. The relations

$$\begin{gathered} Q(g_1)Q(g_2) = Q(g_1g_2)\,, \qquad P(h_1)P(h_2) = \delta_{h_1,h_2}\cdot P(h_2)\,, \\ Q(g)P(h) = P(ghg^{-1})Q(g)\,, \end{gathered} \tag{1.5}$$

that can be obtained directly from (1.2–1.3), define the algebra $\mathscr{D}(G)$, the double of G. The same algebra occurs in an apparently quite different context, in orbifold constructions of conformal field theories [DVVV,B1]. In our context $\mathscr{D}(G)$ is the generalization of $G \times \hat{G}$ to non-Abelian groups and can be interpreted as the order-disorder symmetry of G-spin models.

$\mathscr{D}(G)$ is a quiasitriangular Hopf algebra and its basic properties were discussed in [DPR, B1]. In one respect $\mathscr{D}(G)$ differs from all quasitriangular Hopf algebras obtained as deformations of Lie algebras or Lie groups [Dr]. This is the existence of a *-operation on $\mathscr{D}(G)$, which makes it a Hopf *-algebra: the coproduct Δ and the counit ε are *-algebra maps and the antipode S also commutes with *. For the coproduct this means that if $\Delta(a) = a^{(1)}\otimes a^{(2)}$, then $\Delta(a^*) = a^{(1)*}\otimes a^{(2)*}$. Furthermore $\mathscr{D}(G)$ possesses an integral and is semisimple.

Once an action of a Hopf algebra H on a Hilbert space $\mathscr{H}$ is given one can define the "adjoint" action γ of H on the algebra $\mathscr{F}$ of operators $F : \mathscr{H} \to \mathscr{H}$ by the formula [M]

$$\gamma_a(F) = U(a^{(1)})FU(Sa^{(2)})\,, \qquad a \in \mathscr{D}(G)\,. \tag{1.6}$$

This action satisfies the following important properties:

$$\gamma_a(F_1F_2) = \gamma_{a^{(1)}}(F_1)\gamma_{a^{(2)}}(F_2)\,, \qquad \gamma_a(F)^* = \gamma_{\bar{a}}(F^*)\,, \tag{1.7}$$

where $\bar{a} = Sa^*$. The first property is known under different names: the coalgebra $\mathscr{D}(G)$ "measures" the algebra $\mathscr{F}$ [S] or as the module algebra property [M]. For us the importance of this relation is that if $\{F^i_\alpha\}$ is a D_α-multiplet, i.e. $\gamma_a(F^i_\alpha) = F^{i'}_\alpha D^{i'i}_\alpha(a)$

for some representations D_α, $\alpha = 1, 2$ of $\mathscr{D}(G)$, then $F^{ij} = F_1^i F_2^j$ is a $(D_1 \times D_2)$-multiplet, where

$$(D_1 \times D_2)(a)^{i'j',ij} := D_1^{i'i}(a^{(1)}) \cdot D_2^{j'j}(a^{(2)}). \tag{1.8}$$

The second property allows one to show that the adjoint $\{F^{i^*}\}$ of a D-multiplet $\{F^i\}$ is a $\bar{D}$-multiplet, where $\bar{D} = D^T \circ S$ is the contragredient representation. For a D-multiplet $\{F^i\}$ the action (1.6) is equivalent to the generalized commutation relations of [BMT]

$$U(a) F^i = F^{i'} D^{i'i}(a^{(1)}) U(a^{(2)}). \tag{1.9}$$

This latter relation can be applied also in the case of quasi Hopf algebras when the measuring relation cannot be true and therefore the adjoint action (1.6) ipso facto cannot be used. For an even more general kind of action of a symmetry algebra see [R1].

Our second motivation for studying G-spin models is to carry out – at least partially – the Doplicher-Haag-Roberts program for exploring the symmetries of the model merely from "observable" data [DHR, DR]. In this approach the internal symmetries are treated as superselected ones therefore one starts from the "$\mathscr{D}(G)$-invariant" subalgebra $\mathscr{A}$ of $\mathscr{F}$,

$$\mathscr{A} := \{A \in \mathscr{F} \mid \gamma_a(A) = \varepsilon(a) A,\ \forall a \in \mathscr{D}(G)\}, \tag{1.10}$$

and interprets it as the algebra of observables. Then the equivalence classes of C^*-representations of $\mathscr{A}$ defines in an abstract way the spectrum of the charge. If the set of representations of $\mathscr{A}$ can be given a monoidal structure then the set of charges becomes the dual of a symmetry group if the spacetime dimension $d \geq 3$ or the dual of a Hopf algebra or something more general [MS1] if $d = 2$. Such a monoidal structure can be given if one considers only those representations π of $\mathscr{A}$ that can be obtained from a fixed faithful irreducible representation π_0 by the application of a localized endomorphism $\varrho : \mathscr{A} \to \mathscr{A}$, that is $\pi = \pi_\varrho \equiv \pi \circ \varrho$.

In concrete models it is technically very difficult to find endomorphisms ϱ that are not automorphisms. The only examples seem to be the chiral Ising model [MS2] and certain generalizations of it [FGV]. In the locally finite dimensional case, that is in lattice models, where all local algebras $\mathscr{A}(\Lambda)$ of observables localized in the finite interval Λ are finite dimensional, this problem turns out to be more than a technicality: there exist no (injective) endomorphisms ϱ localized in Λ that are not automorphisms. Since G-spin models belong to this class of models and the presence of an action of $\mathscr{D}(G)$ suggests a fairly non-Abelian superselection structure, the question naturally arises: what kind of generalization of endomorphisms can create the $\mathscr{D}(G)$-charges?

The answer is to use amplifying homomorphisms $\mu : \mathscr{A} \to M_n(\mathscr{A})$, that is C^*-algebra homomorphisms from $\mathscr{A}$ to the finite matrix amplification $M_n(\mathscr{A}) = \mathscr{A} \otimes M_n$. Throughout the paper we use the notation M_n for the algebra of $n \times n$ complex matrices and $M_n(\mathscr{A})$ for the algebra of $n \times n$ matrices with entries in $\mathscr{A}$. Given an amplifying homomorphism μ we can define the representation $\pi_\mu = (\pi_0 \otimes \mathrm{id}) \circ \mu$ of $\mathscr{A}$ on the Hilbert space $\mathscr{H}_0 \otimes \mathbf{C}^n$, where $\mathscr{H}_0$ is the representation space of π_0. The monoidal structure on such representations is defined by the product $\pi_\mu \times \pi_\nu := \pi_{\mu \times \nu}$ with a natural product $\mu \times \nu$. In G-spin models we show that for a special class of amplyifying homomorphisms this monoidal structure reproduces the product of representations of $\mathscr{D}(G)$. This special class of amplifying homomorphisms consists of μ-s that have the form $\mu = \mu_F$, where

$$\mu_F^{ij}(A) := \sum_k F^{ik} A F^{jk^*}, \tag{1.11}$$

for some multiplet $F^{ij} \in \mathscr{F}$, $i, j = 1, \ldots, n$. F is called a multiplet matrix if for each fixed value of i, $\{F^{ij}\}$ is a multiplet under the action of γ and the following relations hold:

$$\sum_k F^{ki^*} F^{kj} = \delta^{ij} \cdot \mathbf{1} = \sum_k F^{ik} F^{jk^*}. \tag{1.12}$$

We call relations (1.12) the F-algebra relations and they play the role for the amplifying homomorphisms which the Cuntz algebra [C] did for endomorphisms [DR]. Since in our case the F-algebra relations lead to a "quantized" symmetry, namely to the Hopf algebra $\mathscr{D}(G)$, relations (1.12) could be called a quantization of the Cuntz algebra.

The concrete fields F found in the G-spin model that satisfy the F-algebra have the form $F = \textit{order} \times \textit{disorder}$ and in this respect are generalizations of the parafermion fields of $Z(N)$ models [FZ] and especially that of the Jordan-Wigner transformation in the Ising model. They satisfy Fröhlich's braided commutation relations [Frö],

$$F^{i_1 j_1}(x) F^{i_2 j_2}(y) = \sum_{j_1' j_2'} F^{i_2 j_2'}(y) F^{i_1 j_1'}(x) B^{\pm j_1 j_2}_{j_2' j_1'}, \quad \text{if} \quad x \lessgtr y, \tag{1.13}$$

with numerical R-matrices $B^\pm$ obtained from the universal R-matrix of $\mathscr{D}(G)$ by applying the representation of $\mathscr{D}(G)$ according to which the F transforms. The representations of the braid group one obtains in this way from $\mathscr{D}(G)$ occurred also in a construction of 3-manifold invariants in [AC].

The paper is organized as follows. In Sect. 2 first we review the properties of $\mathscr{D}(G)$, then we define the non-local field algebra $\mathscr{F}$ generated by order and disorder fields and the action γ of $\mathscr{D}(G)$ on $\mathscr{F}$. Special multiplet matrices obeying the F-algebra and Fröhlich's braid relation are constructed here. From Sect. 3 we start analyzing the model from the DHR point of view. In Sect. 3 the local net structure of the observable algebra $\mathscr{A}$ is investigated. We find algebraic generators for $\mathscr{A}$, study the inclusions of local observable algebras $\mathscr{A}(\Lambda)$, prove Haag duality and triviality of the relative commutant of $\mathscr{A}$ in $\mathscr{F}$. in Sect. 4 the notions of a multiplet matrix F and the associated special amplifying homomorphism μ_F are introduced. The main result here is the equivalence of two braided monoidal C^*-categories with subobjects, direct sums and conjugates: the category $\mathbf{Rep}_0\,\mathscr{A}$ of representations π_μ of $\mathscr{A}$ with $\mu = \mu_F$ for some multiplet matrix F on the one hand and the category $\mathbf{Rep}\,\mathscr{D}(G)$ of representations of $\mathscr{D}(G)$ on the other hand. In Sect. 5 the notions of the statistics operator, the left inverse, and the statistics parameter are discussed in the general "amplified" circumstances. Applying them to G-spin models we compute the statistical dimension and statistics phase in all the $\mathscr{D}(G)$-sectors for an arbitrary finite group G. We reproduce Longo's result [L] that the index is equal to the square of the statistical dimension. Finally we point out that the representations of the modular group defined in [R2] and [B2] coincide in G-spin models.

In the end let us call the reader's attention to some points that are not contained in this paper. If we take for f in formula (1.1) a character on the group then for non-Abelian groups the model has a larger symmetry than the one generated by Q-s and P-s. There will be analogous symmetry operators Q^R and P^R that act by right translation on the group and measuring right twist $\sigma_{-\infty}^{-1}\sigma_\infty$, respectively.

The full symmetry is then an amalgamation of two, left and right, copies of $\mathscr{D}(G)$. The observable algebra becomes smaller and the structure of the sectors is more complicated. In this paper we study only the left $\mathscr{D}(G)$ symmetry. There is another direction where our discussion could be generalized. In the chiral Ising model a non-Abelian sector exists [MS2] which has to have an analogue in the lattice Ising model. This kind of sectors, however, are out of the scope of the present paper because $\mathscr{D}(Z(2))$ is the group algebra of $Z(2) \times Z(2)$. Last but not least we would like to warn the reader that our discussion of superselection sectors is purely kinematical. We have not proved for any particular Hamiltonian or transfer matrix that all of the $\mathscr{D}(G)$-sectors actually exist. All the information on the dynamics is comprised in the assumption that the vacuum representation π_0 obeys Haag duality which is the typical condition for unbroken symmetries. Experience with $Z(N)$ models suggests that such a representation π_0 should be found at intermediate temperatures where neither the order nor the disorder symmetries are broken and this phenomenon is accompanied with criticality. To our knowledge critical points in non-Abelian spin models are not yet known. There is an indication, however, that the integrable dynamics found in [SV] in the S_3-spin model is critical and has $\mathscr{D}(S_3)$ symmetry.

2. The Field Algebra of G-Spin Models

After summarizing the main properties of the quasitriangular C^*-Hopf algebra $\mathscr{D}(G)$ we analyze G-spin models in the spirit of traditional quantum field theory. We define its field algebra $\mathscr{F}$ then the action of the symmetry algebra $\mathscr{D}(G)$ on $\mathscr{F}$. The observable algebra $\mathscr{A}$ is obtained as the $\mathscr{D}(G)$-invariant subalgebra of $\mathscr{F}$ and its charged representations can be found in the reduction of a vacuum representation π of $\mathscr{F}$. Special fields, called non-Abelian parafermions, satisfying the F-algebra relations are introduced, the existence of which will be important in later sections when we will analyze the model from the DHR theory point of view.

2.1. The Double $\mathscr{D}(G)$ of a Finite Group G

Let $\mathbf{C}(G)$ denote the algebra of complex functions on G and $\mathbf{C}G$ be the group algebra. Then $\mathscr{D}(G)$ as an algebra is defined as the crossed product of $\mathbf{C}(G)$ and $\mathbf{C}G$ with respect to the adjoint action of the latter on the former. Using the basis elements $(g,h) \equiv P(g)Q(h)$ the multiplication rule $m\colon\mathscr{D}(G)\otimes\mathscr{D}(G) \to \mathscr{D}(G)$ is the following:

$$m((g_1,h_1)\otimes(g_2,h_2)) \equiv (g_1,h_1)\cdot(g_2,h_2) = \delta_{g_1h_1,h_1g_2}\cdot(g_1,h_1h_2)\,. \tag{2.1}$$

The unit element of $\mathscr{D}(G)$ is $\mathbf{1} = (E,e) \equiv \sum_g (g,e)$, where E and e are the unit elements of $\mathbf{C}(G)$ and $\mathbf{C}G$, respectively. $\mathscr{D}(G)$ becomes a unital *-algebra by defining the *-operation as $(g,h)^* = (h^{-1}gh,h^{-1})$ on the basis elements and extending antilinearly to $\mathscr{D}(G)$.

The coproduct $\Delta\colon\mathscr{D}(G) \to \mathscr{D}(G)\otimes\mathscr{D}(G)$, the counit $\varepsilon\colon\mathscr{D}(G) \to \mathbf{C}$ and the antipode $S\colon\mathscr{D}(G) \to \mathscr{D}(G)$ are defined on the basis elements as

$$\begin{aligned}\Delta(g,h) &= \sum_{f\in G}(f,h)\otimes(f^{-1}g,h) \equiv (g,h)^{(1)}\otimes(g,h)^{(2)}\,,\\ \varepsilon(g,h) &= \delta_{g,e}\,,\qquad S(g,h) = (h^{-1}g^{-1}h,h^{-1})\end{aligned} \tag{2.2}$$

and are linearly extended to $\mathscr{D}(G)$. One proves that Δ and ε are *-algebra homomorphisms, S is a linear *-algebra antihomomorphism, $S^2 = \mathrm{id}_{\mathscr{D}(G)}$. Moreover, the coproduct is coassociative and the counit and the antipode obey the relations:

$$\lambda \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = \lambda \circ (\mathrm{id} \otimes \varepsilon) \circ \Delta\,, \tag{2.3}$$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta\,, \tag{2.4}$$

where λ denotes the map of multiplication by scalars and $\eta : \mathbf{C} \to \mathscr{D}(G)$ is the unit preserving algebra homomorphism. Due to these properties $\mathscr{D}(G)$ becomes a Hopf *-algebra [S].

If G is non-Abelian, the Hopf algebra $\mathscr{D}(G)$ is neither commutative nor cocommutative. But a weaker cocommutativity holds, namely, $\mathscr{D}(G)$ is a quasitriangular Hopf algebra [M]. Indeed, there exists an invertible universal R-matrix $R \in \mathscr{D}(G) \otimes \mathscr{D}(G)$,

$$R = \sum_{g \in G} (g, e) \otimes (E, g)\,, \qquad R^{-1} = \sum_{g \in G} (g, e) \otimes (E, g^{-1})\,, \tag{2.5}$$

with the properties

$$\Delta'(a) = R \cdot \Delta(a) \cdot R^{-1}\,, \qquad \forall a \in \mathscr{D}(G)\,, \tag{2.6a}$$

$$(\Delta \otimes \mathrm{id}) R = R_{13} \cdot R_{23}\,, \qquad (\mathrm{id} \otimes \Delta) R = R_{13} \cdot R_{12}\,, \tag{2.6b}$$

where Δ' is the coproduct with interchanged tensor product factors.

An integral $z \in H$ in a Hopf algebra H is defined by the property $a \cdot z = \varepsilon(a) \cdot z$, $a \in H$. For finite dimensional Hopf algebras the linear space of integrals is one-dimensional. The semisimplicity of a finite dimensional Hopf algebra is equivalent to the statement $\varepsilon(z) \neq 0$ [S]. The element $z = \frac{1}{|G|} \sum_g (e, g)$ is an integral in $\mathscr{D}(G)$ with the property $\varepsilon(z) = 1$. As a consequence $\mathscr{D}(G)$ is semisimple, that is, all the left $\mathscr{D}(G)$-modules are completely reducible. Therefore using standard results [CR] one concludes that every simple left $\mathscr{D}(G)$-module occurs in the left regular $\mathscr{D}(G)$-module with multiplicity equal to its dimension.

2.1. Proposition. *Let $C_{\tilde g} = \{h \in G \mid h\tilde g = \tilde g h\} \subset G$ be the centralizer subgroup of $\tilde g \in G$ and let $f_1, f_2, \ldots, f_N$ be representatives of the left cosets of $C_{\tilde g}$ with $f_1 = e$ and $N = |G : C_{\tilde g}|$. Let $\{v_1, v_2, \ldots, v_n\} \subset \mathbf{C}C_{\tilde g}$ be basis vectors of an irreducible representation π of $C_{\tilde g}$ in $\mathbf{C}C_{\tilde g}$ with $n = \dim \pi$. Then*

i) *a linear basis of the irreducible subrepresentation $D_{(\tilde g, \pi)}$ can be given as*

$$\{(f_i \tilde g f_i^{-1}, f_i v_t) \mid i = 1, \ldots, N; t = 1, \ldots, n\} \subset \mathscr{D}(G)\,, \tag{2.7}$$

ii) *the matrix elements in this basis and the character of the representation $D_{(\tilde g, \pi)}$ are*

$$D^{g_1 t_1, g_2 t_2}_{(\tilde g, \pi)}(g, h) = \delta_{g, g_1} \cdot \delta_{g, h g_2 h^{-1}} \cdot \pi^{t_1, t_2}(f_1^{-1} h f_2)\,, \quad f_i \tilde g f_i^{-1} = g_i,\ i = 1, 2\,; \tag{2.8a}$$

$$\Phi_{(\tilde g, \pi)}(g, h) = \delta_{g \in A_{\tilde g}} \cdot \delta_{h \in C_g} \cdot \chi_\pi(f^{-1} h f)\,, \qquad f \tilde g f^{-1} = g\,, \tag{2.8b}$$

where the index pairs $g_i t_i$, $i = 1, 2$ refer to the vectors $(g_i, f_i v_{t_i})$ of (2.7), χ_π denotes the character of the $C_{\tilde g}$-representation π, and $A_{\tilde g}$ is the conjugacy class of $\tilde g$ in G.

iii) *The irreducible representations of $\mathscr{D}(G)$ are characterized by a conjugacy class A of G and by an irreducible representation π of the centralizer subgroup $C_{\tilde g}$ of $\tilde g \in A$ in G.*

Proof. Left to the reader. $\square$

The minimal central idempotent in $\mathscr{D}(G)$ corresponding to the irreducible representation $r = (A, \pi)$ of dimension $n_r = |A| \cdot \dim \pi$ is

$$M_r = \frac{n_r}{|G|} \sum_{g \in A} \sum_{h \in C_g} \bar{\chi}_\pi(f^{-1} h f) \cdot (g, h), \qquad f \tilde{g} f^{-1} = g, \tag{2.9}$$

where $\tilde{g} \in A$ is fixed, but arbitrary. Notice that the linear map $\mu : \mathscr{D}(G) \to \mathbf{C}$ defined by $\mu(g, h) = \delta_{h,e}$ determines a symmetric associative non-degenerate bilinear form $\beta : \mathscr{D}(G) \times \mathscr{D}(G) \to \mathbf{C}$ through $\beta(a, b) := \mu(a \cdot b)$ which in turn determines a scalar product $\langle\, , \rangle : \mathscr{D}(G) \times \mathscr{D}(G) \to \mathbf{C}$ by $|G| \cdot \langle a, b \rangle = \beta(a^*, b)$, $a, b \in \mathscr{D}(G)$. Since the dual of the basis element (g, h) with respect to β is just $(g, h)^*$, this scalar product is consistent with the *-operation on $\mathscr{D}(G)$. In terms of coefficients with respect to the chosen basis the scalar product looks like

$$\langle a^1, a^2 \rangle = \frac{1}{|G|} \sum_{g,h \in G} \bar{a}^1_{g,h} \cdot a^2_{g,h}, \qquad a^i = \sum_{g,h \in G} a^i_{g,h} \cdot (g, h), \qquad i = 1, 2, \tag{2.10}$$

where bar means complex conjugation. This scalar product makes $\mathscr{D}(G)$ and its dual $\mathscr{D}(G)^*$ to be a Hilbert space. Considering the elements of $\mathscr{D}(G)$ as operators on $\mathscr{D}(G)$ by left multiplication they acquire an induced operator norm. This latter norm makes $\mathscr{D}(G)$ a C^*-algebra and this is the unique such norm consistent with the given *-operation.

The trivial representation of $\mathscr{D}(G)$ is the counit ε. The integral z is just the central projector of the trivial representation. The contragredient representation $\bar{D}$ of the representation D is defined by the help of the antipode $S : \bar{D}(a) = D(Sa)^T$, $a \in \mathscr{D}(G)$, where T refers to the transposed matrix. A representation D is unitary if it is a *-representation, i.e. if $D(a)^* = D(a^*)$, $a \in \mathscr{D}(G)$. In the sequel a representation of $\mathscr{D}(G)$ will always mean a *-representation. The set of equivalence classes of irreducible (unitary) representations of $\mathscr{D}(G)$ is denoted by $\widehat{\mathscr{D}(G)}$.

Using the matrix elements given in (2.8) and orthogonality relations of matrix elements for finite groups, one verifies the following orthogonality relations:

$$\langle D_r^{ij}, D_{r'}^{i'j'} \rangle = \frac{1}{n_r} \cdot \delta_{r,r'} \cdot \delta_{i,i'} \cdot \delta_{j,j'}, \qquad \langle \Phi_r, \Phi_{r'} \rangle = \delta_{r,r'}, \tag{2.11}$$

$$\frac{1}{|G|} \sum_{r \in \widehat{\mathscr{D}(G)}} \sum_{i,j=1}^{n_r} n_r \cdot \overline{D_r^{ij}(g_1, h_1)} D_r^{ij}(g_2, h_2) = \delta_{g_1,g_2} \cdot \delta_{h_1,h_2}, \tag{2.12}$$

where bar means complex conjugation.

The product $D_1 \times D_2$ of two representations D_1 and D_2 is defined by the help of the comultiplication in $\mathscr{D}(G) : (D_1 \times D_2)(a) = D_1(a^{(1)}) \otimes D_2(a^{(2)})$. Using orthogonality relations of irreducible characters one can decompose product representations into a direct sum of irreducible ones:

$$D_{r_1} \times D_{r_2} = \bigoplus_{r_3 \in \widehat{\mathscr{D}(G)}} N^{r_3}_{r_1 r_2} D_{r_3}, \qquad N^{r_3}_{r_1 r_2} = \langle \Phi_{r_3}, \Phi_{r_1} \times \Phi_{r_2} \rangle. \tag{2.13}$$

For all representations D of $\mathscr{D}(G)$ we have $D \times \varepsilon = D = \varepsilon \times D$. Moreover, the trivial representation ε occurs in the product of two irreducible representations if and only if they are contragredient to each other. In that case the multiplicity of ε in the product representation is one. To prove this one notes that if $r_i = (A_i, \pi_i)$, $i = 1, 2$

then using the explicit form of the product character and the counit one gets that $N^0_{r_1 r_2} = \delta_{A_1, A_2^{-1}} \cdot \delta_{\pi_1, \bar{\pi}_2}$, where $\bar{\pi}$ is the contragredient representation of π. Now the statement follows from the relation valid for the contragredient $\bar{\Phi}$ of a character Φ:

$$\bar{\Phi}_{(A,\pi)}(g,h) \equiv \Phi_{(A,\pi)}(S(g,h)) = \Phi_{(A,\pi)}(g^{-1},h^{-1}) = \Phi_{(A^{-1},\bar{\pi})}(g,h)\,. \tag{2.14}$$

This equation also implies the equivalence between the contragredient representation $\bar{D}_r$ of D_r and the representation $D_{\bar{r}}$ given by $\bar{r} = (A^{-1}, \bar{\pi})$. The equivalence is induced by the map given on the basis elements of the representation spaces as

$$(g_i, v_t) \mapsto (g_i^{-1}, v_t)\,; \qquad i = 1, \ldots, |A|\,,\ t = 1, \ldots, \dim \pi\,. \tag{2.15}$$

The set of finite dimensional matrix representations of $\mathscr{D}(G)$ are the objects of a category $\mathbf{Rep}\,\mathscr{D}(G)$ in which the set of morphisms from D_2 to D_1 is the space

$$(D_1 \mid D_2) = \{t \in \mathrm{Mat}(n_1 \times n_2 \mid \mathbf{C}) \mid D_1(a)t = tD_2(a), a \in \mathscr{D}(G)\} \tag{2.16}$$

of intertwiners from D_2 to D_1. This category is a strict monoidal braided C^*-category with direct sums, subobjects and conjugates [DR, M]. The star operation and the conjugation are contravariant functors from $\mathbf{Rep}\,\mathscr{D}(G)$ to itself acting on the objects and morphisms respectively as follows: ${}^*\!: D \mapsto D$, $t \mapsto t^*$, ${}^-\!: D \mapsto \bar{D}$, $t \mapsto t^T$. The strict monoidal structure is given by the covariant functor $\times\!: \mathbf{Rep}\,\mathscr{D}(G) \times \mathbf{Rep}\,\mathscr{D}(G) \to \mathbf{Rep}\,\mathscr{D}(G)$, $D_1 \times D_2 = (D_1 \otimes D_2) \circ \Delta$, $t_1 \times t_2 = t_1 \otimes t_2$. The functors * and $^-$ are monoidal: $^* \circ \times = \times \circ (^*,^*)$, $^- \circ \times \sim \times \circ (^-,^-)$, where the natural equivalence $\sim$ is given by $(D_1, D_2) \mapsto (\bar{D}_1 \otimes \bar{D}_2)(R)$, with the universal R-matrix R. The braiding structure is given by the natural equivalence $(D_1, D_2) \mapsto B(D_1, D_2) = P_{12} \cdot (D_1 \otimes D_2)(R)$ between the functors $\times$ and $\times^{\mathrm{op}}$, where P_{12} interchanges the factors in the tensor product of the representation spaces of D_1 and D_2, and $\times^{\mathrm{op}}$ is the product in the opposite order.

2.2. *The Definition of the Field Algebra*

For a finite chain of length n the state space $\mathscr{H}_n$ is the tensor product of n copies of $\mathbf{C}(G)$. The vectors $\{|\sigma\rangle \mid \sigma\!: \{1, \ldots, n\} \to G\}$ form an orthonormal basis in $\mathscr{H}_n$. The full operator algebra on $\mathscr{H}_n$ is generated by order parameters $\delta_g(x)$, $g \in G$, $x \in \{1, \ldots, n\}$ and disorder or kink creating operators $\varrho_g(l)$, $g \in G$, $l \in \{\frac{1}{2}, \ldots, n - \frac{1}{2}\}$ defined as follows:

$$\delta_g(x)\,|\sigma\rangle = \delta_{g,\sigma_x} \cdot |\sigma\rangle\,, \qquad \varrho_g(l)\,|\sigma\rangle = |\sigma_1, \ldots, \sigma_{l-\frac{1}{2}}, g\sigma_{l+\frac{1}{2}}, \ldots, g\sigma_n\rangle\,. \tag{2.17}$$

Notice that $\sum_g \delta_g(x) = \mathbf{1} = \varrho_e(l)$. The multiplication and commutation relations of these operators lead us to the following

2.2. Definition. The local field algebra $\mathscr{F}_{\mathrm{loc}}$ of a G-spin model is a unital associative algebra over $\mathbf{C}$ given by the following presentation: the generators are the unit element $\mathbf{1}$ and the elements of the set $\{\delta_h(x), \varrho_g(l) \mid h, g \in G; x \in \mathbf{Z}, l \in \mathbf{Z} + \frac{1}{2}\}$. The relations are

$$\delta_g(x)\delta_h(x) = \delta_{g,h} \cdot \delta_h(x)\,, \qquad \varrho_g(l)\,\varrho_h(l) = \varrho_{gh}(l)\,, \tag{2.18a}$$

$$\sum_{g\in G} \delta_g(x) = \mathbf{1} = \varrho_e(l)\,, \tag{2.18b}$$

$$\delta_g(x)\delta_h(x') = \delta_h(x')\delta_g(x)\,, \tag{2.18c}$$

$$\begin{aligned} \varrho_g(l)\delta_h(x) &= \begin{cases} \delta_{gh}(x)\varrho_g(l)\,, & l < x\,; \\ \delta_h(x)\varrho_g(l)\,, & l > x\,; \end{cases} \\ \varrho_g(l)\,\varrho_h(l') &= \begin{cases} \varrho_h(l')\,\varrho_{h^{-1}gh}(l)\,, & l > l' \\ \varrho_{ghg^{-1}}(l')\,\varrho_g(l)\,, & l < l' \end{cases} \end{aligned} \tag{2.18d}$$

for $x, x' \in \mathbf{Z}$; $l, l' \in \mathbf{Z} + \frac{1}{2}$ and $g, h \in G$.

The *-operation is defined on the generators as $\delta_h^*(x) = \delta_h(x)$, $\varrho_g^*(l) = \varrho_{g^{-1}}(l)$ and is extended antilinearly and antimultiplicatively. In this way $\mathscr{F}_{\mathrm{loc}}$ becomes a unital *-algebra. $\mathscr{F}_{\mathrm{loc}}$ can be extended to a C^*-algebra $\mathscr{F}$, called the field algebra, in the following way. First, for any finite subset $\Lambda \subset \frac{1}{2}\mathbf{Z}$ we define the subalgebra $\mathscr{F}(\Lambda)$ of $\mathscr{F}_{\mathrm{loc}}$ as

$$\mathscr{F}(\Lambda) = \langle \delta_g(x), \varrho_g(l) \mid x, l \in \Lambda, g \in G \rangle\,. \tag{2.19}$$

In particular, we consider the case when $\Lambda = \Lambda_{a,b} = \left\{ s \in \frac{1}{2}\mathbf{Z} \,\middle|\, a \le s \le b \right\}$ is an interval, $a, b \in \frac{1}{2}\mathbf{Z}$, $a \le b$. $\Lambda_{a,b}$ is called open (closed) from the left – and similarly from the right – if a is half-integer (integer). Now let us consider an increasing sequence of intervals $\Lambda_n \equiv \Lambda_{l_n, x_n}$, $n \in \mathbf{N}$, that are open from the left and are closed from the right and have the recursion relations

$$\Lambda_{n+1} = \begin{cases} \Lambda_{l_n, x_n+1}\,, & n \in 2\mathbf{N} - 1 \\ \Lambda_{l_n - 1, x_n}\,, & n \in 2\mathbf{N} \end{cases} \tag{2.20}$$

with $x_1 = 0$, $l_1 = -\frac{1}{2}$. The corresponding subalgebras $\mathscr{F}(\Lambda_n)$, $n \in \mathbf{N}$ are full matrix algebras, they can be identified with $M_{|G|^n}$ using (2.17). Moreover, acting on the finite dimensional Hilbert space $\mathscr{H}_n$ a norm is induced on $\mathscr{F}(\Lambda_n)$,

$$\|F\| = \sup_{\|\psi\|=1} \|F\psi\|\,, \qquad \psi \in \mathscr{H}_n\,, \;\; F \in \mathscr{F}(\Lambda_n)\,. \tag{2.21}$$

Therefore $\mathscr{F}(\Lambda_n)$, $n \in \mathbf{N}$ become finite dimensional C^*-algebras. The natural embeddings $\iota_n : \mathscr{F}(\Lambda_n) \to \mathscr{F}(\Lambda_{n+1})$, $n \in \mathbf{N}$, that identify the δ and ϱ generators, are norm preserving.

2.3. Definition. The field algebra $\mathscr{F}$ of a G-spin model is the C^*-algebra given by the C^*-inductive limit

$$\mathscr{F} = \overline{\bigcup_{n\in\mathbf{N}} \mathscr{F}(\Lambda_n)}\,.$$

2.3. The Action of $\mathscr{D}(G)$ on the Field Algebra

An action γ of $\mathscr{D}(G)$ on the field algebra $\mathscr{F}$ has to be compatible with the algebraic structure of $\mathscr{F}$. In ordinary cases when the symmetry algebra is a group algebra or a Lie algebra this requirement means that a product of two field multiplets, both carrying a representation of the symmetry algebra, is transformed by the tensor product representation. In case of Hopf algebras the product of representations is given by the help of the coproduct. Therefore this rule has to govern the transformation properties of products of field multiplets.

2.4. Proposition. *The map $\gamma: \mathscr{D}(G) \times \mathscr{F}_{\text{loc}} \to \mathscr{F}_{\text{loc}}$ given on the generators of the field algebra $\mathscr{F}_{\text{loc}}$ as*

$$\gamma_{(g,h)}(\delta_f(x)) = \delta_{g,e} \cdot \delta_{hf}(x), \qquad \gamma_{(g,h)}(\varrho_f(l)) = \delta_{g,hfh^{-1}} \cdot \varrho_{hfh^{-1}}(l), \tag{2.22}$$

for $x \in \mathbf{Z}$, $l \in \mathbf{Z} + \frac{1}{2}$ and $g, h, f \in G$, extended for products of generators inductively in the number of generators by the rule

$$\gamma_{(g,h)}(f \cdot F) = \gamma_{(g,h)^{(1)}}(f) \cdot \gamma_{(g,h)^{(2)}}(F), \tag{2.23}$$

where f is one of the generators in $\mathscr{F}_{\text{loc}}$ and F is a finite product of them, finally, linearly extended both in $\mathscr{D}(G)$ and $\mathscr{F}_{\text{loc}}$, defines an automorphic action of $\mathscr{D}(G)$ on $\mathscr{F}_{\text{loc}}$, that is:
i) *$\mathscr{F}_{\text{loc}}$ is a left $\mathscr{D}(G)$-module algebra with respect to the map γ, which means that γ is a bilinear map satisfying the relations*

$$\gamma_a(F_1 \cdot F_2) = \gamma_{a^{(1)}}(F_1) \cdot \gamma_{a^{(2)}}(F_2), \qquad a \in \mathscr{D}(G), \; F_1, F_2 \in \mathscr{F}_{\text{loc}}, \tag{2.24}$$

$$\gamma_{a \cdot a'}(F) = \gamma_a(\gamma_{a'}(F)), \qquad a, a' \in \mathscr{D}(G), \; F \in \mathscr{F}_{\text{loc}}. \tag{2.25}$$

ii) *Let $\bar{a} = S(a^*)$. The action γ obeys the conjugation property*

$$\gamma_a(F^*) = \gamma_{\bar{a}}(F)^*, \qquad a \in \mathscr{D}(G), \; F \in \mathscr{F}_{\text{loc}}. \tag{2.26}$$

Proof. i) is quite elementary. To prove ii) we note that (2.26) fulfills for the generators of $\mathscr{F}$. Therefore the general statement can be proved inductively in the number of generators using the relation

$$\begin{aligned} \gamma_a((F_1F_2)^*) &= \gamma_a(F_2^* F_1^*) = \gamma_{a^{(1)}}(F_2^*)\gamma_{a^{(2)}}(F_1^*) = \gamma_{\overline{a^{(1)}}}(F_2)^*\gamma_{\overline{a^{(2)}}}(F_1)^* \\ &= [\gamma_{\overline{a^{(2)}}}(F_1)\gamma_{\overline{a^{(1)}}}(F_2)]^* = [\gamma_{\bar{a}^{(1)}}(F_1)\gamma_{\bar{a}^{(2)}}(F_2)]^* = \gamma_{\bar{a}}(F_1F_2)^*. \end{aligned} \tag{2.27}$$

Here we used the identity $(S \otimes S) \circ \Delta' = \Delta \circ S$ valid for the antipode S in a Hopf algebra [S] and the fact that the coproduct is a *-algebra homomorphism. $\square$

We extend the action γ to the field algebra $\mathscr{F}$ by continuity because we have the

2.5. Lemma. *The maps $\gamma_a: \mathscr{F}_{\text{loc}} \to \mathscr{F}_{\text{loc}}$, $a \in \mathscr{D}(G)$ are continuous.*

Proof. Let $F \in \mathscr{F}_{\text{loc}}$. There exists a finite half open, half closed interval Λ such that $F \in \mathscr{F}(\Lambda)$. Let $\Lambda^0 = \Lambda \cap \mathbf{Z}$, $\Lambda^1 = \Lambda \cap \mathbf{Z} + \frac{1}{2}$, and $\tau, \tau': \Lambda^0 \to G$, then for an appropriate $\sigma: \Lambda^1 \to G$ the elements

$$E_{\tau'\tau}(\Lambda) = \overrightarrow{\prod_{l \in \Lambda^1}} \varrho_{\sigma_l}(l) \cdot \prod_{x \in \Lambda_0} \delta_{\tau_x}(x) = \prod_{x \in \Lambda_0} \delta_{\tau'_x}(x) \cdot \overrightarrow{\prod_{l \in \Lambda^1}} \varrho_{\sigma_l}(l) \tag{2.28}$$

obey the algebra of matrix units for $\mathscr{F}(\Lambda) \cong M_{|G|^{|\Lambda^0|}}$. Since

$$\gamma_{(g,h)}(E_{\tau'\tau}(\Lambda)) = \delta_{g,hg_\sigma h^{-1}} \cdot E_{h\tau',h\tau}(\Lambda), \qquad g_\sigma = \prod_{l\in\Lambda^1}^{\rightarrow} \sigma_l, \qquad (h\tau)_x = h\tau_x, \tag{2.29}$$

it follows that $\|\gamma_{(g,h)}(F)\| \leq \|F\|$ independently on the size of the interval Λ, which implies that γ_a is continuous. □

2.4. The Observable Algebra and its Charged Representations

Using the orthogonal projector properties of the primitive central idempotents M_r, $r \in \widehat{\mathscr{D}(G)}$ one finds that the field algebra $\mathscr{F}$ – as a linear space – can be decomposed into a direct sum:

$$\mathscr{F} = \bigoplus_{r\in\widehat{\mathscr{D}(G)}} \mathscr{F}_r, \qquad \mathscr{F}_r = \gamma_{M_r}(\mathscr{F}). \tag{2.30}$$

The subspace $\mathscr{F}_0 \subset \mathscr{F}$ corresponds to the trivial representation, where M_0 is just the integral z. Due to the property (2.3) of the counit, one obtains that $\mathscr{F}_0$ is not only a subspace, but also a subalgebra of $\mathscr{F}$. Since we treat $\mathscr{D}(G)$ as the symmetry algebra of the G-spin models, the observables are the elements of the "$\mathscr{D}(G)$-invariant" subspace of $\mathscr{F}$. Therefore we give the following

2.6. Definition. The algebra of observables $\mathscr{A}$ of a G-spin model is defined as the subalgebra of $\mathscr{F}$ corresponding to the trivial representation of $\mathscr{D}(G)$, that is $\mathscr{A} = \mathscr{F}_0$.

2.7. Proposition. *The observable algebra $\mathscr{A} = \gamma_z(\mathscr{F})$ is a C^*-algebra.*

Proof. Since $\gamma_z(\mathscr{F})$ is already a subalgebra of $\mathscr{F}$ we only have to prove that it is a self-adjoint and closed subspace of $\mathscr{F}$. Let $A \in \gamma_z(\mathscr{F})$, then using the conjugation property (2.26) of γ and the $\bar{z} = z$ relation for the integral z one obtains that

$$\gamma_z(A^*) = \gamma_{\bar{z}}(A)^* = \gamma_z(A)^* = A^*, \tag{2.31}$$

therefore $\gamma_z(\mathscr{F})$ is a self-adjoint subspace. To prove that it is closed with respect to the norm topology of $\mathscr{F}$ first we note that the relation

$$\begin{aligned} \gamma_z(F^*F) &= \gamma_{z^{(1)}}(F^*)\gamma_{z^{(2)}}(F) = \gamma_{\overline{z^{(1)}}}(F)^*\gamma_{z^{(2)}}(F) \\ &\equiv \frac{1}{|G|}\sum_{g,h\in G} \gamma_{(g,h)}(F)^*\gamma_{(g,h)}(F) \end{aligned} \tag{2.32}$$

valid for arbitrary $F \in \mathscr{F}$, implies that γ_z is a positive map on $\mathscr{F}$. Since $\gamma_z(\mathbf{1}) = \mathbf{1}$, too, the norm of $\gamma_z : \mathscr{F} \to \mathscr{F}$ is 1 [BR]. Now an easy ε-argument shows that $\gamma_z(\mathscr{F})$ is closed. □

Realizations of the symmetry algebra emerge every time a GNS representation π of $\mathscr{F}$ is given associated to a "$\mathscr{D}(G)$-invariant" state on $\mathscr{F}$.

2.8. Theorem. *Let π be an irreducible representation of $\mathscr{F}$ on the Hilbert space $\mathscr{H} = \pi(\mathscr{F})\Omega$ with a vacuum vector Ω giving rise to a $\mathscr{D}(G)$-invariant state:*

$$(\Omega, \pi(\gamma_a(F))\Omega) = \varepsilon(a)(\Omega, \pi(F)\Omega), \qquad a \in \mathscr{D}(G), \ F \in \mathscr{F}. \tag{2.33}$$

Then there exists a unique C^*-homomorphism $U : \mathscr{D}(G) \to \mathscr{B}(\mathscr{H})$, with the properties:

i) *U implements the action γ on $\pi(\mathscr{F})$ in the adjoint way:*

$$U(a^{(1)})\pi(F)U(Sa^{(2)}) = \pi(\gamma_a(F)), \qquad a \in \mathscr{D}(G), \; F \in \mathscr{F}, \tag{2.34}$$

ii) *the vacuum vector Ω is invariant: $U(a)\Omega = \varepsilon(a)\Omega$, $a \in \mathscr{D}(G)$.*

iii) *U obeys the equalities*

$$U(\mathscr{D}(G))' = \pi(\mathscr{A})^-, \qquad U(\mathscr{D}(G)) = \pi(\mathscr{A})', \tag{2.35}$$

where bar means weak closure and prime denotes commutant in $\mathscr{B}(\mathscr{H})$.

Proof. Let the action of $U(a)$ be defined on the dense subset $\pi(\mathscr{F})\Omega \subset \mathscr{H}$ by

$$U(a)\pi(F)\Omega = \pi(\gamma_a(F))\Omega, \qquad a \in \mathscr{D}(G), \; F \in \mathscr{F}. \tag{2.36}$$

This definition is meaningful since if $\pi(F)\Omega = 0$ then using (2.33) the relation

$$\begin{aligned}
&(\pi(G)\Omega, \pi(\gamma_a(F))\Omega) \\
&= (\Omega, \pi(G^*\gamma_a(F))\Omega) = (\Omega, \pi(G^*\gamma_{a^{(1)}}(F))\Omega)\cdot\varepsilon(Sa^{(2)}) \\
&= (\Omega, \pi(\gamma_{Sa^{(2)}}(G^*\gamma_{a^{(1)}}(F)))\Omega) = (\Omega, \pi(\gamma_{a^{(3)*}}(G)^*\gamma_{Sa^{(2)}\,\cdot\,a^{(1)}}(F))\Omega) \\
&= (\Omega, \pi(\gamma_{a^{(3)*}}(G))^*\pi(\gamma_{(Sa)^{(1)}\,\cdot\,S(Sa)^{(2)}}(F))\Omega) = (\pi(\gamma_{a^*}(G))\Omega, \pi(F)\Omega)
\end{aligned} \tag{2.37}$$

holds for $a \in \mathscr{D}(G)$, $F, G \in \mathscr{F}$, which implies that $\pi(\gamma_a(F))\Omega = 0$, $a \in \mathscr{D}(G)$, too. One checks that the properties of homomorphism and of the implementation

$$\begin{aligned}
&U(a^{(1)})\pi(F)U(Sa^{(2)})\pi(G)\Omega \\
&= \pi(\gamma_{a^{(1)}}(F)\gamma_{a^{(2)}\,\cdot\,Sa^{(3)}}(G))\Omega = \pi(\gamma_{a^{(1)}}(F)\gamma_{\varepsilon(a^{(2)})}(G))\Omega \\
&= \pi(\gamma_a(F))\pi(G)\Omega, \qquad a \in \mathscr{D}(G), \; F, G \in \mathscr{F},
\end{aligned} \tag{2.38}$$

fulfill for a dense subset $\pi(\mathscr{F})\Omega$ in $\mathscr{H}$. Moreover, (2.37) shows that the implementation U is unitary, that is U is a *-representation. It follows [Di] that $\|U(a)\| \leq \|a\|$, $a \in \mathscr{D}(G)$. Therefore one extends U with the desired properties to the whole representation space $\mathscr{H}$ by continuity. Statement ii) and uniqueness of U is obvious. Finally, in the first equality of (2.35) $U(\mathscr{D}(G))' \subset \pi(\mathscr{A})^-$ is trivial, the reverse containment can be proven using

$$\begin{aligned}
\gamma_{a^{(1)}}(F)U(a^{(2)}) &= U(a^{(1)})FU(S(a^{(2)}))\cdot U(a^{(3)}) \\
&= U(a^{(1)})F\varepsilon(a^{(2)}) = U(a)F, \\
&= \varepsilon(a^{(1)})FU(a^{(2)}) = FU(a).
\end{aligned} \tag{2.39}$$

The second equality is obtained from the first taking the commutants in $\mathscr{B}(\mathscr{H})$ and using that $U(\mathscr{D}(G))$, being finite dimensional, is weakly closed. $\square$

As a corollary we get that the irreducible subrepresentations of $\pi|_{\mathscr{A}}$ are in one-to-one correspondence with the irreducible representations of $\mathscr{D}(G)$ because $\pi(\mathscr{A})$ and $U(\mathscr{D}(G))$ have the common center. To see how an irreducible subrepresentation of $\pi|_{\mathscr{A}}$ emerges let us decompose the Hilbert space $\mathscr{H}$ using the minimal central idempotents M_r, $r \in \widehat{\mathscr{D}(G)}$ of $\mathscr{D}(G)$:

$$\mathscr{H} = \bigoplus_{r \in \widehat{\mathscr{D}(G)}} \mathscr{H}_r, \qquad \mathscr{H}_r = U(M_r)\mathscr{H}. \tag{2.40}$$

Since the $\mathscr{D}(G)$-modules are completely reducible $\mathscr{H}_r$ can be written as a tensor product $\mathscr{H}_r \cong \mathscr{K}_r \otimes V_r$ induced by the bijection of bases $\mathscr{H}_r \ni F_r^{i\alpha}\Omega \mapsto \varphi_r^i \otimes e_r^\alpha \in \mathscr{K}_r \otimes V_r$, where V_r is an irreducible $\mathscr{D}(G)$-module with basis elements e_r^α, $\alpha = 1, \ldots, n_r$ and $i \in \mathbf{N}$ is a multiplicity index. The multiplet fields $F_r^{i\alpha} \in \mathscr{F}_r$, $\alpha = 1, \ldots, n_r$ carry the same matrix representation of $\mathscr{D}(G)$ for all i. Then the implementation operators $U(a)$, $a \in \mathscr{D}(G)$ act as

$$U(a)\,(\varphi^{ir} \otimes e_r^\alpha) = \sum_{\alpha'=1}^{n_r} \varphi_r^i \otimes e_r^{\alpha'} \cdot D_r^{\alpha'\alpha}(a)\,. \tag{2.41}$$

Since an observable acts only on the multiplicity index the action of observables is given by $\pi(A)\,(\varphi_r^i \otimes e_r^\alpha) = \pi_r(A)\,\varphi_r^i \otimes e_r^\alpha$. This defines the irreducible representation π_r of $\mathscr{A}$ on $\mathscr{K}_r$.

2.5. The Non-Abelian Parafermion Fields

Here we shall explicitly construct charged multiplet fields that are linear subspaces of operators in $\mathscr{F}$ carrying irreducible representations of $\mathscr{D}(G)$. Since $\mathscr{D}(G)$ is an internal symmetry, that is the action γ commutes with translations, the simplest choice is to consider the subspace spanned by products of disorder and order operators with a fixed $l \in \mathbf{Z} + \frac{1}{2}$ and $x \in \mathbf{Z}$.

In the Ising model, $G = Z(2)$, this product gives rise to anticommuting Majorana fermion field, while in case of $Z(N)$ models these products are parafermion fields having an Abelian braid type commutation relation. In a general G-spin model we have the following.

Let $\mathscr{F}(l, x)$, $l \in \mathbf{Z} + \frac{1}{2}$, $x \in \mathbf{Z}$ denote the linear subspace in $\mathscr{F}$ that is spanned by the basis elements $\varrho_g(l) \cdot \delta_h(x)$, $g, h \in G$. Then $\mathscr{F}(l, x)$ carries the left regular representation of $\mathscr{D}(G)$ with respect to the action γ, therefore there must be exactly as many linearly independent D_r-multiplets in $\mathscr{F}(l, x)$ for each $r \in \widehat{\mathscr{D}(G)}$ as the dimension n_r of D_r is. Define

$$F_r^{ik}(l, x) = \sum_{g,h \in G} D_r^{ik}((g, h)^*) \cdot \varrho_g(l)\,\delta_h(x)\,, \qquad i, k = 1, \ldots, n_r\,, \tag{2.42}$$

where the matrix elements D_r^{ij} of the irreducible representation D_r have been given in (2.8a). Then we have the following

2.9. Theorem. *The $F_r(l, x)$ multiplets obey the following properties*
i) *for each fixed i the $F_r^{ik}(l, x)$, $k = 1, \ldots, n_r$ form a D_r-multiplet:*

$$\gamma_{(g,h)}(F_r^{ik}(l, x)) = \sum_{k'=1}^{n_r} F_r^{ik'}(l, x) \cdot D_r^{k'k}(g, h)\,, \tag{2.43}$$

ii) *"orthogonality":*

$$\sum_{i=1}^{n_r} F_r^{ik^*}(l, x)\,F_r^{ik'}(l, x) = \delta^{kk'} \cdot \mathbf{1}\,, \tag{2.44}$$

iii) *"completeness"*:

$$\sum_{k=1}^{n_r} F_r^{ik}(l,x) F_r^{jk^*}(l,x) = \delta^{ij} \cdot \mathbf{1}\,, \tag{2.45}$$

iv) *the adjoint basis elements* $F_r^{ik^*}(l,x)$, *for each* i *transform according to the contragredient representation* $\bar{D}_r$, *and one has the relation*

$$F_r^{ik^*}(l,x) = \begin{cases} F_{\bar{r}}^{\bar{i}\bar{k}}(l,x)\,, & l > x\,, \\ \bar{\omega}_r \cdot F_{\bar{r}}^{\bar{i}\bar{k}}(l,x)\,, & l < x\,, \end{cases} \tag{2.46}$$

where $i \mapsto \bar{i}$, $k \mapsto \bar{k}$ *denote the map* (2.15), *and the phase* $\bar{\omega}_r = 1/n_\pi \cdot \mathrm{tr}[\pi(\tilde{g}^{-1})]$ *is the value of the centrum element* $\sum_{g\in G} (g, g^{-1}) \in \mathscr{D}(G)$ *in the representation* $r \equiv (A_{\tilde{g}}, \pi)$.

v) *the fields* $F_r(l,x)$ *obey Fröhlich's braided commutation relations: If* $\{l_1, x_1\} < \{l_2, x_2\}$ *then*

$$\begin{aligned} &F_{r_1}^{i_1k_1}(l_1,x_1) F_{r_2}^{i_2k_2}(l_2,x_2) \\ &\quad = \sum_{k_1'=1}^{n_1} \sum_{k_2'=1}^{n_2} F_{r_2}^{i_2k_2'}(l_2,x_2) F_{r_1}^{i_1k_1'}(l_1,x_1) (D_{r_1}^{k_1'k_1} \otimes D_{r_2}^{k_2'k_2})(R)\,. \end{aligned} \tag{2.47}$$

vi) *the set* $\{F_r^{ij}(l, l+l') \mid r \in \widehat{\mathscr{D}(G)}, i,j = 1, \ldots, n_r, l \in \mathbf{Z} + \frac{1}{2}\}$ *generates* $\mathscr{F}$ *for arbitrary* $l' \in \mathbf{Z} + \frac{1}{2}$,

vii) *the multiplets* $F_{r_1}(l,x)$, $F_{r_2}(l,x)$, $r_1, r_2 \in \widehat{\mathscr{D}(G)}$ *in the case of* $l > x$ *obey the following operator product expansion:*

$$\begin{aligned} &F_{r_1}^{i_1k_1}(l,x) \cdot F_{r_2}^{i_2k_2}(l,x) \\ &\quad = \sum_{r\in\widehat{\mathscr{D}(G)}} \sum_{i,j=1}^{n_r} \langle n_r \cdot D_r^{ik}, (D_{r_1} \times D_{r_2})^{i_1i_2,k_1k_2} \rangle \cdot F_r^{ik}(l,x)\,. \end{aligned} \tag{2.48}$$

Proof. By straightforward computation. □

One recognizes that properties ii–iii) of the multiplets $F_r(l,x)$ generalize the Cuntz algebra [C]. We shall see in Sect. 4 that these relations help to construct amplifying homomorphisms of the observable algebra in a similar way as the Cuntz algebra leads to the canonical endomorphism [C, DR].

3. The Structure of the Observable Algebra

In the previous section we defined the observable algebra as the $\mathscr{D}(G)$-invariant subalgebra of $\mathscr{F}: \mathscr{A} = \gamma_z(\mathscr{F})$. Here we shall study the question how to give a local net structure to $\mathscr{A}$, which satisfies Haag duality. This will be achieved by finding algebraic generators for $\mathscr{A}$ with local commutation relations. We will also discuss the inclusions of the local observable algebras and find the associated Temperley-Lieb algebra.

3.1. The Local Net Structure of the Observables

3.1. Propositon. *Let $\Lambda_{a,b} \subset \frac{1}{2}\mathbf{Z}$ be a finite interval. The $\mathscr{D}(G)$-invariant subalgebra of $\mathscr{F}(\Lambda_{a,b})$ is generated by the operators*

$$\begin{aligned} v_g(x) &:= \sum_{h\in G} \varrho_{hg^{-1}h^{-1}}\left(x-\tfrac{1}{2}\right)\delta_h(x)\,\varrho_{hgh^{-1}}\left(x+\tfrac{1}{2}\right), \\ w_g(l) &:= \sum_{h\in G} \delta_h\left(l-\tfrac{1}{2}\right)\delta_{hg}\left(l+\tfrac{1}{2}\right), \end{aligned} \tag{3.1}$$

while $x, l \in \Lambda_{a+1/2,b-1/2}$ and $g \in G$. That is

$$\gamma_z(\mathscr{F}(\Lambda_{a,b})) = \langle v_g(x), w_g(l) \mid x, l \in \Lambda_{a+\frac{1}{2},b-\frac{1}{2}}, g \in G\rangle . \tag{3.2}$$

We note that the meaning of the oprator $v_g(x)$ is right translation of the spin σ_x by g while $w_g(l)$ projects to those states that have right twist on the boundary of the link l equal to $g : \sigma_{l-1/2}^{-1}\sigma_{l+1/2} = g$.

Proof of 3.1. Let us write the integral of $\mathscr{D}(G)$ as a product $z = z \cdot (e,e)$. Then the invariant subalgebra can be computed in two steps $\gamma_z(\mathscr{F}(\Lambda_{a,b})) = \gamma_z(\gamma_{(e,e)}(\mathscr{F}(\Lambda_{a,b})))$. $\gamma_{(e,e)}$ is the projection to operators with trivial twist. Thus $\gamma_{(e,e)}(\mathscr{F}(\Lambda_{a,b}))$ is generated by the δ-s and v-s. Since $v_g(x)$ is already invariant under γ_z, we are ready if we can show that $\gamma_z(\delta_{h_1}(1)\dots\delta_{h_n}(n))$ can be expressed in terms of w-s:

$$\begin{aligned} \gamma_z(\delta_{h_1}(1)\cdot\ldots\cdot\delta_{h_n}(n)) &= \frac{1}{|G|}\sum_{g\in G}\delta_{gh_1}(1)\dots\delta_{gh_n}(n) \\ &= \frac{1}{|G|}\cdot \sum_{g_1,\dots,g_{n-1}\in G}\prod_{x=1}^{n-1}\delta_{g_x h_x}(x)\delta_{g_x h_{x+1}}(x+1) \\ &= \frac{1}{|G|}\cdot w_{h_1^{-1}h_2}\left(\tfrac{3}{2}\right) w_{h_2^{-1}h_3}\left(\tfrac{5}{2}\right)\dots w_{h_{n-1}^{-1}h_n}\left(n-\tfrac{1}{2}\right). \quad \square \end{aligned}$$

3.2. Definition. The algebra of observables localized in $\Lambda \subset \frac{1}{2}\mathbf{Z}$ is the C^*-algebra

$$\mathscr{A}(\Lambda) = \langle v_g(x), w_g(l) \mid x, l \in \Lambda, g \in G\rangle . \tag{3.3}$$

The correspondence $\Lambda \subset \frac{1}{2}\mathbf{Z} \mapsto \mathscr{A}(\Lambda) \in \{C^*\text{-subalgebras of } \mathscr{A}\}$ satisfies

i) *isotony:* $\Lambda_1 \subset \Lambda_2 \Rightarrow \mathscr{A}(\Lambda_1) \subset \mathscr{A}(\Lambda_2)$,

ii) *locality:* $\operatorname{dist}(\Lambda_1, \Lambda_2) \geq 1 \Rightarrow \mathscr{A}(\Lambda_1) \subset \mathscr{A}(\Lambda_2)'$,

iii) $\mathscr{A} = \overline{\bigcup_\Lambda \mathscr{A}(\Lambda)}$, where the union is taken over finite intervals Λ and the bar denotes uniform closure.

To see ii) it is enough to compute the commutation relations of the v, w generators:

$$v_{g_1}(x)\,v_{g_2}(x) = v_{g_1g_2}(x), \qquad w_{h_1}(l)w_{h_2}(l) = \delta_{h_1,h_2}\cdot w_{h_2}(l), \tag{3.4a}$$

$$\begin{aligned} v_g(x)w_h\left(x+\tfrac{1}{2}\right) &= w_{gh}\left(x+\tfrac{1}{2}\right)v_g(x), \\ v_g(x)w_h\left(x-\tfrac{1}{2}\right) &= w_{hg^{-1}}\left(x-\tfrac{1}{2}\right)v_g(x). \end{aligned} \tag{3.4b}$$

Other pairs of v and/or w fields commute. Property iii) follows from Proposition 3.1 and the continuity of the projection γ_z. If $A \in \mathscr{A}$ and $\varepsilon > 0$ then $A = \gamma_z(A)$ and $\exists B \in \mathscr{F}(\Lambda_{a,b})$ with $\|A-B\| < \varepsilon$. Then $\|A-\gamma_z(B)\| = \|\gamma_z(A-B)\| \leq \|A-B\| < \varepsilon$ and $\gamma_z(B) \in \mathscr{A}(\Lambda_{a+\frac{1}{2},b-\frac{1}{2}})$.

Properties i)–iii) establish the local net structure of our observable algebra.

3.2. The Types and Inclusions of the Local Algebras $\mathscr{A}(\Lambda)$

For any finite subset $\Lambda \subset \frac{1}{2}\mathbf{Z}$ and maps $\sigma: \Lambda^0 \to G$, $\tau: \Lambda^1 \to G$ let us define the operators

$$Q^R_\Lambda(\sigma) = \prod_{x\in\Lambda^0} v_{\sigma_x}(x)\,, \qquad P^R_\Lambda(\tau) = \prod_{l\in\Lambda^1} w_{\tau_l}(l)\,. \tag{3.5}$$

They satisfy

$$Q^R_\Lambda(\sigma)P^R_\Lambda(\tau) = P^R_\Lambda(\tau^\sigma)Q^R_\Lambda(\sigma)\,, \tag{3.6}$$

where $(\tau^\sigma)_l = \sigma_{l-\frac{1}{2}}\tau_l\sigma^{-1}_{l+\frac{1}{2}}$ with the convention $\sigma_{l\pm\frac{1}{2}} = e$ if $l \pm \frac{1}{2} \notin \Lambda$.

If Λ is an interval which is closed from one side and open from the other then for arbitrary $\tau, \tau': \Lambda^1 \to G$ there exists a unique $\sigma: \Lambda^0 \to G$ such that $\tau' = \tau^\sigma$. With this σ the operators

$$E_{\tau'\tau}(\Lambda) = Q^R_\Lambda(\sigma)P^R_\Lambda(\tau) = P^R_\Lambda(\tau')Q^R_\Lambda(\sigma) \tag{3.7}$$

satisfy the algebra of matrix units:

$$E_{\tau'_1\tau_1}(\Lambda)E_{\tau'_2\tau_2}(\Lambda) = \delta_{\tau_1,\tau'_2}E_{\tau'_1\tau_2}(\Lambda)\,, \qquad E_{\tau'\tau}(\Lambda)^* = E_{\tau\tau'}(\Lambda)\,. \tag{3.8}$$

This shows that $\mathscr{A}(\Lambda)$ is a simple algebra, namely $M_{|G|^n}$, $n = |\Lambda^1|$ for this kind of intervals. If Λ is an open interval then $\tau' = \tau^\sigma$ has a solution for σ iff

$$\overrightarrow{\prod_{l\in\Lambda^1}} \tau_l = \overrightarrow{\prod_{l\in\Lambda^1}} \tau'_l\,, \tag{3.9}$$

in which case the solution is unique. Equation (3.9) expresses the fact that the total right twist along the interval is unchanged by any operator $A \in \mathscr{A}(\Lambda)$. The matrix units can be defined like in (3.7) but now τ and τ' are subjected to the condition (3.9). Therefore $\mathscr{A}(\Lambda)$ becomes a sum of full matrix algebras. If $\Lambda = \Lambda_{a,b}$ is a closed interval then $\tau' = \tau^\sigma$ always has a solution but it is unique only if we fix the value of σ (let us say) at the right endpoint to be $\sigma_b = e$. Using this solution formula (3.7) again defines a full matrix algebra in $\mathscr{A}(\Lambda)$ but it does not generate the whole. What are missing are related to the global right multiplication. For $g \in G$, $\tau: \Lambda^1 \to G$ let $\gamma_x[\tau, g]$ be the parallel transport of g from b to x in the presence of the "gauge field" τ, that is

$$\gamma_x[\tau, g] = \tau_{x+\frac{1}{2}} \cdot \ldots \cdot \tau_{b-\frac{1}{2}} \cdot g \cdot \tau^{-1}_{b-\frac{1}{2}} \cdot \ldots \cdot \tau^{-1}_{x+\frac{1}{2}}\,. \tag{3.10}$$

Then

$$Q^R_\Lambda(g) = \sum_{\tau:\Lambda^1\to G} Q^R_\Lambda(\gamma[\tau, g])P^R_\Lambda(\tau) \tag{3.11}$$

is a unitary operator commuting with all matrix units and such that $g \in G \mapsto Q^R_\Lambda(g)$ is a homomorphism. Hence $\mathscr{A}(\Lambda)$ is a full matrix algebra tensored with the group algebra $\mathbf{C}G$.

What we have found can be summarized in the following

3.3. Theorem. i) *If Λ is a finite interval in $\frac{1}{2}\mathbf{Z}$ then $\mathscr{A}(\Lambda)$ is isomorphic to the following finite dimensional C^*-algebra:*

$$\mathscr{A}(\Lambda) \cong \begin{cases} \mathrm{Mat}(|G|^{n-1}, \mathbf{C}) \otimes \mathbf{C}(G)\,, & \text{if } \Lambda \text{ is open,} \\ \mathrm{Mat}(|G|^{n}, \mathbf{C}) \otimes \mathbf{C}G\,, & \text{if } \Lambda \text{ is closed,} \\ \mathrm{Mat}(|G|^{n}, \mathbf{C})\,, & \text{otherwise,} \end{cases} \tag{3.12}$$

where n is the length of the interval, $n = |\Lambda^1|$.
ii) *Consider the tower of local observable algebras*

$$\mathscr{A}(\Lambda_{0,0}) \subset \mathscr{A}(\Lambda_{0,\frac{1}{2}}) \subset \ldots \subset \mathscr{A}(\Lambda_{0,n-\frac{1}{2}}) \subset \mathscr{A}(\Lambda_{0,n}) \subset \ldots .$$

The inclusion matrix for $\mathscr{A}(\Lambda_{0,n-\frac{1}{2}}) \subset \mathscr{A}(\Lambda_{0,n})$ is that of the unit preserving inclusion $\mathbf{C} \subset \mathbf{C}G$. The inclusion matrix for $\mathscr{A}(\Lambda_{0,n}) \subset \mathscr{A}(\Lambda_{0,n+\frac{1}{2}})$ is the transposed of that. Similar statements hold for the tower

$$\mathscr{A}(\Lambda_{\frac{1}{2},\frac{1}{2}}) \subset \mathscr{A}(_{\frac{1}{2},1}) \subset \ldots \subset \mathscr{A}(\Lambda_{\frac{1}{2},n}) \subset \mathscr{A}(\Lambda_{\frac{1}{2},n+\frac{1}{2}}) \subset \ldots .$$

The Bratteli diagram for $\mathscr{A}(\Lambda_{\frac{1}{2},n}) \subset \mathscr{A}(\Lambda_{\frac{1}{2},n+\frac{1}{2}})$ is that of $\mathbf{C} \subset \mathbf{C}(G)$.

3.4. Remark. The above mentioned Bratteli diagrams offer two ("dually" related) presentations of our observable algebra as the operator algebra of a graph-IRF model [P]. In either case there is a Temperley-Lieb algebra generated by the projections

$$e_s = \begin{cases} \dfrac{1}{|G|} \displaystyle\sum_{g \in G} v_g\big(s + \tfrac{1}{2}\big)\,, & s \in \mathbf{Z} + \tfrac{1}{2}\,, \\ w_e\big(s + \tfrac{1}{2}\big)\,, & s \in \mathbf{Z}\,. \end{cases} \tag{3.13}$$

Let $\mathscr{A}_s = \mathscr{A}(\Lambda_{s_0,s})$, $s_0 = 0$ or 1/2, $s > s_0$, then $e_s \in \mathscr{A}_{s+\frac{1}{2}} \cap \mathscr{A}'_{s-\frac{1}{2}}$, $e_s^2 = e_s = e_s^*$ with

$$e_s e_{s\pm\frac{1}{2}} e_2 = \frac{1}{|G|}\, e_s\,, \qquad e_s e_r = e_r e_s\,, \qquad |r - s| \geq 1\,. \tag{3.14}$$

The e_s-s induce the conditional expectations $\varepsilon_s : \mathscr{A}_s \to \mathscr{A}_{s-1}$, since $e_s A e_s = \varepsilon_s(A) e_s$, $A \in \mathscr{A}_s$, where

$$\varepsilon_x(A) = \frac{1}{|G|} \sum_{g \in G} v_g(x) A v_{g^{-1}}(x)\,, \qquad \varepsilon_l(A) = \sum_{h \in G} w_h(l) A w_h(l)\,. \tag{3.15}$$

The coefficient $1/|G|$ in (3.14) is in accordance with the Perron-Frobenius eigenvalues of both Bratteli diagrams being $\sqrt{|G|}$.

3.3. Haag Duality

Turning to the problem of Haag duality for the observable algebra we generalize formulae (3.15) for arbitrary finite sets Λ. For $A \in \mathscr{A}$ let

$$\varepsilon_\Lambda(A) = \frac{1}{|G|^{|\Lambda^0|}} \sum_{\sigma:\Lambda^0 \to G} \sum_{\tau:\Lambda^1 \to G} Q^R_\Lambda(\sigma) P^R_\Lambda(\tau) A P^R_\Lambda(\tau) Q^R_\Lambda(\sigma)^{-1}. \tag{3.16}$$

3.5. Lemma. *ε_Λ obeys the properties*
i) $\varepsilon_{\Lambda_1} \circ \varepsilon_{\Lambda_2} = \varepsilon_{\Lambda_1 \cup \Lambda_2}$, $\varepsilon_\Lambda \circ {}^* = {}^* \circ \varepsilon_\Lambda$, $\varepsilon \geq 0$;
ii) $\varepsilon_\Lambda : \mathscr{A} \to \mathscr{A}(\Lambda)'$,
iii) $\varepsilon_\Lambda(\mathbf{1}) = \mathbf{1}$, $\varepsilon_\Lambda(ABC) = A\varepsilon_\Lambda(B)C$, $A, C \in \mathscr{A}(\Lambda)'$, $B \in \mathscr{A}$.

Proof. Left to the reader. □

The above lemma implies that ε_Λ is a continuous projection onto $\mathscr{A}(\Lambda)'$. Point iii) even states that ε_Λ is a conditional expectation.

3.6. Lemma. *Let I be a closed interval,* Int I *be its interior (*i.e. *the largest open interval contained in I), and $\Lambda \subset$* Int I *an arbitrary interval. Then*

$$\varepsilon_{I\setminus\bar\Lambda}(\mathscr{A}(\operatorname{Int} I)) = \mathscr{A}(\Lambda), \qquad \bar\Lambda := \left\{ s \in \tfrac{1}{2}\mathbf{Z} \,\middle|\, \operatorname{dist}(s,\Lambda) \leq \tfrac{1}{2} \right\}. \tag{3.17}$$

Proof. At first we prove the statement for open intervals Λ. Let $\tau, \tau' : I^1 \to G$ have the same total right twist (3.9). Then $E_{\tau'\tau}(\operatorname{Int} I)$ runs over a basis of $\mathscr{A}(\operatorname{Int} I)$. $I\setminus\bar\Lambda$ decomposes into two intervals I_- and I_+ that are neither closed nor open. Let $\tau_+, \tau_-, \bar\tau$ and $\tau'_+, \tau'_-, \bar\tau'$ denote the restrictions of τ resp. τ' onto I_+, I_- and $\bar\Lambda$. Then we have

$$\varepsilon_{I\setminus\bar\Lambda}(E_{\tau'\tau}(\operatorname{Int} I)) = \varepsilon_{I_-}(\varepsilon_{I_+}(E_{\tau'\tau}(\operatorname{Int} I))) = \delta_{\tau'_-,\tau_-}\delta_{\tau'_+,\tau_+} E_{\bar\tau'\bar\tau}(\Lambda). \tag{3.18}$$

Since $E_{\bar\tau'\bar\tau}(\Lambda)$ runs over a basis of $\mathscr{A}(\Lambda)$, (3.17) is proven for open Λ. If Λ is not open it is obtained from an open one by discarding one or two of its boundary links. Then one or two extra projections ε_x [see (3.15)] should be applied on both sides of (3.18) and the result follows from the fact that ε_x is the appropriate conditional expectation. □

Now we can formulate the Haag duality for the observable algebra of G-spin models:

3.7. Theorem. *Let Λ be a finite interval in $\frac{1}{2}\mathbf{Z}$ and $\Lambda^c := \left\{ s \in \frac{1}{2}\mathbf{Z} \,\middle|\, \operatorname{dist}(s,\Lambda) \geq 1 \right\}$. Then $\mathscr{A}(\Lambda^c)' = \mathscr{A}(\Lambda)$.*

Proof. Obviously $\mathscr{A}(\Lambda) \subset \mathscr{A}(\Lambda^c)'$. Let $A \in \mathscr{A}(\Lambda^c)'$ and $\varepsilon > 0$. Then there exists a closed finite interval I and $B \in \mathscr{A}(\operatorname{Int} I)$ such that $\|A - B\| \leq \varepsilon$. We may choose I so large that I contains Λ in its interior. Since $I\setminus\bar\Lambda$ is a finite subset of Λ^c, $\varepsilon_{I\setminus\bar\Lambda}(A) = A$ thus

$$\|A - \varepsilon_{I\setminus\bar\Lambda}(B)\| = \|\varepsilon_{I\setminus\bar\Lambda}(A - B)\| \leq \|A - B\| < \varepsilon.$$

Since $\varepsilon_{I\setminus\bar\Lambda}(B) \in \mathscr{A}(\Lambda)$ by Lemma 3.6, $A \in \overline{\mathscr{A}(\Lambda)} = \mathscr{A}(\Lambda)$. □

We note that it was essential to restrict ourselves to intervals. Theorem 3.7 is not true for arbitrary finite subsets Λ.

We end this section by proving the triviality of the relative commutant of $\mathscr{A}$ in $\mathscr{F}$.

3.8. Lemma. *Let I be a closed interval. Then $\varepsilon_I(\mathscr{F}(I)) = \mathbf{C}\cdot\mathbf{1}$.*

Proof. Let $I = \Lambda_{a,b}$, $a,b\in\mathbf{Z}$. $\mathscr{F}(I)$ is generated by the fields

$$\prod_{x\in I^0}\delta_{\sigma_x}(x)\cdot\prod_{x\in I^0}v_{\sigma_x^{-1}\pi_x}(x)\cdot\varrho_g\big(b+\tfrac{1}{2}\big) =: \mathscr{E}_{\sigma,\pi}\cdot\varrho_g\big(b+\tfrac{1}{2}\big)\,, \tag{3.19}$$

while σ and π run over the set of functions $I^0\to G$ such that $\sigma_a=\pi_a$ and $g=\sigma_b\pi_b^{-1}$. The $\mathscr{E}_{\sigma,\sigma'}$ obey the algebra of matrix units. For $\sigma,\sigma',\sigma'':I^0\to G$, $\tau:I^1\to G$,

$$Q_I^R(\sigma)\,P_I^R(\tau) = \sum_{\sigma',\sigma''}\mathscr{E}_{\sigma'\sigma''}(I)\cdot\delta_{\sigma'',\sigma'\sigma}\cdot\delta_{d\sigma'',\tau}\,, \tag{3.20}$$

where $(d\sigma'')_l=(\sigma''_{l-\frac{1}{2}})^{-1}\sigma''_{l+\frac{1}{2}}$, $l\in\mathbf{Z}+\frac{1}{2}$. Thus we have

$$\varepsilon_I(F)=\frac{1}{|G|^{|I^0|}}\sum_{\sigma',\sigma''}\sum_{\pi',\pi''}\mathscr{E}_{\sigma'\sigma''}(I)F\mathscr{E}_{\pi''\pi'}(I)\cdot\delta_{d\sigma'',d\pi''}\cdot\delta_{\sigma'^{-1}\sigma'',\pi'^{-1}\pi''}\,,$$

$$\varepsilon_I\big(\mathscr{E}_{\sigma,\pi}(I)\,\varrho_{\sigma_b\pi_b^{-1}}\big(b+\tfrac{1}{2}\big)\big)=\delta_{d\sigma,d\pi}\frac{1}{|G|^{|I^0|}}\sum_{\psi:I^0\to G}\mathscr{E}_{\sigma\psi,\pi\psi}(I)\,\varrho_{\sigma_b\pi_b^{-1}}\big(b+\tfrac{1}{2}\big)\,.$$

If $\sigma_a=\pi_a$ then $d\sigma=d\pi$ on I^1 implies that $\sigma=\pi$. Especially $\sigma_b=\pi_b$. Therefore

$$\varepsilon_I\big(\mathscr{E}_{\sigma,\pi}(I)\,\varrho_{\sigma_b\pi_b^{-1}}\big(b+\tfrac{1}{2}\big)\big)=\delta_{\sigma,\pi}\cdot\mathbf{1}\,.$$ □

3.9. Theorem. *The commutant of $\mathscr{A}$ in $\mathscr{F}$ is the set of scalars: $\mathscr{A}'=\mathbf{C}\cdot\mathbf{1}$.*

Proof. For an increasing sequence $\{I_n\}_{n=1}^{\infty}$ of closed intervals define $\varepsilon:\mathscr{F}_{\mathrm{loc}}\to\mathscr{F}_{\mathrm{loc}}$ by

$$\varepsilon(F):=\lim_{n\to\infty}\varepsilon_{I_n}(F)\,. \tag{3.21}$$

This ε is positive and thus can be extended to $\mathscr{F}$ by continuity. The extension ε also satisfies (3.21) for all $F\in\mathscr{F}$. Since $\varepsilon_{I_n}(\mathscr{F})=\mathscr{A}(I_n)'$ is a monotone decreasing sequence of closed sets in $\mathscr{F}$,

$$\mathscr{A}'=\bigcap_n\mathscr{A}(I_n)'=\bigcap_n\varepsilon_{I_n}(\mathscr{F})=\varepsilon(\mathscr{F})=\overline{\varepsilon(\mathscr{F}_{\mathrm{loc}})}\doteq\mathbf{C}\cdot\mathbf{1}\,, \tag{3.22}$$

where in the last equation we used $\varepsilon(\mathscr{F}_{\mathrm{loc}})=\mathbf{C}\cdot\mathbf{1}$ which follows from Lemma 3.8. □

This theorem implies that $\mathscr{A}$ has trivial center which would not be the case if we quantized the G-spin model on a periodic chain. (Cf. the chiral Ising model [MS2] or [Fre].)

4. The Amplifying Homomorphisms

The aim of this section is to show – on the example of the G-spin model – that localized amplifying homomorphisms of the type $\mu: \mathscr{A} \to M_n(\mathscr{A})$ are capable of describing non-Abelian superselection sectors even in the case when localized endomorphisms $\varrho: \mathscr{A} \to \mathscr{A}$ allow only Abelian sectors because all the local algebras $\mathscr{A}(\Lambda)$ are finite dimensional. The key for finding amplifying homomorphisms μ in the G-spin model is the existence of multiplet matrices F_r satisfying the F-algebra defined below. After learning how these morphisms behave and create charged representations of the observable algebra we shall discuss the equivalence of the representation theory of the symmetry algebra $\mathscr{D}(G)$ and (a subcategory of) the representation theory of $\mathscr{A}$.

4.1. Multiplet Matrices and the F-Algebra

4.1. Definition. Let F be an $m \times n$ matrix with entries $F^{ij} \in \mathscr{F}_{\mathrm{loc}}$ and let D be an n-dimensional (unitary) representation of $\mathscr{D}(G)$. F is called a D-multiplet matrix if the following two relations hold:

$$\gamma_a(F^{ij}) = \sum_{k=1}^{n} F^{ik} \cdot D^{kj}(a)\,, \qquad a \in \mathscr{D}(G)\,,\ i = 1, \ldots, m\,; \tag{4.1}$$

$$\sum_{i=1}^{m} F^{ij^*} F^{ik} = \delta^{jk} \cdot \mathbf{1}\,, \qquad j,k = 1, \ldots, n\,. \tag{4.2}$$

If D is irreducible F is called an irreducible multiplet matrix. F is called non-degenerate if in addition to (4.1) and (4.2) F satisfies

$$\sum_{j=1}^{n} F^{ij} F^{kj^*} = \delta^{ik} \cdot \mathbf{1}\,, \qquad i,k = 1, \ldots, m\,. \tag{4.3}$$

Relations (4.2–4.3) will be referred to as the F-algebra, (4.2) alone as the weak F-algebra.

The special fields $F^{ij} = F_r^{ij}(l, x)$ introduced in Sect. 2.5 are examples of irreducible non-degenerate multiplet matrices. The set of multiplet matrices is closed under the following two operations. The *product* of the $m_1 \times n_1$ matrix F_1 and the $m_2 \times n_2$ matrix F_2 is the $m_1 m_2 \times n_1 n_2$ matrix $F_1 \times F_2$ with entries $(F_1 \times F_2)^{i_1 i_2, j_1 j_2} = F_1^{i_1 j_1} F_2^{i_2 j_2}$. The *direct sum* of F_1 and F_2 is the $(m_1 + m_2) \times (n_1 + n_2)$ block diagonal matrix with diagonal blocks F_1 and F_2. The product or direct sum of non-degenerate multiplet matrices is again non-degenerate. If F_1 is a D_1-multiplet matrix, F_2 is a D_2-multiplet matrix then $F_1 \times F_2$ and $F_1 \oplus F_2$ are $D_1 \times D_2$ and $D_1 \oplus D_2$ multiplet matrices, respectively. Examples of degenerate multiplet matrices can be obtained by taking a product $F_1 \times \ldots \times F_s$ of irreducible multiplet matrices $F_\alpha = F_{r_\alpha}(l_\alpha, x_\alpha)$, $\alpha = 1, \ldots, s$ and then multiplying it from the right by a c-number intertwiner matrix t, which intertwines from a representation D to the representation $D_{r_1} \times \ldots \times D_{r_s}$. If t is appropriately normalized then this F satisfies the weak F-algebra but not the F-algebra in general.

The next lemma sheds some light on the general form of multiplet matrices.

4.2. Lemma. *Let D_1, D_2 be equivalent representations with* $\dim D_1 = n = \dim D_2$ *and let $u \in (D_1 \mid D_2)$ be a unitary intertwiner from D_2 to D_1. If F_1 and F_2 are D_1- and D_2-multiplet matrices of sizes $m_1 \times n$ and $m_2 \times n$, respectively, then*

$$U = F_1 u F_2^* \in \mathrm{Mat}(m_1 \times m_2, \mathscr{A}_{\mathrm{loc}}) \tag{4.4}$$

is a partial isometry with initial and final projections $U^*U = F_2F_2^*$ and $UU^* = F_1F_1^*$, respectively. Furthermore we have $UF_2u^* = F_1$ and $U^*F_1u = F_2$.

Proof. Using the multiplet properties of the F-s one checks that $\gamma_a(U^{i_1i_2}) = \varepsilon(a) \cdot U^{i_1i_2}$. This implies that $U \in \mathrm{Mat}(m_1 \times m_2, \mathscr{A}_{\mathrm{loc}})$. To prove the further identities one has to use only the weak F-algebra relation in matrix form: $F_1^*F_1 = \mathbf{1} \otimes I_n = F_2^*F_2$. □

From physical grounds we would like to consider two multiplet matrices equivalent if they create the same charge. This leads to

4.3. Definition. Let F_1 and F_2 be D_1- and D_2-multiplet matrices, respectively. F_1 is equivalent to F_2, $F_1 \sim F_2$, if D_1 is equivalent to D_2.

4.4. Proposition. *Given an arbitrary multiplet matrix $\tilde{F}$ and given $x \in \mathbf{Z}$ and $l \in \mathbf{Z} + \frac{1}{2}$ there exists a non-degenerate multiplet matrix F equivalent to $\tilde{F}$ with entries $F^{ij} \in \mathscr{F}(l,x)$. Furthermore $\tilde{F} = UFu^*$ for some c-number unitary matrix u and for some observable partial isometry matrix U.*

Proof. Let $\tilde{F}$ be a D-multiplet matrix and $D \cong D_{r_1} \oplus \ldots \oplus D_{r_s}$ for a sequence $r_1, \ldots, r_s \in \widehat{\mathscr{D}(G)}$. Let $u \in (D \mid D_{r_1} \otimes \ldots \oplus D_{r_s})$ be a unitary intertwiner. Construct the direct sum $F = F_{r_1}(l,x) \oplus \ldots \oplus F_{r_s}(l,x)$, which is a non-degenerate multiplet matrix, that is $FF^* = \mathbf{1} \otimes I_n (n = n_{r_1} + \ldots + n_{r_s})$, then $F \sim \tilde{F}$. Applying Lemma 4.2 we construct the partial isometry $U = \tilde{F}uF^* \in \mathrm{Mat}(\mathscr{A}_{\mathrm{loc}})$ and obtain $\tilde{F} = UFu^*$. □

4.2. Amplifying Homomorphisms Generated by Multiplet Matrices

For an arbitrary multiplet matrix F of size $m \times n$ we define a map $\mu_F : \mathscr{A} \to M_m(\mathscr{A}) \equiv \mathscr{A} \otimes M_m$ via the formula

$$\mu_F^{ij}(A) = \sum_{k=1}^{n} F^{ik} A F^{jk^*}, \qquad i,j = 1, \ldots, m; \tag{4.5}$$

or in matrix notation $\mu_F(A) = F(A \otimes I_n)F^*$. μ_F will be called the amplifying homomorphism (or amplimorphism, for short) generated by F since we have

4.5. Lemma. *$\mu_F : \mathscr{A} \to M_m(\mathscr{A})$ is a C^*-homomorphism, that is*
i) *$\mu_F^{ij}(A) \in \mathscr{A}$, $\mu_F^{ij}(A)^* = \mu_F^{ji}(A^*)$ for all $A \in \mathscr{A}$,*
ii) *$\sum_j \mu_F^{ij}(A)\mu_F^{jk}(B) = \mu_F^{ik}(AB)$ for all $A, B \in \mathscr{A}$,*
iii) *$\mu_F(\mathbf{1}) = FF^*$ and μ_F is unit preserving* iff *F is non-degenerate.*

Proof. i) Analogous to the proof of Lemma 4.2. ii) follows from the weak F-algebra. iii) μ_F is unit preserving iff $\mu_F(\mathbf{1}) = \mathbf{1} \otimes I_m$, that is iff F satisfies the F-algebra. □

4.6. Definition. Let $\mu:\mathscr{A} \to M_m(\mathscr{A})$ be a C^*-algebra map and $\Lambda \subset \frac{1}{2}\mathbf{Z}$ be a finite interval. μ is called localized in Λ if $\mu(A) = \mu(\mathbf{1})(A \otimes I_m)$, $A \in \mathscr{A}(\Lambda^c)$. We say that μ is localized if it is localized in some finite interval Λ.

4.7. Lemma. *If $\mu:\mathscr{A} \to M_m(\mathscr{A})$ is a C^*-algebra map, Λ is an interval, and μ is localized in Λ then*

$$\mu(\mathscr{A}(\Lambda)) \subset M_m(\mathscr{A}(\Lambda)). \tag{4.6}$$

Proof. If $A \in \mathscr{A}$ is hermitian $\mu^{ij}(\mathbf{1})A = A\mu^{ij}(\mathbf{1})$ because μ is localized. Since every $A \in \mathscr{A}(\Lambda^c)$ is a linear combination of two hermitian elements, $\mu^{ij}(\mathbf{1}) \in \mathscr{A}(\Lambda^c)' = \mathscr{A}(\Lambda)$ by Haag duality. Let $A \in \mathscr{A}(\Lambda^c)$, $B \in \mathscr{A}(\Lambda)$. Then

$$A\mu(B) = (A \otimes I_m)\mu(\mathbf{1})\mu(B) = \mu(AB) = \mu(BA) = \mu(B)\mu(\mathbf{1})(A \otimes I_m) = \mu(B)A.$$

Hence $\mu^{ij}(B) \in \mathscr{A}(\Lambda^c)' = \mathscr{A}(\Lambda)$ by Haag duality again. $\square$

Comparing formulae (3.2) and (3.3) we see that $\gamma_z(\mathscr{F}(\bar{\Lambda})) = \mathscr{A}(\Lambda)$ for any interval Λ. Therefore if F is a multiplet matrix with entries $F^{ij} \in \mathscr{F}(\bar{\Lambda})$ then μ_F is an amplifying homomorphism localized in $\bar{\Lambda}$. But it leaves $\mathscr{A}(\Lambda)$ "invariant", too: $\mu_F(\mathscr{A}(\Lambda)) \subset M_m(\mathscr{A}(\Lambda))$. When $F = F_r\left(x+\frac{1}{2}, x\right)$ the concrete form of the action of μ_F on the generators of $\mathscr{A}$ is

$$\mu_F^{ij}(v_g(y)) = \begin{cases} \delta^{ij} v_g(y), & y \neq x, \\ D_r^{ij}(E,g)v_g(x), & y = x, \end{cases} \tag{4.7a}$$

$$\mu_F^{ij}(w_h(l)) = \begin{cases} \delta^{ij} w_h(l), & l \neq x+\frac{1}{2}, \\ \sum\limits_{g\in G} D_r^{ij}(g,e)w_{gh}\left(x+\frac{1}{2}\right), & l = x+\frac{1}{2}. \end{cases} \tag{4.7b}$$

4.3. Representations of $\mathscr{A}$ Created by μ_F

The physical meaning of μ_F defined in (4.5) is that it creates a charge equal to the charge of F^*. To make this precise let π_0 be, once and for all, a fixed faithful irreducible representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}_0$. π_0 is thought to be a vacuum representation with respect to some dynamics not discussed in this paper. The only assumption on the dynamics is Haag duality for the vacuum representation,

$$\pi_0(\mathscr{A}(\Lambda^c))' = \pi_0(\mathscr{A}(\Lambda)), \tag{4.8}$$

which encodes in some way the absence of symmetry breaking. We assume also that π_0 is a subrepresentation of the restriction to $\mathscr{A}$ of an irreducible representation π of $\mathscr{F}$. The other subrepresentations of $\pi|_{\mathscr{A}}$ are the charged representations π_r introduced after Theorem 2.8.

Given an amplimorphism $\mu:\mathscr{A} \to M_m(\mathscr{A})$ we define a representation π_μ of $\mathscr{A}$ on the Hilbert space $\mathscr{H}_0 \otimes \mathbf{C}^m$:

$$\pi_\mu := (\pi_0 \otimes \mathrm{id}) \circ \mu. \tag{4.9}$$

That is for $\Phi \in \mathscr{H}_0$ and $\{e_1, \ldots, e_m\}$ a fixed orthonormal basis in $\mathbf{C}^m$,

$$\pi_\mu(A)(\Phi \otimes e_j) = (\pi_0 \otimes \mathrm{id})\left(\sum_{ik} \mu^{ik}(A) \otimes e_{ik}\right)(\Phi \otimes e_j) = \sum_i \pi_0(\mu^{ij}(A))\Phi \otimes e_i,$$

where the $\{e_{ik}\}$ is the system of matrix units in M_m associated to the basis $\{e_i\}$.

4.8. Theorem. *Let $\mu = \mu_F$ be the amplimorphism generated by the irreducible multiplet matrix $F = F_r(l,x)$ of charge $r \in \widehat{\mathscr{D}(G)}$. Then π_μ is unitarily equivalent to $\pi_{\bar{r}}$.*

Proof. Using the notations introduced after Theorem 2.8 if $\Phi \in \mathscr{H}_0$ then $\pi(F_r^{i\alpha^*})\Phi \in \mathscr{H}_{\bar{r}} \otimes V_{\bar{r}}$. Let $\{v^\alpha\}$ be the orthonormal basis in V_r associated to the matrix representation D_r according to which the $F_r^{i\alpha}$ transforms. Then

$$\pi(F_r^{i\alpha^*})\Phi = \frac{1}{\sqrt{n_r}}\Psi^i(\Phi) \otimes v^\alpha \tag{4.10}$$

defines a linear map $\Psi^i : \mathscr{H}_0 \to \mathscr{H}_{\bar{r}}$. Using F-algebra relations one finds for $\Phi_1, \Phi_2 \in \mathscr{H}_0$ that $\delta^{ij} \cdot (\Phi_1, \Phi_2) = (\Psi^i(\Phi_1), \Psi^j(\Phi_2))$, therefore $\Psi^i(\mathscr{H}_0) \perp \Psi^j(\mathscr{H}_0)$ if $i \neq j$ and Ψ^i is an isometry onto its image. Now the equivalence map $S : \mathscr{H}_0 \otimes \mathbf{C}^{n_r} \to \mathscr{H}_{\bar{r}}$ is constructed as follows. For $\Phi \in \mathscr{H}_0$, $u = \sum_i u^i e_i \in \mathbf{C}^{n_r}$ let

$$S(\Phi \otimes u) = \sum_{i=1}^{n_r} \Psi^i(\Phi) u^i . \tag{4.11}$$

S is an isometry onto its image because

$$(S(\Phi_1 \otimes u_1), S(\Phi_2 \otimes u_2)) = \sum_{i,j} \overline{u_1^i} (\Psi^i(\Phi_1), \Psi^j(\Phi_2)) u_2^j = (\Phi_1, \Phi_2)(u_1, u_2) .$$

To see that it is an intertwiner compute

$$S\pi_\mu(A)(\Phi \otimes u) = S\Big(\sum_{ij} \pi_0(\mu^{ij}(A))\Phi \otimes e_{ij} u\Big) = \sum_{ij} \Psi^i(\pi_0(\mu^{ij}(A))\Phi) u^j .$$

Since

$$\sum_i \pi(F_r^{i\alpha^*}) \pi_0(\mu^{ij}(A))\Phi = \pi(A)\pi(F_r^{j\alpha^*})\Phi = \frac{1}{\sqrt{n_r}} \pi_{\bar{r}}(A)\Psi^j(\Phi) \otimes v^\alpha ,$$

it follows that $\sum_i \Psi^i(\pi_0(\mu^{ij}(A))\Phi) = \pi_{\bar{r}}(A)\Psi^j(\Phi)$ thus

$$S\pi_\mu(A)(\Phi \otimes u) = \sum_j \pi_{\bar{r}}(A)\Psi^j(\Phi)u^j = \pi_{\bar{r}}(A) S(\Phi \otimes u) , \qquad A \in \mathscr{A} .$$

Therefore S is an intertwiner from π_μ to $\pi_{\bar{r}}$ thus $\operatorname{Im} S$ is an invariant subspace in $\mathscr{H}_{\bar{r}}$ under the action of $\pi_{\bar{r}}(\mathscr{A})$. Since $\pi_{\bar{r}}$ is irreducible by construction and $S \neq 0$, it follows that $\operatorname{Im} S = \mathscr{H}_{\bar{r}}$ therefore S is a unitary equivalence. □

The above theorem gives us the right to interpret μ_F, when $F = F_r(l,x)$, as a morphism creating charge $\bar{r}$. Of course one expects that for arbitrary multiplet matrix F μ_F creates a charge equal to that of F^*. What is more, the whole representation theory of $\mathscr{A}$ based on representations of the form (4.9) should be equivalent to the representation theory of the symmetry algebra $\mathscr{D}(G)$. More precisely there is an equivalence between a full subcategory $\mathbf{Rep}_0\, \mathscr{A}$ of the category of C^*-representations of $\mathscr{A}$ and the category $\mathbf{Rep}\, \mathscr{D}(G)$ of C^*-representations of $\mathscr{D}(G)$ as we shall see later.

4.4. The Essential Dimension of μ

4.9. Definition. Let $\mu : \mathscr{A} \to M_m(\mathscr{A})$ and $\nu : \mathscr{A} \to M_n(\mathscr{A})$ be amplimorphisms. The space of intertwiners from ν to μ is

$$(\mu \mid \nu) = \{T \in \mathrm{Mat}(m \times n, \mathscr{A}_{\mathrm{loc}}) \mid \mu(A)T = T\nu(A), A \in \mathscr{A}, \\ \mu(\mathbf{1})T = T = T\nu(\mathbf{1})\}; \tag{4.12}$$

μ and ν are called equivalent, $\mu \sim \nu$, if $\exists U \in (\mu \mid \nu)$ partial isometry such that $UU^* = \mu(\mathbf{1})$ and $U^*U = \nu(\mathbf{1})$. Such a U is a called an equivalence from ν to μ.

The amplimorphism μ is called transportable if $\alpha_x \circ \mu \circ \alpha_{-x} \sim \mu$, $\forall x \in \mathbf{Z}$, where α_x denotes the translation automorphism of $\mathscr{A}$.

In order to see the relation between the equivalence of multiplet matrices and that of the generated amplimorphisms we need the following

4.10. Lemma. *Let F_s be a D_s-multiplet matrix of size $m \times n_s$ for $s = 1, 2$. Then $\mu_{F_1} = \mu_{F_2}$ if and only if $n_1 = n_2$ and $\exists u \in (D_1 \mid D_2)$ c-number unitary such that $F_2 = F_1 u$.*

Proof. The "if" statement is obvious. Assume $\mu_{F_1} = \mu_{F_2}$. Then

$$F_1^* F_2 (A \otimes I_{n_2}) = F_1^* \mu_{F_2}(A) F_2 = F_1^* \mu_{F_1}(A) F_2 = (A \otimes I_{n_1}) F_1^* F_2 \,,$$

thus $u^{i_1 i_2} := (F_1^* F_2)^{i_1 i_2} \in \mathscr{A}' = \mathbf{C} \cdot \mathbf{1}$ by Theorem 3.9. Using identities for the coproduct, counit, antipode, and contragredient representation one proves that $u \in (D_1 \mid D_2)$. Thus $\mu_{F_1} = \mu_{F_2} \Rightarrow F_1 F_1^* = F_2 F_2^* \Rightarrow F_1 u = F_1 F_1^* F_2 = F_2 F_2^* F_2 = F_2$. Since $u^* u = F_2^* F_1 F_1^* F_2 = F_2^* F_2 F_2^* F_2 = I_{n_2}$ and $uu^* = I_{n_1}$, u is unitary and $n_1 = n_2$. $\square$

4.11. Proposition. *Let F_1 and F_2 be multiplet matrices. Then $\mu_{F_1} \sim \mu_{F_2}$ if and only if $F_1 \sim F_2$.*

Proof. If $F_1 \sim F_2$ then from Lemma 4.2 $F_2 = U^* F_1 u$ follows for some observable partial isometry matrix U and c-number unitary matrix u. Since $UU^* = F_1 F_1^*$, $U^*U = F_2 F_2^*$, $U \in (\mu_{F_1} \mid \mu_{F_2})$ is a partial isometry with initial and final projections $\mu_{F_2}(\mathbf{1})$ and $\mu_{F_1}(\mathbf{1})$, respectively. I.e. $\mu_{F_1} \sim \mu_{F_2}$.

If $\mu_{F_1} \sim \mu_{F_2}$ let $U \in (\mu_{F_1} \mid \mu_{F_2})$ be a partial isometry with $UU^* = F_1 F_1^*$, $U^*U = F_2 F_2^*$. Then $F_1' := U F_2$ satisfies $\mu_{F_1'} = \mu_{F_1}$. By Lemma 4.10 $F_1' = F_1 u$ with a c-number unitary $u \in (D_1 \mid D_2)$. Hence $F_2 = U^* F_1 u$ and $F_1 \sim F_2$. $\square$

4.12. Lemma. *μ_F is transportable for every multiplet matrix F.*

Proof. $\alpha_x \circ \mu \circ \alpha_{-x} = \mu_{F_x}$, where $F_x = \alpha_x(F)$. Since γ_a commutes with α_x for $a \in \mathscr{D}(G)$, $x \in \mathbf{Z}$, F_x is a multiplet matrix which is transformed by the same matrix representation of $\mathscr{D}(G)$ as F. Hence $F_x \sim F$ and using Proposition 4.11 $\mu_{F_x} \sim \mu_F$. $\square$

The so-called charge transporter that realizes the equivalence of μ_F and its translate μ_{F_x} is the partial isometry $U = F_x F^* \in M_m(\mathscr{A})$. U is unitary if F is non-degenerate, i.e. if μ_F is unit preserving.

4.13. Theorem. *Let $\mu : \mathscr{A} \to M_m(\mathscr{A})$ be an arbitrary localized amplimorphism (not necessarily generated by a multiplet matrix). Then there exists a unit preserving amplimorphism $\nu : \mathscr{A} \to M_n(\mathscr{A})$ which is equivalent to μ. The number n, called* *the essential dimension of μ, is uniquely determined by the equivalence class of μ. If μ is generated by a multiplet matrix then ν can be chosen to be such too. The essential dimension of μ is equal to the number of columns in the multiplet matrix generating μ.*

Proof. Since μ is localized, we can choose an interval $\Lambda = \Lambda_{a,b}$ with $a \in \mathbf{Z}$, $b \in \mathbf{Z}+\frac{1}{2}$ such that μ is localized in Λ. Then by Lemma 4.7 $\mu(\mathscr{A}(\Lambda)) \subset M_m(\mathscr{A}(\Lambda))$ and $\mathscr{A}(\Lambda)$ is isomorphic to a full matrix algebra M_N by Theorem 3.3. Therefore the projection $\mu(\mathbf{1})$ can be represented by a hermitian projection matrix $P \in M_m(M_N) = M_{mN}$. In M_{mN} two projections are equivalent iff they have the same dimension. P is equivalent to $\mathbf{1} \otimes p$ with some projection $p \in M_m$ iff $\dim \operatorname{Range} P = \operatorname{tr} P$ is divisible by N. $\mu|_{\mathscr{A}(\Lambda)}$ is uniquely determined by the projection $p_1 = \mu(e_{11})$ and by $u_a = \mu\left(e_{a1} + e_{1a} + \sum_{b\neq 1,a} e_{bb}\right)$, where e_{ab} are the matrix units in M_N. Since

$$\begin{aligned}\operatorname{tr} P &= \sum_{a=1}^{N} \operatorname{tr} \mu(e_{aa}) = \sum_{a=1}^{N} \operatorname{tr} u_a p_1 u_a^* = \sum_{a=1}^{N} \operatorname{tr} \mu(\mathbf{1}) p_1 \\ &= N \operatorname{tr} p_1 \equiv Nn\,, \qquad n \in \mathbf{Z}_+\,,\end{aligned}$$

there exists $V \in M_m(\mathscr{A}(\Lambda))$ unitary such that

$$\mu(\mathbf{1}) = V\left(\mathbf{1} \otimes \begin{pmatrix} I_n & \\ & 0 \end{pmatrix}\right) V^{-1}\,. \tag{4.13}$$

Let $v^{ij} = V^{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then $v^* v = \mathbf{1} \otimes I_n$, $vv^* = \mu(\mathbf{1})$ and the formula $\nu(A) = v^* \mu(A) v$ defines an amplimorphism ν such that $\nu \sim \mu$ and $\nu(\mathbf{1}) = \mathbf{1} \otimes I_n$.

In order to show that the number n is independent of the choice of Λ let Λ' be another half-closed, half-open interval, $\Lambda' \supset \Lambda$. Considering the above unitary V as an element of $M_m(\mathscr{A}(\Lambda'))$, Eq. (4.10) becomes an identity in $M_m(\mathscr{A}(\Lambda'))$. Thus the number n is independent of Λ. If $\mu' \sim \mu$ is another amplimorphism from the equivalence class of μ then for a large enough Λ both $\mu(\mathbf{1})$ and $\mu'(\mathbf{1})$ belong to $M_m(\mathscr{A}(\Lambda))$ and $\operatorname{tr} \mu'(\mathbf{1}) = \operatorname{tr} \mu(\mathbf{1}) = N \cdot n$. Hence n depends only on the equivalence class of μ.

If $\mu = \mu_F$ with an $m \times n'$ multiplet matrix F then $\nu = \mu_{F'}$, where $F' = v^* F$ is non-degenerate: $F'^* F' = F^* vv^* F = \mathbf{1} \otimes I_{n'}$, $F' F'^* = v^* \mu(\mathbf{1}) v = \mathbf{1} \otimes I_n$. But this is possible only for $n' = n$. □

4.5. The Category **Amp** $\mathscr{A}$

The category **Amp** $\mathscr{A}$ is defined as follows. The objects of **Amp** $\mathscr{A}$ are the localized, transportable amplimorphisms μ of $\mathscr{A}$. The set $(\mu \mid \nu)$ of morphisms from the object ν to the object μ is the Banach space of intertwiners defined in (4.12). If $\mu: \mathscr{A} \to M_m(\mathscr{A})$, $\nu: \mathscr{A} \to M_n(\mathscr{A})$, $\lambda: \mathscr{A} \to M_l(\mathscr{A})$ are objects in **Amp** $\mathscr{A}$ then the composition of $T \in (\mu \mid \nu)$ and $S \in (\nu \mid \lambda)$ is $TS \in (\mu \mid \lambda)$ with matrix elements $(TS)^{ik} = \sum_{j=1}^{n} T^{ij} S^{jk}$. This composition is associative and the identity of the object ν is $\nu(\mathbf{1}) \in (\nu \mid \nu)$ satisfying $\nu(\mathbf{1}) S = S$, $T = T\nu(\mathbf{1})$ for $T \in (\mu \mid \nu)$, $S \in (\nu \mid \lambda)$.

Notice that equivalence in the sense of Definition 4.9 is the same as equivalence of two objects in **Amp** $\mathscr{A}$ in the category theoretical sense.

Amp $\mathscr{A}$ has *subobjects*: If $E \in (\mu \mid \mu)$ is a projection then $\exists$ and object ν and a partial isometry $U \in (\mu \mid \nu)$ with $UU^* = E$, $U^*U = \nu(\mathbf{1})$: let $\nu(A) = E\mu(A)E$ and $U = E$.

Amp $\mathscr{A}$ has *direct sums*: Given the objects μ, ν in **Amp** $\mathscr{A}$ there exists an object λ in **Amp** $\mathscr{A}$ and partial isometries $V \in (\lambda \mid \mu)$, $W \in (\lambda \mid \nu)$ such that $VV^* + WW^* = \lambda(\mathbf{1})$, $V^*V = \mu(\mathbf{1})$, $W^*W = \nu(\mathbf{1})$: let $\lambda(A) = \mu(A) \oplus \nu(A)$ and V, W be the obvious partial isometries.

Thus **Amp** $\mathscr{A}$ is a C^*-category with subobjects and direct sums.

The product $F_1 \times F_2$ of multiplet matrices suggests the following definiton. The *product* of two amplimorphisms $\mu: \mathscr{A} \to M_m(\mathscr{A})$, $\nu: \mathscr{A} \to M_n(\mathscr{A})$ is the amplimorphism $\mu \times \nu: \mathscr{A} \to M_{mn}(\mathscr{A})$ defined by

$$(\mu \times \nu)^{ik,jl}(A) = \mu^{ij}(\nu^{kl}(A)). \tag{4.14}$$

The multiplication rule (4.14) has the interpretation as the "addition" of charges. It makes our category to be a strict monoidal category: the composition is an associative operation with unit ($=$ id $: \mathscr{A} \to M_1(\mathscr{A})$) and for $T \in (\mu_1 \mid \mu_2)$, $S \in (\nu_1 \mid \nu_2)$ there is an intertwiner

$$T \times S := \mu_1(S)(T \otimes I_{n_2}) = (T \otimes I_{n_1})\mu_2(S) \in (\mu_1 \times \nu_1 \mid \mu_2 \times \nu_2), \tag{4.15}$$

satisfying the identities

$$\mathbf{1} \times T = T = T \times \mathbf{1}, \qquad T_1R_1 \times T_2R_2 = (T_1 \times T_2)(R_1 \times R_2), \tag{4.16a}$$

$$(T \times S)^* = T^* \times S^*, \qquad (T \times S) \times R = T \times (S \times R), \tag{4.16b}$$

whenever they are defined. Further structures on **Amp** $\mathscr{A}$ such as braiding and conjugation will be discussed in Sect. 5.

Amp$_0$ $\mathscr{A}$ is defined to be the full subcategory of **Amp** $\mathscr{A}$ the objects of which are the amplimorphisms generated by multiplet matrices.

4.6. The Category **Rep** $\mathscr{A}$

Let the category **Rep** $\mathscr{A}$ be defined as follows. Its objects are the representations π_μ where μ runs over the set of transportable localized amplimorphisms. The set of morphisms from π_ν to π_μ is the space of intertwiners

$$(\pi_\mu \mid \pi_\nu) := \{\hat{T}: \mathscr{H}_0 \otimes \mathbf{C}^n \to \mathscr{H}_0 \otimes \mathbf{C}^m \mid \pi_\mu(A)\hat{T} = \hat{T}\pi_\nu(A), A \in \mathscr{A}, \\ \pi_\mu(\mathbf{1})\hat{T} = \hat{T} = \hat{T}\pi_\nu(\mathbf{1})\}.$$

The conditions $\pi_\mu(\mathbf{1})\hat{T} = \hat{T} = \hat{T}\pi_\nu(\mathbf{1})$ stem from the fact that π_μ is a degenerate representation if μ is not unit preserving. As a matter of fact for every $A \in \mathscr{A}$ $\pi_\mu(A)$ is zero on the subspace orthogonal to the range of the projection $\pi_\mu(\mathbf{1})$. $\pi_\mu(\mathbf{1})$ is nothing else but the identity morphism of the object μ. **Rep**$_0$ $\mathscr{A}$ is defined to be the full subcategory of **Rep** $\mathscr{A}$ the objects of which are the π_μ-s with $\mu = \mu_F$, where F runs over the multiplet matrices.

The equivalence of **Rep**$_0$ $\mathscr{A}$ and **Rep** $\mathscr{D}(G)$ will be established in two steps. At first we show – using Haag duality – that **Rep**$_0$ $\mathscr{A}$ is equivalent to the category **Amp**$_0$ $\mathscr{A}$

of amplifying homomorphisms generated by multiplet matrices. The equivalence of $\mathbf{Amp}_0\,\mathscr{A}$ and $\mathbf{Rep}\,\mathscr{D}(G)$ will be proven in Theorem 4.16.

4.14. Proposition. *The vacuum representation π_0 of $\mathscr{A}$ determines a covariant functor $\hat{\pi}_0$, of C^*-categories with direct sums and subobjects, between the categories $\mathbf{Amp}_0\,\mathscr{A}$ and $\mathbf{Rep}_0\,\mathscr{A}$ (and also between $\mathbf{Amp}\,\mathscr{A}$ and $\mathbf{Rep}\,\mathscr{A}$) according to the rules*

$$\hat{\pi}_0 : \mu \mapsto \pi_\mu = (\pi_0 \otimes \mathrm{id}) \circ \mu\,, \qquad \hat{\pi}_0 : T \mapsto \hat{T} = (\pi_0 \otimes \mathrm{id})\,(T)\,. \tag{4.17}$$

This functor is bijective both on the objects and on the morphisms and establishes the isomorphism of $\mathbf{Rep}_0\,\mathscr{A}$ and $\mathbf{Amp}_0\,\mathscr{A}$.

Proof. By definition of $\mathbf{Rep}_0\,\mathscr{A}$, $\hat{\pi}_0 : \mu \mapsto \pi_\mu$ is surjective. It is also injective since the faithfulness of π_0 implies that of $\pi_0 \otimes \mathrm{id}$, therefore $\pi_{\mu_1} = \pi_{\mu_2}$ only if $\mu_1 = \mu_2$.

If $T \in (\mu \mid \nu)$ then $\hat{\pi}_0(T) \in (\pi_\mu \mid \pi_\nu)$ obviously. Furthermore $T \mapsto (\pi_0 \otimes \mathrm{id})\,(T)$ is injective, since π_0 is faithful. To prove that it is also surjective let $\hat{T} \in (\pi_\mu \mid \pi_\nu)$ and suppose that Λ is an interval such that both μ and ν are localized in Λ. Then for $A \in \mathscr{A}(\Lambda^c)$,

$$\hat{T}(\pi_0(A) \otimes I_n) = \hat{T}\pi_\nu(A) = \pi_\mu(A)\hat{T} = (\pi_0(A) \otimes I_m)\hat{T}\,,$$

hence $\hat{T}^{ij} \in \pi_0(\mathscr{A}(\Lambda^c))' = \pi_0(\mathscr{A}(\Lambda))$ using Haag duality for the vacuum representation π_0. Thus there exists $T \in \mathscr{A}_{\mathrm{loc}}$ such that $\hat{T} = (\pi_0 \otimes \mathrm{id})\,(T)$ and surjectivity is proven.

It remained to show that the bijective map $\hat{\pi}_0$ is a functor. The identity morphisms at μ and at π_μ respectively are $\mu(\mathbf{1}) \in (\mu \mid \mu)$ and $\pi_\mu(\mathbf{1}) \in (\pi_\mu \mid \pi_\mu)$. $\hat{\pi}_0$ maps $\mu(\mathbf{1})$ precisely into $\pi_\mu(\mathbf{1})$. Finally, $\hat{\pi}_0$ also preserves composition of intertwiners, $\hat{\pi}_0(TS) = \hat{\pi}_0(T)\hat{\pi}_0(S)$, since π_0 is an algebra map. The proof of that $\hat{\pi}_0$ preserves the C^*-structure, direct sums, and subobjects is left to the reader. The same proof applies for $\hat{\pi}_0$ as a functor from $\mathbf{Amp}\,\mathscr{A}$ to $\mathbf{Rep}\,\mathscr{A}$. □

$\mathbf{Amp}\,\mathscr{A}$ is of course a richer category than $\mathbf{Rep}\,\mathscr{A}$. It also has a monoidal product, a braiding, and a notion of the conjugate. Since these notions do not exist a priori for the category $\mathbf{Rep}\,\mathscr{A}$ one can use Proposition 4.14 to transfer these structures to the category $\mathbf{Rep}\,\mathscr{A}$. The functor $\hat{\pi}_0$ will then identify $\mathbf{Rep}\,\mathscr{A}$ and $\mathbf{Amp}\,\mathscr{A}$ in all respects that a representation theory can desire. The same holds for the subcategories $\mathbf{Rep}_0\,\mathscr{A}$ and $\mathbf{Amp}_0\,\mathscr{A}$.

The definition of $\mathbf{Rep}\,\mathscr{A}$ contained in an essential way the notion of the amplimorphisms. The question naturally arises whether one can find a selection criterion which inherently characterizes a representation π as being an object of $\mathbf{Rep}\,\mathscr{A}$. The answer is the following

4.15. Theorem. *Let π be a representation of $\mathscr{A}$. Assume that there exists an interval Λ and a positive integer n such that*

$$\pi|_{\mathscr{A}(\Lambda^c)} \cong n \cdot \pi_0|_{\mathscr{A}(\Lambda^c)}\,. \tag{4.18}$$

That is, when restricted to $\mathscr{A}(\Lambda^c)$, π is equivalent to a finite multiple of π_0. Then there exists a unit preserving amplimorphisms $\mu : \mathscr{A} \to M_n(\mathscr{A})$ localized in Λ such that $\pi \cong \pi_\mu$. If π is space translation covariant then μ is transportable.

Proof. Let $V : \mathscr{H}_\pi \to \mathscr{H}_0 \otimes \mathbf{C}^n$ be an isometry such that $V\pi(A) = (\pi_0(A) \otimes I_n)V$ for all $A \in \mathscr{A}(\Lambda^c)$. Let us define $\mu : \mathscr{A} \to M_n(\mathscr{A})$ by the formula

$$(\pi_0 \otimes \mathrm{id})\,(\mu(A)) = V\pi(A)V^{-1}\,, \qquad A \in \mathscr{A}\,. \tag{4.19}$$

This definition makes sense if we show that $V\pi(A)V^{-1} \in (\pi_0 \otimes \mathrm{id})(M_n(\mathcal{A}_{\mathrm{loc}}))$ for $A \in \mathcal{A}_{\mathrm{loc}}$. Let Λ_1 be chosen in such a way that $\Lambda_1 \supset \Lambda$ and $A \in \mathcal{A}(\Lambda_1)$. Then for $B \in \mathcal{A}(\Lambda_1^c)$,

$$\begin{aligned} V\pi(A)V^{-1}(\pi_0(B) \otimes I_n) &= V\pi(AB)V^{-1} = V\pi(BA)V^{-1} \\ &= (\pi_0(B) \otimes I_n)V\pi(A)V^{-1}. \end{aligned}$$

Hence each entry of the matrix $V\pi(A)V^{-1}$ belongs to $\pi_0(\mathcal{A}(\Lambda_1^c))' = \pi_0(\mathcal{A}(\Lambda_1))$ which we wanted to show. The so defined map μ is obviously a *-homomorphism and $\mu(A) = A \otimes I_n$ for $A \in \mathcal{A}(\Lambda^c)$. Hence μ is localized in Λ and is unit preserving. Now (4.19) implies that $\pi \cong (\pi_0 \otimes \mathrm{id}) \circ \mu$.

If π is space translation covariant then let U_x^π be the unitary implementing the translation automorphism α_x on $\mathcal{H}_\pi$. If U_x denotes the respective implementation operator on $\mathcal{H}_0$ then it is easy to see that the unitary $(U_x \otimes I_n)VU_{-x}^\pi V^{-1}$ on $\mathcal{H}_0 \otimes \mathbf{C}^n$ is an equivalence between π_μ and π_{μ_x}, where $\mu_x = \alpha_x \circ \mu \circ \alpha_{-x}$. Now Proposition 4.14 implies that $\mu_x \sim \mu$, too. $\square$

The selection criterion could be weakened by allowing a certain multiple of the zero representation on the RHS of (4.18). This would then give account for all π equivalent to a π_μ with μ being possibly not unit preserving. We have seen, however, that every amplimorphism μ is equivalent to a unit preserving one (Theorem 4.13).

We do not know any inherent characterization of the representations belonging to the subcategory $\mathbf{Rep}_0\,\mathcal{A}$. The reason might be that all localized transportable amplimorphisms are generated by multiplet matrices, so $\mathbf{Rep}_0\,\mathcal{A}$ is actually equivalent to $\mathbf{Rep}\,\mathcal{A}$. If this is the case then our main theorem below, together with Proposition 4.9, implies that $\mathcal{D}(G)$ is the symmetry algebra of all superselection sectors of $\mathcal{A}$ satisfying the selection criterion formulated by the conditions of Theorem 4.15.

4.7. Reconstruction of the Category $\mathbf{Rep}\,\mathcal{D}(G)$

The sectors of (or equivalence classes in) $\mathbf{Amp}_0\,\mathcal{A}$ were created by field operators that were $\mathcal{D}(G)$ multiplets. Therefore one expects that $\mathcal{D}(G)$ is the symmetry algebra working behind the sectors of $\mathbf{Amp}_0\,\mathcal{A}$. Theorem 4.16 below shows that the symmetry algebra $\mathcal{D}(G)$ can really be recovered merely from the structure of the category $\mathbf{Amp}_0\,\mathcal{A}$. What we mean by "recovering" is that $\mathbf{Rep}\,\mathcal{D}(G)$ – as a strict monoidal braided C^*-category with subobjects, direct sums, and conjugates – can be reconstructed from $\mathbf{Amp}_0\,\mathcal{A}$ modulo isomorphisms between such categories, namely because $\mathbf{Amp}_0\,\mathcal{A}$ and $\mathbf{Rep}\,\mathcal{D}(G)$ are isomorphic.

The isomorphism is established if we can find functors $\tau\colon \mathbf{Amp}_0\,\mathcal{A} \to \mathbf{Rep}\,\mathcal{D}(G)$ and $\alpha\colon \mathbf{Rep}\,\mathcal{D}(G) \to \mathbf{Amp}_0\,\mathcal{A}$ such that $\tau \circ a$ and $a \circ \tau$ are naturally equivalent to the corresponding identity functors. Furthermore these functors should be bijections between the intertwiner spaces and preserve all structures given on these categories.

There exists, however, no natural choice for these functors. A functor $\tau = \tau_f$ can be defined for each map f that associates to an object μ of $\mathbf{Amp}_0\,\mathcal{A}$ a multiplet matrix $f(\mu)$ generating μ, i.e. $\mu_{f(\mu)} = \mu$. If $f(\mu)$ is a D-multiplet matrix then define $\tau_f(\mu) := D$. If $T \in (\mu \mid \nu)$ then define $\tau_f(T) := f(\mu)^* T f(\nu)$. Similarly if φ is a map associating to each object D of $\mathbf{Rep}\,\mathcal{D}(G)$ a D-multiplet matrix $\varphi(D)$ then a functor $a = a_\varphi$ can be defined as follows. Let $a_\varphi(D) := \mu_{\varphi(D)}$ and for $t \in (D_1 \mid D_2)$ let $a_\varphi(t) := \varphi(D_1) t \varphi(D_2)^*$.

The functors τ_f for different choices of the map f are all naturally equivalent. Similarly all a_φ are naturally equivalent.

4.16. Theorem. *The categories* $\mathbf{Amp}_0\mathscr{A}$ *and* $\mathbf{Rep}\,\mathscr{D}(G)$, *considered as strict monoidal braided* C^**-categories with subobjects, direct sums, and conjugates, are isomorphic. The isomorphism is provided by the functors* $\tau_f: \mathbf{Amp}_0\mathscr{A} \to \mathbf{Rep}\,\mathscr{D}(G)$ *and* $a_\varphi: \mathbf{Rep}\,\mathscr{D}(G) \to \mathbf{Amp}_0\mathscr{A}$ *satisfying*

$$\tau_f \circ a_\varphi \sim \mathrm{id}_{\mathbf{Amp}_0\mathscr{A}}, \qquad a_\varphi \circ \tau_f \sim \mathrm{id}_{\mathbf{Rep}\,\mathscr{D}(G)},$$

and the properties listed below.

Let $\mu, \mu_1, \mu_2, \mu_1', \mu_2'$ *be objects of* $\mathbf{Amp}_0\mathscr{A}$, $T_{11'} \in (\mu_1 \mid \mu_1')$, $T_{22'} \in (\mu_2 \mid \mu_2')$ *and let* D, D_1, D_2, D_1', D_2' *be objects of* $\mathbf{Rep}\,\mathscr{D}(G)$, $t_{11'} \in (D_1 \mid D_1')$, $t_{22'} \in (D_2 \mid D_2')$. *Then*

i) $\mu_1 \sim \mu_2 \Leftrightarrow \tau_f(\mu_1) \sim \tau_f(\mu_2)$, $D_1 \sim D_2 \Leftrightarrow a_\varphi(D_1) \sim a_\varphi(D_2)$;

ii) $\tau_f : (\mu_1 \mid \mu_2) \to (\tau_f(\mu_1) \mid \tau_f(\mu_2))$ *and* $a_\varphi : (D_1 \mid D_2) \to (a_\varphi(D_1) \mid a_\varphi(D_2))$ *are linear isomorphisms;*

iii) $\exists$ *equivalences* $u(\mu_1, \mu_2) \in (\tau_f(\mu_1) \times \tau_f(\mu_2) \mid \tau_f(\mu_1 \times \mu_2))$ *such that*

$$\tau_f(T_{11'} \times T_{22'}) = u(\mu_1, \mu_2)^* (\tau_f(T_{11'}) \times \tau_f(T_{22'})) u(\mu_1', \mu_2'),$$

$\exists$ *equivalences* $U(D_1, D_2) \in (a_\varphi(D_1) \times a_\varphi(D_2) \mid a_\varphi(D_1 \times D_2))$ *such that*

$$a_\varphi(t_{11'} \times t_{22'}) = U(D_1, D_2)^* (a_\varphi(t_{11'}) \times a_\varphi(t_{22'})) U(D_1', D_2');$$

iv) $\tau_f(\varepsilon(\mu_1, \mu_2)) = u(\mu_2, \mu_1)^* B(\tau_f(\mu_1), \tau_f(\mu_2)) u(\mu_1, \mu_2)$ *and* $a_\varphi(B(D_1, D_2)) = U(D_2, D_1)^* \varepsilon(a_\varphi(D_1), \varepsilon(a_\varphi(D_2)) U(D_1, D_2)$ *with the same natural equivalences* u *and* U *as in* iii);

v) $\tau_f(T^*) = \tau_f(T)^*$, $a_\varphi(t^*) = a_\varphi(t)^*$, $\|\tau_f(T)\| = \|T\|$, $\|a_\varphi(t)\| = \|t\|$;

vi) *if* $E \in (\mu \mid \mu)$ *is a projection and* ν *is the corresponding subobject then* $\exists$ *an equivalence* $v(\mu, E) \in (\tau_f(\mu) \mid \tau_f(\nu))$ *such that*

$$\tau_f(\nu)(b) = v(\mu, E)^* \tau_f(E) \tau_f(\mu)(b) \tau_f(E) v(\mu, E), \qquad b \in \mathscr{D}(G),$$

and analogue statement for a_φ;

vii) $\tau_f(\mu_1 \oplus \mu_2) \sim \tau_f(\mu_1) \oplus \tau_f(\mu_2)$, $a_\varphi(D_1 \oplus D_2) \sim a_\varphi(D_1) \oplus a_\varphi(D_2)$;

viii) $\exists$ *equivalences* $w(\mu) \in (\tau_f(\bar{\mu}) \mid \overline{\tau_f(\mu)})$, $W(D) \in (a_\varphi(\bar{D}) \mid \overline{a_\varphi(D)})$ *such that*

$$\tau_f(\bar{T}) = w(\nu) \overline{\tau_f(T)} w(\mu)^*, \qquad T \in (\mu \mid \nu),$$
$$a_\varphi(\bar{t}) = W(D_2) \overline{a_\varphi(t)} W(D_1)^*, \qquad t \in (D_1 \mid D_2).$$

The interpretation of i–viii) is the following. i–ii) together mean that τ_f and a_φ are equivalences from one category to the other. Properties iii–viii) express the fact that τ_f and a_φ preserve the monoidal structure, braiding, the C^*-structure, subobjects, direct sums, and conjugates, respectively. Although braiding ε and conjugation $^-$ on $\mathbf{Amp}_0\mathscr{A}$ will only be introduced in Sect. 5 we included points iv) and viii) for sake of completeness.

Proof. i) According to Proposition 4.11 and Definition 4.3 $\mu_1 \sim \mu_2 \Leftrightarrow f(\mu_1) \sim f(\mu_2) \Leftrightarrow \tau_f(\mu_1) \sim \tau_f(\mu_2)$, $D_1 \sim D_2 \Leftrightarrow \varphi(D_1) \sim \varphi(D_2) \Leftrightarrow a_\varphi(D_1) \sim a_\varphi(D_2)$.

ii) Since $t \mapsto f(\mu_1) t f(\mu_2)^*$ is an inverse for τ_f and $T \mapsto \varphi(D_1)^* T \varphi(D_2)$ is an inverse for a_φ, the statements follow.

iii) The formulae can be verified by setting $u(\mu_1, \mu_2) = [f(\mu_1) \times f(\mu_2)]^* f(\mu_1 \times \mu_2)$ and $U(D_1, D_2) = [\varphi(D_1) \times \varphi(D_2)]\varphi(D_1 \times D_2)^*$.

iv) This follows from Eq. (5.16) and from the above expressions for u and U.

v) τ_f and a_φ obviously commute with *. The norm of $T \in (\mu \mid \nu)$ is $\|T\| := \|T^*T\|^{1/2}$ and similarly for $t \in (D_1 \mid D_2)$. Since $\tau_f : (\nu \mid \mu) \to (\tau_f(\nu) \mid \tau_f(\nu))$ is a linear isomorphism by i), it is multiplicative because τ_f is a functor, and finally it commutes with the *-operation, it is a *-algebra isomorphism. Therefore it preserves the norm, $\|\tau_f(T^*T)\| = \|T^*T\|$, hence $\|\tau_f(T)\| = \|T\|$. $\|a_\varphi(t)\| = \|t\|$ can be proven analogously.

vi) Use Lemma 4.10 to conclude that since $f(\nu)$ and $Ef(\mu)$ both generate ν,$\exists v(\mu, E)$ such that $f(\nu) = Ef(\mu)v(\mu, E)$. The rest is an easy computation. We note that degenerate representations in **Rep** $\mathscr{D}(G)$ must be included in order to fit the corresponding structure in **Amp** $\mathscr{A}$.

vii) Trivial.

viii) $w(\mu) = f(\bar{\mu})^* \overline{f(\mu)}$ and $W(D) = \varphi(\bar{D}) \cdot [f(a_\varphi(D))^* \varphi(D)]^T \cdot \overline{f(a_\varphi(D))}^*$ can be shown to fulfill the requirements.

Notice that we used the same map f in the definition of the conjugate as in τ_f. Since the functors τ_f are all naturally equivalent, this can be done without loss of generality. $\square$

5. Statistics and Conjugation

The statistics operator as it was defined by Doplicher, Haag and Roberts in [DHR] realizes the concept of interchanging identical particles in the framework of local quantum field theory. Besides describing the statistics of partices, the statistics operator plays a crucial role in reconstructing the internal symmetry group [DR]. Fredenhagen, Rehren and Schroer have shown [FRS] that in two spacetime dimensions, where the statistics can no longer be analyzed in terms of the permutation group, the statistics operator gives rise to a correspondence between superselection sectors and equivalence classes of representations of the braid group. They show, furthermore, that the left inverse of an endomorphism determines a positive Markov trace on the braid group.

In this section we extend these results to G-spin models where the sectors are created by amplifying homomorphisms. We find that all the notions of the theory such as the statistics operator, statistics parameter, left inverse and the associated Markov trace and link invariant work also in these "amplified" circumstances. We compute the values of the statistical dimensions and statistics phases as explicitly as possible for an arbitrary finite group G. Finally, we relate the representation of the modular group based on the general theory of superselection sectors [R2], on the one hand, and the one based on the representation theory of $\mathscr{D}(G)$ [B2], on the other hand.

5.1. The Statistics Operator

Let $\Lambda_1, \Lambda_2 \subset \frac{1}{2}\mathbf{Z}$ finite sets. Λ_1 is said to lie in the left (right) complement of Λ_2, $\Lambda_1 \prec \Lambda_2 (\Lambda_1 \succ \Lambda_2)$, if $\Lambda_1 \subset \Lambda_2^c$ and $\Lambda_1 < \Lambda_2 (\Lambda_1 > \Lambda_2)$ hold. For the amplimorphisms μ_1, μ_2, the relations $\mu_1 \prec \mu_2$ or $\mu_1 \succ \mu_2$ mean the corresponding statements for their localization regions Λ_1, Λ_2.

5.1. Definition. Let $\mu_i : \mathcal{A} \to M_{m_i}(\mathcal{A})$, $i = 1,2$ be localized transportable amplifying homomorphisms. Choose equivalences $U_i \in (\tilde{\mu}_i \mid \mu_i)$ from μ_i to an equivalent $\tilde{\mu}_i$, $i = 1,2$, such that $\tilde{\mu}_1 \prec \tilde{\mu}_2$. Then the statistics operator of μ_1 and μ_2 is defined to be

$$\begin{aligned}\varepsilon_{\prec}(\mu_1, U_1; \mu_2, U_2) &= (U_2^* \times U_1^*) P_{12} (U_1 \times U_2) \\ &= \mu_2(U_1^*)(U_2^* \otimes I_1) P_{12} (U_1 \otimes I_2) \mu_1(U_2) \,, \end{aligned} \tag{5.1}$$

where $P_{12} \in (\tilde{\mu}_2 \times \tilde{\mu}_1 \mid \tilde{\mu}_1 \times \tilde{\mu}_2)$ has matrix elements $P_{12}^{i_2 i_1, j_1 j_2} = \mu_1^{i_1 j_1}(\mu_2^{i_2 j_2}(\mathbf{1}))$.

We note that instead of the statistics operator $\varepsilon_{\prec}$ one can introduce the operator $\varepsilon_{\succ}$, $\varepsilon_{\succ}(\mu_1, U_1; \mu_2, U_2) = \varepsilon_{\prec}(\mu_2, U_2; \mu_1, U_1)^*$ on equal right. From now on the statistics operator ε means $\varepsilon_{\prec}$.

5.2. Proposition. i) *The statistics operator is an equivalence from $\mu_1 \times \mu_2$ to $\mu_2 \times \mu_1$:*

$$\mu_2 \times \mu_1(A) \cdot \varepsilon(\mu_1, U_1; \mu_2, U_2) = \varepsilon(\mu_1, U_1; \mu_2, U_2) \cdot \mu_1 \times \mu_2(A) \,, \qquad A \in \mathcal{A} ; \tag{5.2}$$

$$\varepsilon(\mu_1, U_1; \mu_2, U_2) \cdot \varepsilon(\mu_1, U_1; \mu_2, U_2)^* = \mu_2 \times \mu_1(\mathbf{1}) \,, \tag{5.3a}$$

$$\varepsilon(\mu_1, U_1; \mu_2, U_2)^* \cdot \varepsilon(\mu_1, U_1; \mu_2, U_2) = \mu_1 \times \mu_2(\mathbf{1}) \,. \tag{5.3b}$$

ii) *$\varepsilon(\mu_1, U_1; \mu_2, U_2)$ is independent of the choice of U_1, U_2 until $\tilde{\mu}_1 \prec \tilde{\mu}_2$ holds, therefore we can write $\varepsilon(\mu_1, \mu_2) := \varepsilon(\mu_1, U_1; \mu_2, U_2)$.*
iii) *Let $\nu_1 \sim \mu_1$, $\nu_2 \sim \mu_2$ and $W_i \in (\nu_i \mid \mu_i)$, $i = 1,2$ be equivalences. Then*

$$\varepsilon(\nu_1, \nu_2) = (W_2 \otimes I_1) \mu_2(W_1) \cdot \varepsilon(\mu_1, \mu_2) \cdot \mu_1(W_2^*)(W_1^* \otimes I_2) \,. \tag{5.4}$$

Proof. i) Repeating the argument in the proof of Lemma 2.2 of [DHR$_2$] we deduce that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ commute:

$$\tilde{\mu}_1^{i_1 j_1}(\tilde{\mu}_2^{i_2 j_2}(A)) = \tilde{\mu}_2^{i_2 j_2}(\tilde{\mu}_1^{i_1 j_1}(A)) \,, \qquad A \in \mathcal{A} \,. \tag{5.5}$$

(Only double cones have to be replaced by intervals $\Lambda \subset \frac{1}{2}\mathbf{Z}$ and the causal complement by $\Lambda \mapsto \Lambda^c$.) Applying this formula for $A = \mathbf{1}$ we conclude that $P_{12} \in (\tilde{\mu}_2 \times \tilde{\mu}_1 \mid \tilde{\mu}_1 \times \tilde{\mu}_2)$ and is an equivalence. Now since the statistics operator is the product of equivalences $U_2^* \times U_1^*$, P_{12} and $U_1 \times U_2$, (5.2–3) follow.

ii) and iii) can be proven in one step. Choose $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\nu}_1, \tilde{\nu}_2$ equivalent to $\mu_1, \mu_2, \nu_1, \nu_2$ respectively in such a way that $\tilde{\mu}_1, \tilde{\nu}_1 \prec \tilde{\mu}_2, \tilde{\nu}_2$. Let $U_i \in (\tilde{\mu}_i \mid \mu_i)$, $V_i \in (\tilde{\nu}_i \mid \nu_i)$, $i = 1,2$ be equivalences. Then

$$S_i := V_i W_i U_i^* \in (\tilde{\nu}_i \mid \tilde{\mu}_i) \,, \qquad i = 1,2 \,. \tag{5.6}$$

The Haag duality argument in the proof of Proposition 4.14 implies that the localization region of S_i is contained in any interval containing the localization regions of $\tilde{\nu}_i$ and $\tilde{\mu}_i$. Therefore $S_1 \prec S_2$ and the matrix elements of S_1 and S_2 commute, i.e. $P_{12}(S_1 \otimes S_2) = (S_2 \otimes S_1) P_{12}$. Since $\tilde{\mu}_1$ is localized, (5.5–6) imply that

$$S_1 \times S_2 = (S_1 \otimes I_2)\tilde{\mu}_1(S_2) = (S_2 \otimes I_2)(\tilde{\mu}_1(\mathbf{1}) \otimes S_2) = S_1 \otimes S_2 \,.$$

Similarly we have $S_2 \times S_1 = S_2 \otimes S_1$. Consequently

$$
\begin{aligned}
&\varepsilon(\nu_1, V_1; \nu_2, V_2) \\
&\quad = (V_2^* \times V_1^*) P_{12} (V_1 \times V_2) \\
&\quad = (W_2 U_2^* S_2^* \times W_1 U_1^* S_1^*) P_{12} (S_1 U_1 W_1^* \times S_2 U_2 W_2^*) \\
&\quad = (W_2 \times W_1)(U_2^* \times U_1^*)(S_2^* \times S_1^*) P_{12} (S_1 \times S_2)(U_1 \times U_2)(W_1^* \times W_2^*) \\
&\quad = (W_2 \times W_1) \varepsilon(\mu_1, U_1; \mu_2, U_2)(W_1^* \times W_2^*) .
\end{aligned}
$$

Choosing $\mu_1 = \nu_1$, $\mu_2 = \nu_2$ this implies ii). One only has to take into account that any two pairs $\{\tilde{\mu}_1, \tilde{\mu}_2\}$, $\{\tilde{\nu}_1, \tilde{\nu}_2\}$ satisfying $\tilde{\mu}_1 \prec \tilde{\mu}_2$, $\tilde{\nu}_1 \prec \tilde{\nu}_2$ can be connected by a sequence of pairs in which the subsequent pairs satisfy $\mu_1', \nu_1' \prec \mu_2', \nu_2'$. Now iii) is obvious. □

5.3. Proposition. i) *For composition of morphisms one has the hexagonal identities:*

$$\varepsilon(\mu_1 \times \mu_2, \mu_3) = (\varepsilon(\mu_1, \mu_3) \otimes I_2) \cdot \mu_1(\varepsilon(\mu_2, \mu_3)) , \tag{5.7a}$$

$$\varepsilon(\mu_1, \mu_2 \times \mu_3) = \mu_2(\varepsilon(\mu_1, \mu_3)) \cdot (\varepsilon(\mu_1, \mu_2) \otimes I_3) . \tag{5.7b}$$

ii) *Let $T_{ca} \in (\mu_c \mid \mu_a)$ and $T_{db} \in (\mu_d \mid \mu_b)$ be arbitrary intertwiners. Then*

$$\varepsilon(\mu_c, \mu_b) \cdot (T_{ca} \otimes I_b) = \mu_b(T_{ca}) \cdot \varepsilon(\mu_a, \mu_b) , \tag{5.8a}$$

$$\varepsilon(\mu_c, \mu_d) \cdot \mu_c(T_{db}) = (T_{db} \otimes I_c) \cdot \varepsilon(\mu_c, \mu_b) . \tag{5.8b}$$

iii) *The statistics operator $\varepsilon_{ab} \equiv \varepsilon(\mu_a, \mu_b)$ obeys the coloured braid relation:*

$$\mu_3(\varepsilon_{12}) \cdot (\varepsilon_{13} \otimes I_2) \cdot \mu_1(\varepsilon_{23}) = (\varepsilon_{23} \otimes I_1) \cdot \mu_2(\varepsilon_{13}) \cdot (\varepsilon_{12} \otimes I_3) . \tag{5.9}$$

Proof. i) Due to the statement ii) of Proposition 5.2 we can use special intertwiners in the statistics operator: $\varepsilon(\mu, \mu_3) = (U_3^* \otimes I) P \mu(U_3)$, $\varepsilon(\mu_1, \mu) = \mu(U_1^*) P (U_1 \otimes I)$. Therefore the right-hand side of (5.7a) can be written as

$$
\begin{aligned}
\mathrm{RHS} &= ((U_3^* \otimes I_1) \cdot P_{13} \cdot \mu_1(U_3)) \otimes I_2) \cdot \mu_1((U_3^* \otimes I_2) \cdot P_{23} \cdot \mu_2(U_3)) \\
&= (U_3^* \otimes I_1 \otimes I_2)(P_{13} \otimes I_2)(I_1 \otimes P_{23}) \mu_1(\mu_2(U_3)) = \varepsilon(\mu_1 \times \mu_2, \mu_3) .
\end{aligned} \tag{5.10}
$$

(5.7b) can be verified similarly.

ii) Let the morphisms $\tilde{\mu}_b = \mathrm{Ad}_{U_b} \circ \mu_b$ and $\tilde{\mu}_c = \mathrm{Ad}_{U_c} \circ \mu_c$ obey the properties $\tilde{\mu}_b \succ \mu_a, \mu_c$ and $\tilde{\mu}_c \prec \mu_b, \mu_d$, respectively. Then

$$
\begin{aligned}
\mu_b(T_{ca}) \cdot \varepsilon_{ab} &\equiv \mu_b(T_{ca})(U_b^* \otimes I_a) P_{ab} \mu_a(U_b) = (U_b^* \otimes I_c) \tilde{\mu}_b(T_{ca}) P_{ab} \mu_a(U_b) \\
&= (U_b^* \otimes I_c)(I_b \otimes T_{ca}) P_{ab} \mu_a(U_b) = (U_b^* \otimes I_c) P_{cb} (T_{ca} \otimes I_b) \mu_a(U_b) \\
&= (U_b^* \otimes I_c) P_{cb} \mu_c(U_b)(T_{ca} \otimes I_b) = \varepsilon_{cb} \cdot (T_{ca} \otimes I_b) ,
\end{aligned} \tag{5.11}
$$

and similarly for (5.8b).

iii) Setting $T_{ca} = \varepsilon_{12}$, $\mu_a = \mu_1 \times \mu_2$, $\mu_b = \mu_3$, $\mu_c = \mu_2 \times \mu_1$ in (5.8a) one obtains that

$$\mu_3(\varepsilon_{12}) \varepsilon(\mu_1 \times \mu_2, \mu_3) = \varepsilon(\mu_2 \times \mu_1, \mu_3) \cdot (\varepsilon_{12} \otimes I_3) . \tag{5.12}$$

Then using (5.7a) the coloured braid relation (5.9) follows. □

5.2. The Braiding Structure on $\mathbf{Amp}\mathcal{A}$

Braiding in the monoidal category $\mathbf{Amp}\mathcal{A}$ means a natural equivalence between the functors $\times$ and $\times^{\mathrm{op}}$ that satisfies the hexagonal identities. Here $\times^{\mathrm{op}}$ denotes the opposite multiplication defined by $\mu_1 \times^{\mathrm{op}} \mu_2 = \mu_2 \times \mu_1$, $T_1 \times^{\mathrm{op}} T_2 = T_2 \times T_1$. A natural equivalence between them is a map $\varepsilon : \mathrm{Ob}(\mathbf{Amp}\mathcal{A} \times \mathbf{Amp}\mathcal{A}) \to \mathrm{Mor}(\mathbf{Amp}\mathcal{A})$ such that $\varepsilon(\mu_1, \mu_2) \in (\mu_1 \times^{\mathrm{op}} \mu_2 \mid \mu_1 \times \mu_2)$ is an equivalence satisfying that if $(T_{11'}, T_{22'}) \in ((\mu_1, \mu_2) \mid (\mu_1', \mu_2'))$, i.e. if $T_{11'} \in (\mu_1 \mid \mu_1')$, $T_{22'} \in (\mu_2 \mid \mu_2')$ then

$$\varepsilon(\mu_1, \mu_2)(T_{11'} \times T_{22'}) = (T_{22'} \times T_{11'})\varepsilon(\mu_1', \mu_2') . \tag{5.13}$$

This relation – called the naturalness of ε – was proven for the statistics operator in Proposition 5.3 since (5.13) is equivalent ot the two Eqs. (5.8a) and (5.8b).

The hexagonal identities (5.7a–b) can also be comprised into one formula namely

$$\begin{aligned} &\varepsilon(\mu_1 \times \mu_2, \mu_1' \times \mu_2') \\ &\quad = [\mu_1'(\varepsilon(\mu_1, \mu_2')) \otimes I_2][\varepsilon(\mu_1, \mu_1') \times \varepsilon(\mu_2, \mu_2')][\mu_1(\varepsilon(\mu_2, \mu_1')) \otimes I_{2'}] . \end{aligned} \tag{5.14}$$

The interpretation of (5.14) in terms of categories is the following. The natural equivalence map ε can be extended to a functor ε: $\mathbf{Amp}\mathcal{A} \times \mathbf{Amp}\mathcal{A} \to \mathrm{MOR}(\mathbf{Amp}\mathcal{A})$, which turns out to be monoidal. The objects of $\mathrm{MOR}(\mathbf{Amp}\mathcal{A})$ are the morphisms T of $\mathbf{Amp}\mathcal{A}$ and its morphisms from $T' \in (\mu' \mid \nu')$ to $T \in (\mu \mid \nu)$ are the pairs (R, S) of morphisms of $\mathbf{Amp}\mathcal{A}$ satisfying $TR = ST'$. The monoidal structure on $\mathrm{MOR}(\mathbf{Amp}\mathcal{A})$ is defined for the objects by $T, T' \mapsto T \times T'$ and for the morphisms by (R, S), $(R', S') \mapsto (R \times R', S \times S')$.

Let us define the functor ε in the following way:

$$\begin{aligned} \varepsilon &: (\mu_1, \mu_2) \mapsto \varepsilon(\mu_1, \mu_2) \in (\mu_2 \times \mu_1 \mid \mu_1 \times \mu_2) . \\ \varepsilon &: (T_{11'}, T_{22'}) \mapsto (T_{11'} \times T_{22'}, T_{22'} \times T_{11'}) \in (\varepsilon(\mu_1, \mu_2) \mid \varepsilon(\mu_1', \mu_2')) . \end{aligned}$$

Then the naturalness of ε is just the condition of ε being the above functor. The comprised hexagon identity (5.14) on the other hand defines a natural equivalence between the functors $\varepsilon \circ (\times, \times)$ and $\times \circ (\varepsilon, \varepsilon)$ from the category $(\mathbf{Amp}\mathcal{A} \times \mathbf{Amp}\mathcal{A}) \times (\mathbf{Amp}\mathcal{A} \times \mathbf{Amp}\mathcal{A})$ to $\mathrm{MOR}(\mathbf{Amp}\mathcal{A})$. That is (5.14) establishes the monoidality of the functor ε.

From the coloured braid relation one easily derives a representation of the braid group $\mathbf{B}_N$, $N = 2, 3, \ldots$ for each amplimorphism μ. Let $\mu : \mathcal{A} \to M_m(\mathcal{A})$ be an object of $\mathbf{Amp}\mathcal{A}$ and let $\sigma_1, \ldots, \sigma_{N-1}$ be the standard generators of $\mathbf{B}_N$. Then

$$\sigma_a \mapsto \beta_\mu^{(N)}(\sigma_a) := \mu^{a-1}(\varepsilon_\mu) \otimes \overbrace{I_m \otimes \ldots \otimes I_m}^{N-1-a} , \qquad a = 1, \ldots, N-1 , \tag{5.15}$$

where $\varepsilon_\mu = \varepsilon(\mu, \mu)$, defines a unitary representation $\beta_\mu^{(N)}$ of $\mathbf{B}_N$ in $M_K(\mathcal{A})$ with $K = m^N$.

The equivalence class of $\beta_\mu^{(N)}$ depends only on the equivalence class $[\mu]$ of μ due to Proposition 5.2. iii). The representations $\beta_\mu^{(N)}$, $N = 2, 3, \ldots$ can be united to a representation β_μ of $\mathbf{B}_\infty$ if we use the obvious inclusions $M_{m^N}(\mathcal{A}) \ni B \mapsto B \otimes I_m \in M_{m^{N+1}}(\mathcal{A})$. In this way β_μ will be a representation in an infinite amplification $\hat{\mathcal{A}}$ of $\mathcal{A}$ (Subsect. 5.6). The equivalence class of β_μ is the statistics of the sector $[\mu]$.

5.2. The Statistics Operator in $\mathbf{Amp}_0\,\mathcal{A}$

In this subsection we give the statistics operator in the case when the amplifying homomorphisms are generated by multiplet matrices.

5.4. Lemma. *Let* $\mu_i \equiv \mu_{F_i}$, $i = 1, 2$. *Then*

$$\varepsilon(\mu_1, \mu_2) = (F_2 \times F_1) \cdot B(D_1, D_2) \cdot (F_1 \times F_2)^*\,, \tag{5.16}$$

where B denotes the braiding in $\mathbf{Rep}\,\mathcal{D}(G)$: $B(D_1, D_2) = P_{12}(D_1 \otimes D_2)(R)$.

Proof. Using charge transfer unitaries U_1, U_2 in the definition of the statistics operator such that $U_i F_i = F_i(l_i, x_i)u_i$, $i = 1, 2$ and the braid commutation relation (2.50) of the special multiplet fields $F_1(l_1, x_1)$ and $F_2(l_2, x_2)$ the calculation is straightforward. □

Let F be an irreducible D_r-multiplet matrix and μ be the corresponding amplimorphism. Using (5.16) the unitary representation $\beta_\mu^{(N)}$ of the braid group $\mathbf{B}_N$ in (5.15) can be written as

$$\begin{aligned}\beta_\mu^{(N)}(\sigma_i) = \underbrace{(F \times \ldots \times F)}_{N} \cdot B_i(D_r, D_r) \\ \times \underbrace{(F \times \ldots \times F)^*}_{N}\,, \qquad i = 1, \ldots, N-1\,,\end{aligned} \tag{5.17a}$$

where

$$B_i(D_r, D_r) = \overbrace{I \otimes \ldots \otimes I}^{i-1} \otimes P_{i,i+1}(D_r \otimes D_r)(R) \otimes \overbrace{I \otimes \ldots \otimes I}^{N-i-1}\,. \tag{5.17b}$$

Formula (5.17a) shows explicitly that $\beta_\mu^{(N)}$ is unitary equivalent to the c-number matrix representation given by (5.17b). We note that for the special multiplet fields $F_r(l, x)$ the braid representation $\beta_\mu^{(N)}$ itself reduces to scalar matrices.

5.4. The Left Inverse and the Statistics Parameter

In this subsection we restrict ourselves to the subcategory $\mathbf{Amp}_0\,\mathcal{A}$.

5.5. Definition. The left inverse of an amplifying homomorphism $\mu: \mathcal{A} \to M_m(\mathcal{A})$ is a unit preserving positive linear map $\phi: M_m(\mathcal{A}) \to \mathcal{A}$ satisfying

$$\phi(\mu(A) B \mu(C)) = A\phi(B)C\,, \qquad A, C \in \mathcal{A}\,,\ B \in M_m(\mathcal{A})\,. \tag{5.18}$$

It follows that $\mu \circ \phi: M_m(\mathcal{A}) \to \mu(\mathcal{A})$ is a conditional expectation and $\phi \circ \mu = \mathrm{id}_{\mathcal{A}}$.

For a μ generated by a multiplet matrix a left inverse exists in the following form:

$$\phi(B) = \frac{1}{n} \cdot \mathrm{tr}(F^* B F) \equiv \frac{1}{n} \sum_{k=1}^{n} \sum_{i,j=1}^{m} F^{ik^*} B^{ij} F^{jk}\,, \tag{5.19}$$

where F is any $m \times n$ multiplet matrix generating μ, i.e. $\mu_F = \mu$. This definition is correct since Lemma 4.10 ensures us that ϕ is independent of the choice of F.

The left inverse helps us to obtain a c-number characterization of the spin-statistics properties of the superselection sectors:

5.6. Definition. The statistical parameter matrix of the amplimorphism μ generated by the multiplet matrix F is defined as

$$\lambda_\mu = \phi(\varepsilon_\mu) \equiv \frac{1}{n} \cdot \mathrm{tr}(F^* \varepsilon_\mu F) \in M_m(\mathscr{A}) . \tag{5.20}$$

We note that $\lambda_\mu \in \mu(\mathscr{A})' \cap M_m(\mathscr{A})$ because the left inverse property of ϕ and the intertwiner property of ε_μ imply the relation $[\phi(\varepsilon_\mu), \mu(A)] = \phi([\varepsilon_\mu, \mu^2(A)]) = 0$. The explicit form of the statistics parameter matrix is given by the next

5.7. Proposition. *Let μ be the amplimorphism induced by the irreducible D_r-multiplet matrix F_r. Then the statistics parameter matrix of μ is*

$$\lambda_\mu = \lambda_r \cdot \mu(\mathbf{1}) , \qquad \lambda_r = \frac{\omega_r}{d_r} , \qquad d_r = n_r , \qquad \omega_r \equiv \omega_{(A,\pi)} = \frac{1}{n_\pi} \cdot \mathrm{tr}[\pi(g)] , \tag{5.21}$$

where λ_r, d_r and ω_r are the statistics parameter, statistical dimension and statistics phase, respectively and π is a n_π-dimensional unitary irreducible representation of the centralizer subgroup $C_g \leq G$ of the element g from the conjugacy class $A \subset G$.

Proof. Using (5.16) one obtains that $\lambda_\mu^{ij} = (1/n_r) \cdot F^{ik} D_r^{kl}(R^{(1)}R^{(2)}) F^{jl^*}$. Since the element $c = R^{(1)}R^{(2)} = \sum_{g \in G} (g, g)$ is in the center of $\mathscr{D}(G)$, moreover it is invertible and $c^{-1} = c^*$, $D_r(c)$ is a phase ω_r times the identity operator in an irreducible representation. Thus $\lambda_\mu = \lambda_r \mu(\mathbf{1})$ follows. The explicit form of ω_r can be obtained by using the irreducible characters of $\mathscr{D}(G)$ given in (2.8b). □

5.5. Conjugation

The conjugation on $\mathbf{Amp}_0 \mathscr{A}$ we want to define should be a contravariant functor $^- : \mathbf{Amp}_0 \mathscr{A} \to \mathbf{Amp}_0 \mathscr{A}$ analogous to the conjugation $^-$ on $\mathbf{Rep}\, \mathscr{D}(G)$. (See the end of Sect. 2.) To achieve this we first define the conjugate of a multiplet matrix.

Let $l \in \mathbf{Z} + \frac{1}{2}$ and $x \in \mathbf{Z}$ be fixed, $x < l$. For an arbitrary representation D let

$$F_D^{ij} := \sum_{g,h \in G} D^{ij}((g,h)^*) \varrho_g(l) \delta_h(x) . \tag{5.22}$$

If F is any D-multiplet matrix then it can be uniquely written in the form $F = UF_D$ by Proposition 4.4. The conjugate of F is the $\bar{D}$-multiplet matrix

$$\bar{F} := UF_{\bar{D}} . \tag{5.23}$$

This conjugation maps equivalent multiplet matrices to equivalent ones. However $F_1 = F_2 u$ does not imply $\bar{F}_1 = \bar{F}_2 u'$. Therefore $\overline{\mu_F} = \mu_{\bar{F}}$ is not a good definition for the conjugate morphism. We have to fix a map f associating to each object μ of $\mathbf{Amp}_0 \mathscr{A}$ a multiplet matrix $f(\mu)$ such that $\mu_{f(\mu)} = \mu$. Then

$$\bar{\mu} := \mu_{\overline{f(\mu)}} \tag{5.24}$$

defines the conjugate of μ. The conjugate of an intertwiner $T \in (\mu \mid \nu)$ is the intertwiner

$$\bar{T} := \overline{f(\nu)}\, [f(\mu)^* T f(\nu)]^T\, \overline{f(\mu)}^* \in (\bar{\nu} \mid \bar{\mu}) . \tag{5.25}$$

In this way conjugation becomes a contravariant functor, which is involutive up to natural equivalence. Since $\overline{F_1 \times F_2} \sim \bar{F}_1 \times \bar{F}_2$, it can be shown that $^-$ is a monoidal functor.

This notion of conjugation, however, has some flaws. It depends on the choice of l and x. In particular it does not commute with translations. A better conjugation could have been defined by taking for $\bar{F}$ the partial isometry part in the polar decomposition of F^{*T}. The reason for not choosing this conjugation is that it presumably leads out from the class of multiplet matrices we are using in this paper.

Of course, a proper conjugation is not an arbitrary involutive monoidal contravariant functor. The conjugate $\bar{\mu}$ has to satisfy that $\bar{\mu} \times \mu$ contains id as a subobject in a special way described below.

5.8. Proposition. *Let $\mu : \mathscr{A} \to M_n(\mathscr{A})$ be an object in* $\mathbf{Amp}_0\mathscr{A}$. *Then there exist intertwiners $R \in (\bar{\mu} \times \mu \mid \mathrm{id})$ and $\bar{R} \in (\mu \times \bar{\mu} \mid \mathrm{id})$ such that*

$$(\bar{R}^* \times \mu(\mathbf{1}))(\mu(\mathbf{1}) \times R) = \mu(\mathbf{1}), \qquad (R^* \times \bar{\mu}(\mathbf{1}))(\bar{\mu}(\mathbf{1}) \times \bar{R}) = \bar{\mu}(\mathbf{1}), \tag{5.26}$$

$$R^* R = n \cdot \mathbf{1} = \bar{R}^* \bar{R}, \tag{5.27}$$

where n is the essential dimension of μ (and of $\bar{\mu}$) defined by Theorem 4.13.

Properties (5.26) describe what is called compactness of the category $\mathbf{Amp}_0\mathscr{A}$ [RT].

Proof. Let F be any multiplet matrix generating μ. Define R and $\bar{R}$ by the formulae

$$R^{ij} = \sum_{k=1}^{n} \bar{F}^{ik} F^{jk}, \qquad \bar{R}^{ij} = \sum_{k=1}^{n} F^{ik} \bar{F}^{jk}. \tag{5.28}$$

Using the weak F-algebra relations for F and $\bar{F}$ one easily obtains Eqs. (5.26–5.27). □

The statistical dimension is usually the c-number $R^* R$. Equation (5.27) therefore tells us that the statistical dimension of μ is equal to its essential dimension, which in turn is equal to the dimension of the representation $D = \tau_f(\mu)$. We shall see that this number is also equal to the square root of the index of μ.

5.9. Proposition. *Using different normalization for R and relating $\bar{R}$ to R let us define $R \in (\bar{\mu} \times \mu \mid \mathrm{id})$ and $\bar{R} \in (\mu \times \bar{\mu} \mid \mathrm{id})$ by*

$$R^{ij} := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \bar{F}^{ik} F^{jk}, \qquad \bar{R} := \varepsilon(\bar{\mu}, \mu) \cdot R, \tag{5.29}$$

when F generates μ. Then $R^* \bar{\mu}(B) R = \phi(B)$, $B \in M_m(\mathscr{A})$, is the left inverse defined in (5.19), *$R^* R = \mathbf{1} = \bar{R}^* \bar{R}$ and*

$$(\bar{R}^* \times \mu(\mathbf{1}))(\mu(\mathbf{1}) \times R) \equiv (\bar{R}^* \otimes I_m)\mu(R) = \frac{1}{n} F D(c) F^*, \tag{5.30a}$$

$$(R^* \times \bar{\mu}(\mathbf{1}))(\bar{\mu}(\mathbf{1}) \times \bar{R}) \equiv (R^* \otimes I_m)\bar{\mu}(\bar{R}) = \frac{1}{n} \bar{F} \bar{D}(\bar{c}) \bar{F}^*, \tag{5.30b}$$

where $c = \sum_{g \in G} (g, g)$ is a central unitary element of $\mathscr{D}(G)$.

Proof. $R^*\bar{\mu}(B)R = \phi(B)$ and unitarity are obvious. To obtain (5.30) we compute $\bar{R}$ using (5.16),

$$\bar{R}^{ij} = \frac{1}{\sqrt{n}}[\varepsilon(\bar{\mu},\mu)(\bar{F}\times F)]^{ij,ss}$$
$$= \frac{1}{\sqrt{n}}[(F\times\bar{F})B(\bar{D},D)]^{ij,ss} = \frac{1}{\sqrt{n}}F^{ik}\bar{F}^{jl}D^{kl}(c^*),$$

where in terms of the universal R-matrix $c^* = c^{-1} = R^{(2)}SR^{(1)}$. Then an easy F-algebra yields (5-30a–b). □

Notice that though the intertwiners R and $\bar{R}$ depend on the choice of the multiplet matrix F, expression (5.30a) does not, because the right-hand side is invariant under the replacement $F \mapsto Fu$. The right-hand side of (5.30a) is nothing else but the statistics parameter matrix of μ_F.

5.6. The Infinite Amplification of $\mathcal{A}$ and the Index of μ

In the case of an endomorphism $\varrho: \mathcal{A} \to \mathcal{A}$ the index of the inclusion $\varrho(\mathcal{A}) \subset \mathcal{A}$ was shown to be equal to the square of the statistical dimension of ϱ [L]. In order to generalize this statement to amplimorphisms $\mu: \mathcal{A} \to M_m(\mathcal{A})$ we need an amplification of $\mathcal{A}$ on which μ acts as an endomorphism.

5.10. Definition. For a fixed positive integer m define the infinite amplification $\hat{\mathcal{A}}$ of the observable algebra as the C^*-inductive limit of the tower

$$\mathcal{A} \subset \mathcal{A}\otimes M_m \subset \mathcal{A}\otimes M_m \otimes M_m \subset \dots$$

with the inclusions of tensoring with the identity matrix I_m from the right. If $\mu: \mathcal{A} \to M_m(\mathcal{A})$ is a homomorphism then μ can act on the subalgebras $M_m^{(k)}(\mathcal{A}) := \mathcal{A}\otimes M_m \otimes \dots \otimes M_m$ (k pieces of M_m) as $\mu(A\otimes a_1 \otimes \dots \otimes a_k) = \mu(A)\otimes a_1 \otimes \dots \otimes a_k \in M_m^{(k+1)}(\mathcal{A})$. The continuous extension of μ to $\hat{\mathcal{A}}$ provides an endomorphism $\mu: \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.

If μ and ν are both m-dimensional amplimorphisms then their monoidal product $\mu \times \nu$ extends to the ordinary composition $\mu\nu = \mu \circ \nu$ of endomorphisms of $\hat{\mathcal{A}}$. If μ possesses a left inverse ϕ then ϕ can also be extended to a unit preserving positive map $\phi: \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ that satisfies

$$\phi(\mu(A)B\mu(C)) = A\phi(B)C\,, \qquad A,B,C \in \hat{\mathcal{A}}. \tag{5.31}$$

Thus $\mu \circ \phi$ is a conditional expectation for the inclusion $\mu(\hat{\mathcal{A}}) \subset \hat{\mathcal{A}}$.

In the next proposition we shall make the following assumptions: Let μ be an irreducible object in **Amp**$\mathcal{A}$, that is $(\mu \mid \mu)$ is one-dimensional. Assume the existence of an irreducible $\bar{\mu}$ and an intertwiner $R \in (\bar{\mu}\mu \mid \mathrm{id})$ such that $R^*R = \mathbf{1}$. Then

$$\phi(B) = R^*\bar{\mu}(B)R \tag{5.32}$$

defines a left inverse for μ and the conditional expectation can be written as

$$\mu \circ \phi(B) = \mu(R)^* \cdot \mu\bar{\mu}(B) \cdot \mu(R)\,, \qquad B \in \hat{\mathcal{A}}. \tag{5.33}$$

5.11. Proposition. *Let μ and R be as above, and $\bar{R}$ be given by* (5.29). *If the statistics parameter λ of μ defined by $\phi(\varepsilon_\mu) = \lambda \cdot \mu(\mathbf{1})$ is non-zero then*

$$\bar{R}^* \mu(R) = \lambda \cdot \mu(\mathbf{1}), \qquad R^* \bar{\mu}(\bar{R}) = \bar{\lambda} \cdot \bar{\mu}(\mathbf{1}), \tag{5.34}$$

every $A \in \hat{\mathscr{A}}$ can be written as

$$A = \frac{1}{|\lambda|^2} \mu(R^*) \bar{E} \mu\bar{\mu}(A) \mu(R), \qquad \bar{E} = \bar{R}\bar{R}^*, \tag{5.35}$$

and the only left inverse of μ is the one defined in (5.32).

Proof. Using naturality and the hexagonal identity of ε amplified to $\hat{\mathscr{A}}$ one derives

$$\begin{aligned} &\mu(R^*)\varepsilon(\bar{\mu}\mu, \mu) = R^* \Rightarrow \mu(R^*)\varepsilon(\bar{\mu}, \mu)\bar{\mu}(\varepsilon_\mu) = R^* \\ &\quad \Rightarrow \varepsilon(\bar{\mu}, \mu)^* \mu(R) = \bar{\mu}(\varepsilon_\mu) R \Rightarrow \bar{R}^* \mu(R) = \phi(\varepsilon_\mu) = \lambda \cdot \mu(\mathbf{1}). \end{aligned}$$

Let λ' be given by $R^* \bar{\mu}(\bar{R}) = \lambda' \cdot \bar{\mu}(\mathbf{1})$ then

$$\begin{aligned} \lambda \mathbf{1} &= \lambda R^* R = R^* \lambda \bar{\mu}\mu(\mathbf{1}) R = R^* \bar{\mu}(\bar{R}^* \mu(R)) R = R^* \bar{\mu}(\bar{R}^*) R R \\ &= R^* [R^* \bar{\mu}(\bar{R})]^* R = \bar{\lambda}' \phi(\mathbf{1}), \end{aligned}$$

so (5.34) is proven. Multiplying the identity $\bar{R}A\bar{R}^* = \bar{E}\mu\bar{\mu}(A)$ from the left by $\mu(R^*)$ and from the right by $\mu(R)$ and then applying (5.34) one obtains formula (5.35).

In the endomorphism case $\mu : \mathscr{A} \to \mathscr{A}$ (5.35) implied that $\mathscr{A}$ is generated by $\mu(\mathscr{A})$ and the projection $\bar{E}$. (See the footnote in [FRS].) In the amplimorphism case this argument does not apply since R is not a square matrix therefore $\mu(R) \notin \mu(\hat{\mathscr{A}})$. Nevertheless formula (5.35) can be used to prove uniqueness of the left inverse.

Let ϕ' be any left inverse of μ. Then $\phi' : M_m(\mathscr{A}) \to \mathscr{A}$ can be extended not only to a map $\hat{\mathscr{A}} \to \hat{\mathscr{A}}$ but also to a map $\phi' : M_m^{(K,L)}(\mathscr{A}) \to M_m^{(K-1,L-1)}(\mathscr{A})$ between the non-square matrices $M_m^{(K,L)}(\mathscr{A}) := \mathrm{Mat}(m^K \times m^L, \mathscr{A})$ by the definition

$$\phi'(B)^{i_2 \dots i_K, j_2 \dots j_L} := \phi'^{i_1 j_1}(B^{i_1 i_2 \dots i_K, j_1 j_2 \dots j_L}),$$

where $\phi'^{ij} : \mathscr{A} \to \mathscr{A}$ are defined by $\phi'(A) = \phi'^{ij}(A^{ij})$, $A \in M_m(\mathscr{A})$. Now it is easy to check that formula (5.31) holds for arbitrary (non-square) matrices of observables provided the products exist. This allows us to apply ϕ' to the identity (5.35) and yields

$$\phi'(A) = \frac{1}{|\lambda|^2} R^* \phi'(\bar{E}) \bar{\mu}(A) R = \frac{1}{|\lambda|^2} \phi'(\bar{E}) \phi(A).$$

The second equation follows from $\phi'(\bar{E}) \in (\bar{\mu} \mid \bar{\mu})$, hence a scalar. Putting $A = \mathbf{1}$, $\phi'(\bar{E}) = |\lambda|^2$ follows and therefore $\phi' = \phi$. $\square$

As a consequence we have a unique conditional expectation $\mu \circ \phi : \hat{\mathscr{A}} \to \mu(\hat{\mathscr{A}})$ and the index of the inclusion $\mu(\hat{\mathscr{A}}) \subset \hat{\mathscr{A}}$ can be defined through the index of this conditional expectation. The latter one is defined through a quasibasis [W]: $\{b_\alpha\} \subset \hat{\mathscr{A}}$ is a quasibasis for $\mu \circ \phi$ if

$$\sum_\alpha b_\alpha \mu \circ \phi(b_\alpha^* A) = A, \qquad A \in \hat{\mathscr{A}}. \tag{5.36}$$

Then the index of the conditional expectation $\mu \circ \phi$ is defined by $\sum_\alpha b_\alpha b_\alpha^*$, which is a central element of $\hat{\mathscr{A}}$ and is independent of the choice of the quasibasis.

5.12. Theorem. *Let μ be as in the above proposition. A quasibasis for $\mu \circ \phi$ is $\{b_{pq} \mid p, q = 1, \ldots, m\}$, where $b_{pq} \in M_m^{(1)}(\mathscr{A})$ has matrix elements*

$$b_{pq}^{ij} = \frac{1}{|\lambda|} \delta^{ip} \bar{R}^{jq^*} . \tag{5.37}$$

Then the index $[\hat{\mathscr{A}}: \mu(\mathscr{A})]$ defined by $\sum_{pq} b_{pq} b_{pq}^* = [\hat{\mathscr{A}}: \mu(\hat{\mathscr{A}})] \cdot \hat{\mathbf{1}}$ is $1/|\lambda|^2$.

Proof. Comparing (5.33) and (5.35) we see that a sufficient condition for $\{b_{pq}\}$ to be a quasibasis is the equation

$$\sum_{pq} b_{pq} \mu(R^*) \mu\bar{\mu}(b_{pq}^*) = \frac{1}{\lambda} \bar{R}^* .$$

Inserting here the ansatz (5.37) a little calculation proves that it is a quasibasis indeed. The index is obtained from

$$\sum_{pq} (b_{pq} b_{pq}^*)^{ij} = \sum_{pq} \frac{1}{|\lambda|^2} \delta^{ip} \bar{R}^{kq^*} \delta^{jp} \bar{R}^{kq} = \frac{1}{|\lambda|^2} \delta^{ij} \cdot \mathbf{1} . \qquad \square$$

Applying this result for $\mu \in \mathrm{Ob}(\mathbf{Amp}_0 \mathscr{A})$ we see that the square root of the index of an irreducible μ is equal not only to the statistical dimension but also to the dimension of the associated irreducible Hopf algebra representation $\tau_f(\mu)$.

5.7. *The Markov Trace and Link Invariant*

For $\mu = \mu_F$ let ϕ be its left inverse and consider its N-th power ϕ^N acting on $(\mu^N \mid \mu^N) \subset M_m^{(N)}(\mathscr{A})$. By Theorem 4.16, ii) and v) the functor τ_f provides a linear isomorphism

$$\tau_f : (\mu^N \mid \mu^N) \ni T \mapsto f(\mu^N)^* T f(\mu^N) \in (D^N \mid D^N) , \qquad D \doteq \tau_f(\mu) ,$$

which commutes with the *-operation. Hence $\tau_f : (\mu^N \mid \mu^N) \to (D^N \mid D^N)$ is a C^*-algebraic isomorphism. The normalized trace $(1/n^N) \cdot \mathrm{tr}$ on $(D^N \mid D^N)$ is obviously a faithful trace state on $(D^N \mid D^N)$. Therefore $(1/n^N) \cdot \mathrm{tr} \circ \tau_f$ is also a faithful trace state on $(\mu^N \mid \mu^N)$.

On the other hand $f(\mu^N)$ and $F \times \ldots \times F$ (N factors) both generate μ^N, thus they differ by a unitary c-number matrix. Hence for $T \in (\mu^N \mid \mu^N)$,

$$\phi^N(T) = \frac{1}{n^N} \cdot \mathrm{tr}(F \times \ldots \times F)^* T (F \times \ldots \times F) = \frac{1}{n^N} \cdot \mathrm{tr}\, \tau_f(T) ,$$

where on both sides tr is the ordinary trace of $n \times n$ matrices. This proves that $\phi^N : (\mu^N \mid \mu^N) \to \mathbf{C}$ is a faithful trace state.

Obviously $\phi^{N+1}(T \otimes I_m) = \phi^N(T)$, that is the powers $\{\phi^N\}$ are compatible with the inclusions $(\mu^N \mid \mu^N) \subset (\mu^{N+1} \mid \mu^{N+1})$. This leads to

5.13. Theorem. i) *The unique left inverse* $\phi: \hat{\mathscr{A}} \to \hat{\mathscr{A}}$ *of* $\mu \in \mathrm{Ob}(\mathbf{Amp}_0 \mathscr{A})$ *defines via*

$$\varphi(T) \cdot \mathbf{1} := \lim_{N \to \infty} \phi^N(T), \qquad T \in \bigcup_N (\mu^N \mid \mu^N)$$

a faithful trace state on the C^**-subalgebra* $\overline{\bigcup_N (\mu^N \mid \mu^N)}$ *of* $\hat{A}$.

ii) *If* μ *is irreducible then* $\psi \equiv \varphi \circ \beta_\mu$ *is a Markov trace on the group algebra of the braid group* $\mathbf{B}_\infty$:

$$(\mathrm{M\,I}): \quad \psi(b_1 b_2) = \psi(b_2 b_1), \qquad b_1, b_2 \in \mathbf{B}_\infty, \tag{5.38}$$

$$(\mathrm{M\,II}): \quad \psi(b\sigma_N) = \lambda \cdot \psi(b), \qquad \psi(b\sigma_N^{-1}) = \bar{\lambda} \cdot \psi(b), \qquad b \in \mathbf{B}_N, \tag{5.39}$$

where λ *is the statistics parameter of* μ. *The strong Markov property,* $\psi(b_1 b_2) = \psi(b_1)\psi(b_2)$ *also holds, where* $b_1 \in \mathbf{B}_N$ *and* b_2 *is a word in* $\sigma_N, \sigma_{N+1}, \ldots, \sigma_M$, $M \geq N$.

Proof. i) was already shown above. (M I) follows since φ is a trace state. (M II) is a consequence of the strong Markov property, which in turn follows from

$$\begin{aligned}
\psi(b_1 b_2) &= \phi^M(\beta_\mu^{(M)}(b_1)\beta_\mu^{(M)}(b_2)) \\
&= \phi^M(\beta_\mu^{(N)}(b_1)\mu^{N-1}(\phi^{N-1}(\beta_\mu^{(M)}(b_2)))) \\
&= \phi^{M-N+1}(\phi^{N-1}(\beta_\mu^{(N)}(b_1))\phi^{N-1}(\beta_\mu^{(M)}(b_2))) \\
&= \phi^{M-N+1}(\psi(b_1)\phi^{N-1}(\beta_\mu^{(M)}(b_2))) = \psi(b_1)\psi_\mu(b_2),
\end{aligned}$$

using that $\phi^{N-1}(\beta_\mu^{(N)}(b)) = \phi^N(\beta_\mu^{(N)}(b)) \cdot \mu(\mathbf{1})$ holds for $b \in \mathbf{B}_N$ since μ is irreducible. $\square$

The Markov trace ψ leads to a link invariant through the same formula as in [FRS].

5.8. Representation of the Modular Group

Modular transformations and representations of the modular group are familiar notions in conformal field theory. Motivated by orbifold models modular transformations on the characters $\{\Phi\}$ of $\mathscr{D}(G)$ was introduced by Bántay [B2],

$$(\hat{S}\Phi)(g,h) = \Phi(h^{-1}, g), \qquad (\hat{T}\Phi)(g,h) = \Phi(g, gh), \qquad g, h \in G. \tag{5.40}$$

In a general two dimensional field theory a representation of the modular group on the superselection sectors was constructed by Rehren [R2]:

$$S_{rs} = \frac{n_r n_s}{|\sigma|}\, \phi_s(\varepsilon(\varrho_r, \varrho_s) \cdot \varepsilon(\varrho_s, \varrho_r)), \qquad T_{rs} = \delta_{r,s} \cdot \omega_r \cdot \left(\frac{\sigma}{|\sigma|}\right)^{1/3}, \tag{5.41}$$

where r, s label the irreducible sectors, ϕ_r is the left inverse of the endomorphism ϱ_r in the class r, ε is the statistics operator, and σ is defined by the help of statistical dimensions d_r and statistics phases $\omega_r : \sigma = \sum_r d_r^2 \omega_r^{-1}$.

The operators S, T are unitary matrix representations of the modular group since in both cases $SS^* = I_N = TT^*$ and the relations

$$(TS)^3 = I_N, \qquad S^2 = C, \qquad TC = CT,$$

fulfill, where $C_{rs} = \delta_{r,\bar{s}}$ is the conjugation matrix, and N is the cardinality of $\widehat{\mathscr{D}(G)}$ or $\{[\varrho_r]\}$. The S diagonalizes the corresponding fusion rules and the fusion coefficients can be expressed in the usual way [V].

Since in the case of G-spin models there is an isomorphism between the categories $\mathbf{Rep}\,\mathscr{D}(G)$ and $\mathbf{Amp}_0\,\mathscr{A}$ one expects that the above mentioned a priori different representations of the modular group should coincide. Indeed, using the form (5.16) of the statistics operator and the fact that $|\sigma|^2 = \sum_r d_r^2 = \sum_r n_r^2 = |G|^2$, with an extra left inverse ϕ_r in S (dummy in the non-amplified case) one obtains

$$
\begin{aligned}
S_{rs} \cdot \mathbf{1} &:= \frac{n_r n_s}{|\sigma|} \cdot \phi_r \phi_s(\varepsilon(\mu_r, \mu_s) \cdot \varepsilon(\mu_s, \mu_r)) \\
&= \frac{n_r}{|G|}\, \phi_r(\mathrm{tr}[F_s^* \cdot (F_s \times F_r) \cdot B(D_r, D_s) \cdot B(D_s, D_r) \cdot (F_s \times F_r)^* \cdot F_s)] \\
&= \frac{n_r}{|G|} \sum_{g,h\in G} \Phi_s(h,g)\, \phi_r(\mathrm{tr}[F_r \cdot D_r(g,h) \cdot F_r^*)] \\
&= \frac{1}{|G|} \sum_{g,h\in G} \Phi_r(g,h) \Phi_s(h,g) \cdot \mathbf{1}\,,
\end{aligned}
$$

thus $S_{rs} = \langle \Phi_r, \hat{S}\Phi_s \rangle$. Since

$$
\sigma \equiv \sum_r d_r^2 \omega_r^{-1} = \sum_r n_r^2 \cdot \frac{\Phi_r(c^{-1})}{n_r} = \Phi_{\mathrm{reg}}(c^{-1}) = |G| = |\sigma|
$$

and $\Phi_r(g, gh) = \Phi_r(c \cdot (g,h)) = \omega_r \cdot \Phi_r(g,h)$, the T-s coincide as well.

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