tiny notes on tiny cubes

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Abstract

Let $X \simeq \mathbb{R}^n$ and let $\mathcal{P}(X)$ be the pair groupoid of X.
Let G_2 be the strict 2-group coming from the crossed module

$$(G \stackrel{\mathrm{Id}}{\to} G) \subset \mathrm{AUT}(G)\,.$$

A smooth pseudofunctor

$$\mathrm{tra} : \mathcal{P}(X) \to \Sigma(G_2)$$

is given by a $\mathrm{Lie}(G)$-valued 1-form A

$$\mathrm{tra} : (\ x \longrightarrow y\ ) \mapsto (\ \bullet \xrightarrow{P\exp\left(\int_x^y A\right)} \bullet\ )$$

and fails to respect strict composition as measured by

$$\mathrm{tra}\ \ : \begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3ex, "\|"] & & x \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & \bullet \arrow[dr, "\mathrm{tra}(y,z)"] & \\ \bullet \arrow[ur, "\mathrm{tra}(x,y)"] \arrow[rr, "\mathrm{tra}(x,z)"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow \mathrm{tra}\,(x,z,y,x)"] & & \bullet \end{tikzcd}$$

Since 2-morphisms in G_2 for given source and target are unique, this compositor necessarily satisfies its coherence law.

$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{tra}(y,z)"] \arrow[dr, phantom, "\Searrow \mathrm{tra}(x,z,y,x)"] & \bullet \arrow[d, "\mathrm{tra}(z,w)"] \\ \bullet \arrow[u, "\mathrm{tra}(x,y)"] \arrow[ur, "\mathrm{tra}(x,z)" description] \arrow[r, "\mathrm{tra}(x,w)"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow \mathrm{tra}(x,w,z,x)"] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, "\mathrm{tra}(y,z)"] \arrow[dr, "\mathrm{tra}(y,w)" description] \arrow[dr, phantom, shift left=3ex, "\Swarrow \mathrm{tra}(y,w,z,y)"] & \bullet \arrow[d, "\mathrm{tra}(z,w)"] \\ \bullet \arrow[u, "\mathrm{tra}(x,y)"] \arrow[r, "\mathrm{tra}(x,w)"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow \mathrm{tra}(x,w,y,x)"] & \bullet \end{tikzcd}\ .$$

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We know that the first order expansion for a little square yields

$$
\mathrm{curv}_A \;:\;
\begin{tikzcd}
x_s \arrow[r, "\gamma_1"] \arrow[d, "\gamma_3"'] \arrow[dr, phantom, "\swArrow_S"] & x_1 \arrow[d, "\gamma_2"] \\
x_2 \arrow[r, "\gamma_4"'] & x_t
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}
\bullet \arrow[r, "1+A(\gamma_1)+\cdots"] \arrow[d, "1+A(\gamma_3)+\cdots"'] \arrow[dr, phantom, "\swArrow 1+F_A(\gamma_3,\gamma_1)+\cdots"] & \bullet \arrow[d, "1+A(\gamma_2)+\cdots"] \\
\bullet \arrow[r, "1+A(\gamma_4)+\cdots"'] & \bullet
\end{tikzcd}
$$

the curvature 2-form F_A. Hence the compositor of our pseudofunctor is given by $\frac{1}{2}F_A$.

Notice how we can write this as the supercommutator

$$
[\mathrm{tra}, \mathrm{tra}]\,(S) =
\begin{tikzcd}
\bullet \arrow[d, "1+A(\gamma_3)+\cdots"'] & \\
\bullet \arrow[r, "1+A(\gamma_4)+\cdots"'] & \bullet
\end{tikzcd}
\;-\;
\begin{tikzcd}
\bullet \arrow[r, "1+A(\gamma_1)+\cdots"] & \bullet \arrow[d, "1+A(\gamma_2)+\cdots"] \\
 & \bullet
\end{tikzcd}
\,.
$$

Compare this to how, when passing from groupoids to algebroids, a connection is a morphism

$$\nabla \equiv d\mathrm{tra} : TX \to TB/G$$

splitting the Atiyah sequence for the trivial G-bundle B

$$0 \to \mathrm{ad}\,(B) \to TB/G \to TX \to 0\,,$$

whose failure to be a morphism of algebroids is measured by its curvature 2-form.

Let us further pass from

$$G_2 = (G \to G) \subset (AUT)(G)$$

to the 3-group

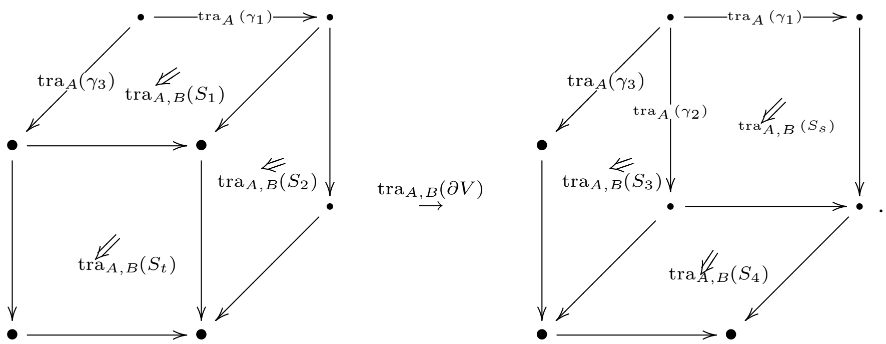
$$G_3 = \mathrm{Inn}(H \to G) \subset \mathrm{AUT}(H \to G)\,.$$

Now a pseudofunctor

$$\mathrm{tra} : \mathcal{P}\,(X) \to \Sigma\,(G_3)$$

is, as before, given by a Lie$\,(G)$-valued 1-form A, but now the compositor contains extra information and satisfies a nontrivial coherence law.

By splitting little n-cubes into $n!$ little n-simplices, we may use our results from $\Sigma\,(G_3)$-2-transport on cubical 2-paths to conclude that the compositor is now given by a Lie(H)-valued 2-form B. The above tetrahedral coherence law is now replaced by a second order compositor filling that tetrahedron, which in cubical notation reads



So this second order compositor is given by the 3-form $H = d_A B$. Its coherence law is the Bianchi identity satisfied by H.

Notice how this may again conveniently be computed from a supercommutator, as displayed in figure 1 (p. 4).

In general, for a pseudofunctor from the pair groupoid to an n-group, we should find a 1-form given by the data on edges, then a series of compositors measuring the failure of respect for composition in the pair groupoid, the highest of which defining the curvature n-form. Finally a coherence law equation, which may be interpreted as the Bianchi identity of the curvature n-form.

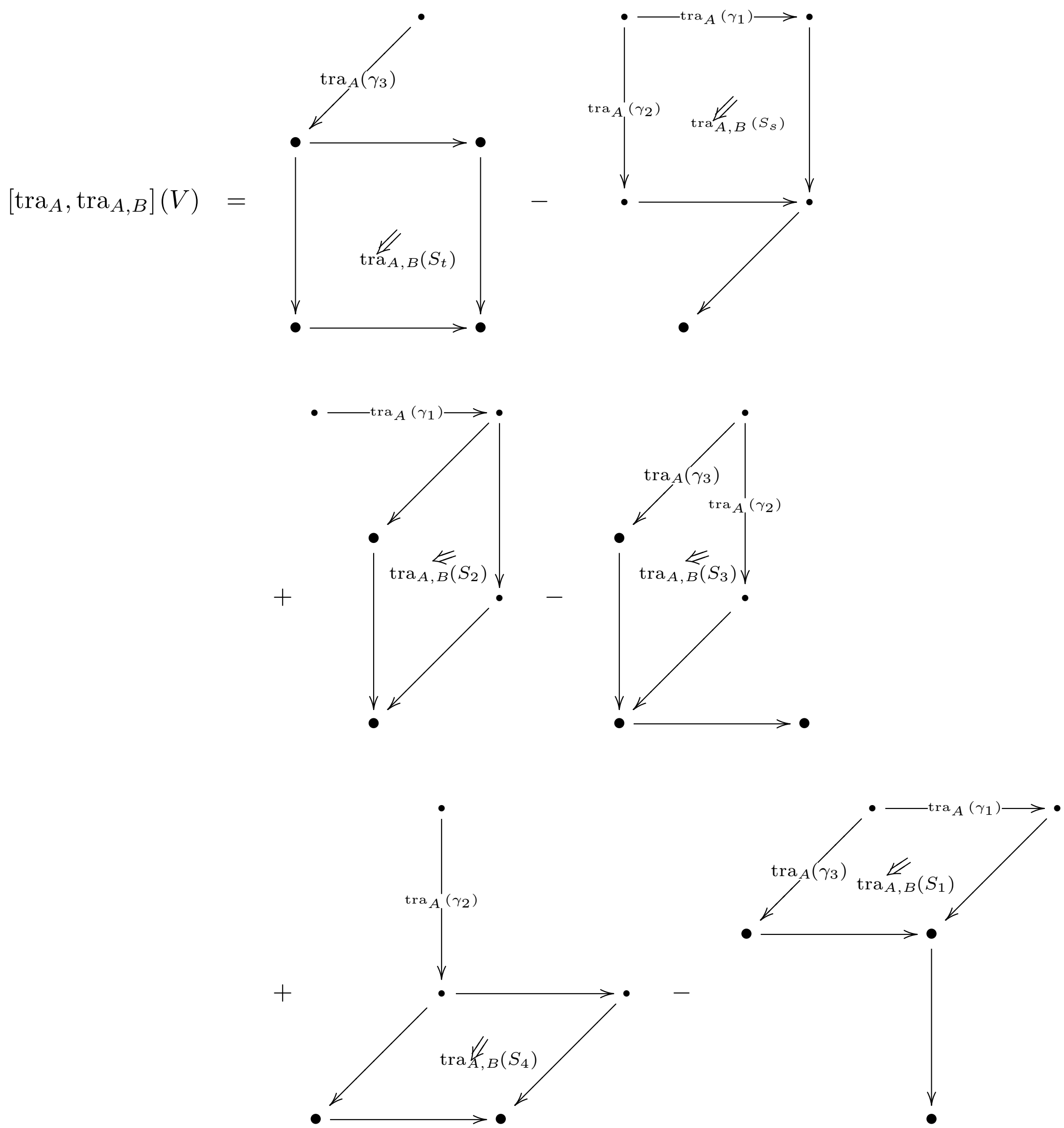


Figure 1: **Curvature 3-form** computed as the graded commutator of a transport 1-functor with a transport 2-functor.