Nuclear Physics B201 (1982) 101–140

GEOMETRIC SUPERGRAVITY IN $D = 11$ AND ITS HIDDEN SUPERGROUP

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Received 11 December 1981

In this paper we address two questions: the geometrical formulation of $D = 11$ supergravity and the derivation of the super Lie algebra it is based on. The solutions of the two problems are intimately related and are obtained via the introduction of the new concept of a Cartan integrable system described in this paper. The previously developed group manifold framework can be naturally extended to a Cartan integrable system manifold approach.

Within this scheme we obtain a geometric action for $D = 11$ supergravity based on a suitable Cartan system. This latter turns out to be a compact description of a two-element class of supergroups containing, besides Lorentz J_{ab}, translation P_a and ordinary supersymmetry Q, the following extra generators: two- and five-index skew-symmetric tensors $Z_{a_1a_2}, Z_{a_1\cdots a_5}$ and a further spinorial charge Q'. Q' commutes with itself and everything else except J_{ab}. It appears in the commutators of Q with P_a, $Z_{a_1a_2}$, $Z_{a_1\cdots a_5}$.

1. Introduction

Simple supergravity in $D = 11$ was introduced by Cremmer, Julia and Scherk in ref. [1] and later formulated by Cremmer and Ferrara in superspace [14]. It is the maximally extended supertheory containing at most spin-2 particles; by dimensional reduction [2] it yields $N = 8$ supergravity in 4 dimensions which is considered, with increasing interest, a possibly viable theory for the unification of all interactions.

An up to now unsolved problem is the identification of the supergroup underlying this theory.

This is no academic question, rather a fundamental one. Indeed, supergravity claims to be the local theory of a suitable supergroup allowing the unification of all truly elementary particles in a single supermultiplet; therefore a supergravity theory whose supergroup is unknown is somehow incomplete. The need for a supergroup was already felt by the inventors of the theory who, in their original paper [1], proposed Osp(32/1) as the most likely candidate. This proposal is based on two facts:

(i) Osp(32/1) is the minimal grading of Sp(32) which, on the other hand, is the maximal bosonic group preserving the Majorana property of a Majorana spinor.

(ii) The generators of Osp(32/1) are, with respect to the Lorentz subgroup $SO(1,10) \subset Osp(32/1)$, the following tensors (or spinors):

$$P_a, \qquad J_{ab}, \qquad Z_{a_1 \cdots a_5}, \qquad Q_\alpha, \tag{1.1}$$

where J_{ab} and $Z_{a_1 \cdots a_5}$ are skew symmetric. J_{ab}, P_a, Q_α can be respectively interpreted as the Lorentz, translation and supersymmetry generators. The 5-index skew-symmetric generator $Z_{a_1 \cdots a_5}$ on the other hand, can be conceived to be associated to the physical $A_{\mu\nu\rho}$ field appearing in $D = 11$ supergravity in the following indirect way. The potential associated to $Z_{a_1 \cdots a_5}$ is a 1-form $B_{a_1 \cdots a_5}$: multiplying $B_{a_1 \cdots a_5}$ by 5 elfbeins $V_{a_1} \wedge \cdots \wedge V_{a_5}$ (the gauge fields of the generator P_a) we obtain a 6-form B:

$$B = B^{a_1 \cdots a_5} \wedge V_{a_1} \wedge \cdots \wedge V_{a_5}. \tag{1.2}$$

Calling $B_{\mu_1 \cdots \mu_6}$ its space-time components and $\mathcal{F}_{\mu_1 \cdots \mu_7}$ their curl,

$$\mathcal{F}_{\mu_1 \cdots \mu_7} = \partial_{[\mu_1} B_{\mu_2 \cdots \mu_7]}, \tag{1.3}$$

it is attractive to assume that $\mathcal{F}_{\mu_1 \cdots \mu_7}$ is related to the curl of $A_{\mu\nu\rho}$ by a duality relation:

$$\mathcal{F}_{\mu_1 \cdots \mu_7} = \text{const} \times \varepsilon_{\mu_1 \cdots \mu_7 \nu_1 \cdots \nu_4} \partial^{\nu_1} A^{\nu_2 \nu_3 \nu_4}. \tag{1.4}$$

If this is the case, then there should be a formulation of $D = 11$ supergravity which utilizes $B_{\mu_1 \cdots \mu_6}$ as a fundamental field instead of $A_{\mu_1 \mu_2 \mu_3}$. Nicolai, Townsend and van Nieuwenhuizen tried to find it [3]. In this respect it must be noted that in the graded Lie algebra of Osp(32/1) the generators $Z_{a_1 \cdots a_5}$ are not abelian and mix, in a non-trivial way, with the space-time symmetries P_a, J_{ab}. Indeed Osp(32/1) is described by the following curvatures:

$$R^{ab} = \mathcal{R}^{ab}(\omega) + \alpha_2 V^a \wedge V^b + \alpha_3 \bar{\psi} \Gamma^{ab} \wedge \psi + \alpha_4 B^{ac_1 \cdots c_4} \wedge B^{b}_{\cdot c_1 \cdots c_4}, \tag{1.5a}$$

$$R^a = \mathcal{D} V^a - \tfrac{1}{2} i \bar{\psi} \Gamma^a \wedge \psi + \alpha_1 \varepsilon^{a b_1 \cdots b_5 c_1 \cdots c_5} B_{b_1 \cdots b_5} \wedge B_{c_1 \cdots c_5}, \tag{1.5b}$$

$$R^{a_1 \cdots a_5} = \mathcal{D} B^{a_1 \cdots a_5} - \tfrac{1}{2} i \bar{\psi} \Gamma^{a_1 \cdots a_5} \wedge \psi + \alpha_5 \varepsilon^{a_1 \cdots a_5 b_1 b_2 b_3 c_1 c_2 c_3} B_{b_1 b_2 b_3 p_1 p_2} \wedge B_{c_1 c_2 c_3}{}^{p_1 p_2}, \tag{1.5c}$$

$$\rho = \mathcal{D}\psi + i\alpha_6 \Gamma_a \psi \wedge V^a + i\alpha_7 \Gamma_{a_1 \cdots a_5} \psi \wedge B^{a_1 \cdots a_5}, \tag{1.5d}$$

where $\mathcal{D}$ denotes the Lorentz covariant derivative and $\mathcal{R}^{ab}$ is defined as

$$\mathcal{R}^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c^{\cdot b},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_7$ are numerical constants, fixed by Jacobi identities [that is integrability conditions ($\mathrm{dd}=0$) of eqs (1.5) at zero curvature]. Because of this property of the algebra, a theory based on Osp(32/1) is bound to violate the Coleman-Mandula theorem [4] since it will provide a non-trivial unification of internal and external symmetries at the bosonic level [5]. Therefore, before looking into a $B_{\mu_1\cdots\mu_6}$ formulation of $D=11$ supergravity it is advisable to perform an Inönu-Wigner contraction of Osp(32/1) by setting

$$\omega^{ab} \to \omega^{ab}, \qquad R^{ab} \to R^{ab}, \tag{1.6a}$$

$$V^a \to eV^a, \qquad R^a \to eR^a, \tag{1.6b}$$

$$B^{a_1\cdots a_5} \to eB^{a_1\cdots a_5}, \qquad R^{a_1\cdots a_5} \to eR^{a_1\cdots a_5}, \tag{1.6c}$$

$$\psi \to \sqrt{e}\,\psi, \qquad \rho \to \sqrt{e}\,\rho, \tag{1.6d}$$

where e is a scaling parameter. In the contraction limit $e \to 0$ one obtains the contracted $\overline{\mathrm{Osp}(32/1)}$ supergroup:

$$R^{ab} = \mathfrak{R}^{ab}, \tag{1.7a}$$

$$R^a = \mathcal{D}V^a - \tfrac{1}{2}i\bar{\psi} \wedge \Gamma^a \psi, \tag{1.7b}$$

$$R^{a_1\cdots a_5} = \mathcal{D}B^{a_1\cdots a_5} - \tfrac{1}{2}i\bar{\psi} \wedge \Gamma^{a_1\cdots a_5}\psi, \tag{1.7c}$$

$$\rho = \mathcal{D}\psi, \tag{1.7d}$$

which is free from the Coleman-Mandula disease since now $Z_{a_1\cdots a_5}$ is abelian. Even with these precautions, however, the result of Nicolai, Townsend and van Nieuwenhuizen was negative. The 6-form formulation of $D=11$ supergravity doesn't seem to exist [3]. As the reader will see, we reach the same conclusion in a totally different set up.

This being the state of the art, the situation we had to force was the following:

(i) $D=4$ and $D=5$ simple supergravities are interpretable as local theories of a suitable supergroup. Their lagrangians can be retrieved in a systematic way using the group manifold approach [6] which utilizes the 1-form potential of the supergroup as the only fundamental field and the geometric operations d (= exterior derivative), $\wedge$ (= wedge product) as the only allowed manipulations in the construction of the action.

(ii) The supergroup interpretation of $D=11$ supergravity and, hence, its geometric formulation within the group manifold approach is not straightforward, essentially because of the following fact: the field $A_{\mu\nu\rho}$ of the Cremmer-Julia-Scherk theory is a

3-form rather than a 1-form and therefore it cannot be interpreted as the potential of a generator in a supergroup.

The solution of the dilemma shows up almost naturally when the problem is formulated in these terms. Since Cremmer, Julia and Scherk's theory contains forms of higher degree, then the physical fields are not 1-form potentials of a super Lie algebra, rather they are p-form potentials of a generalized Cartan integrable system. The notion of Cartan integrable system (CIS), discussed in sect. 2, is a natural generalization to the case of p-forms of the Maurer-Cartan equations defining a (super) Lie algebra. All the concepts advocated by the group manifold framework, namely curvature, covariant exterior derivative, cosmo-cocycle condition for the existence of the vacuum solution and rheonomy can be almost trivially extended to the case of a CIS manifold. In this paper we first introduce the notion of Cartan integrable system and then, after showing the existence of a specific CIS in $D=11$ we construct supergravity as a geometric theory on this CIS manifold. Later, once the theory has been obtained, we address the question whether our CIS is equivalent to an ordinary supergroup; namely, whether our 3-form A can be viewed as a polynomial in a set of ordinary 1-forms in such a way that, giving the exterior derivatives of these latter, we recover the exterior derivative of the former (A).

The answer is yes and we actually get a dichotomic solution: there are two different supergroups whose 1-form potentials can be alternatively used to parameterize the 3-form A.

Both in establishing the integrability of our CIS and in solving the cosmo-cocycle condition for the linear part of the lagrangian a central role is played by Fierz identities. Because of that, in sect. 3 we study the systematics of $D=11$ Fierz identities following the group theoretical technique fully explained in ref. [7]. In this respect we want to point out that Fierz identities in $D=11$ and also the specific CIS we use were already derived by D'Adda and Regge in some unpublished notes [8] which were very inspiring for us.

The structure of the paper is the following:

Sect. 2 describes the notion of Cartan integrable system and the related concepts for the construction of a geometric theory on a CIS manifold.

In sect. 3 we give the systematics of Fierz identities for Majorana spinor 1-forms in eleven dimensions and we introduce the specific CIS we shall use in the sequel.

Sect. 4 deals with the construction of the lagrangian of $D=11$ supergravity utilizing the cosmo-cocycle closure equation (vacuum condition) to fix the linear part and the 3-form gauge-invariance principle to determine the quadratic term coefficients.

Sect. 5 deals with the equations of motion and the rheonomy property.

In sect. 6 we discuss the supergroup problem, deriving the equivalence of our CIS to two different ordinary supergroups whose 1-form potentials can be alternatively used to parametrize the physical 3-form A.

Sect. 7 contains our conclusions.

2. Cartan integrable systems

It is very well known that a (super) Lie algebra can be described in two equivalent ways. The first is provided by the familiar commutation relations among the generators (GCR). One starts with a set of operators T_A forming the basis of the tangent space T(M) to a manifold M. If we can write a set of commutation relations

$$[T_A, T_B\} = C^{\mathrm{L}}_{AB} T_{\mathrm{L}}, \tag{2.1}$$

where C^{L}_{AB} are structure constants satisfying the Jacobi identities,

$$[T_A,[T_B,T_C\}\} + (-)^{A(B+C)}[T_B,[T_C,T_A\}\} + (-)^{B(C+A)}[T_C,[T_A,T_B\}\} = 0, \tag{2.2}$$

then the manifold M is a (super) Lie group and (2.1) is its (super) Lie algebra. The Jacobi identities (2.2) is all we have to check in order to be sure that (2.1) defines a viable (super) Lie algebra. The second description of a (super) group, equally well known but, just for historical reasons, less used in the physical literature, consists of the Maurer-Cartan equations.

In this set up one considers a manifold M and its cotangent space CT(M): CT(M) is the vector space of 1-forms on the manifold M. Given a basis σ^A of CT(M), the exterior derivative $\mathrm{d}\sigma^A$ is a 2-form and can be decomposed in the basis provided by $\sigma^B \wedge \sigma^C$,

$$\mathrm{d}\sigma^A = F^A{}_{BC}\sigma^B \wedge \sigma^C. \tag{2.3}$$

If we can find a set $\{\sigma^A\}$ such that $F^A{}_{BC}$ are constants,

$$F^A{}_{BC} = -\tfrac{1}{2}C^A_{\cdot BC}, \tag{2.4}$$

consistent with the integrability condition $\mathrm{dd} = 0$; namely, if we can set

$$\mathrm{d}\sigma^A + \tfrac{1}{2}\sigma^B \wedge \sigma^C C^A_{\cdot BC} = 0, \tag{2.5}$$

and, using, (2.5) we automatically get out

$$\mathrm{dd}\sigma^A = -C^A_{\cdot BC}\mathrm{d}\sigma^B \wedge \sigma^C = -\tfrac{1}{2}C^A_{\cdot BC}C^B_{\cdot RS}\sigma^R \wedge \sigma^S \wedge \sigma^C = 0, \tag{2.6}$$

then M is a (super) Lie group and (2.5) are its Maurer-Cartan equations. The (super) Lie algebra of M is obtained via the introduction of a dual basis in the tangent space T(M): indeed if $\{T_A\}$ is a set of tangent vectors such that

$$\sigma^A(T_B) = \delta^A_B, \tag{2.7}$$

eq. (2.5) implies eq. (2.1) and vice versa. In the same way eq. (2.6) implies Jacobi identities (2.2) and vice versa. Therefore, all we have to do in order to be sure that eq. (2.5) defines a true (super) Lie group is to check whether eq. (2.6) holds. Eq. (2.6) is the integrability condition of the Maurer-Cartan equations (2.5).

As we have already pointed out, the two ways of describing a Lie algebra are totally equivalent, yet the first is more customary in physics. Dealing with gravity and supergravity theories, however, the second approach is more appropriate for the following reason. Since the ultimate goal is the construction of an action integral for the (super) group potentials, if we start with the Maurer-Cartan equations (2.5) the transition to the potentials is simply performed via the replacement of the 1-forms σ^A satisfying (2.5) (left-invariant 1-forms) with a set of 1-forms μ^A which do not satisfy (2.5) (soft forms or supergroup potentials). The 2-forms

$$R^A = R^A[\mu] = \mathrm{d}\mu^A + \tfrac{1}{2} C^A_{\cdot BC} \mu^B \wedge \mu^C \tag{2.8}$$

expressing the deviation from the Maurer-Cartan equations are called the curvatures of μ^A. The physical action is the integral of a polynomial (in the exterior algebra sense) in μ^A and R^A with the eventual addition of some 0-forms. The rules of this game, which goes under the name of group manifold approach, are discussed for example in [6] or with more details in [9]: all supergravity theories so far examined fit nicely into this framework.

The notion of Cartan integrable system appears to be a most natural generalization of the concept of (super) Lie group if we adopt the language of the Maurer-Cartan equations as the primary description of the group structure.

Suppose that we have a manifold M whose dimension, however, is not, at this point, fixed. (In the case of the proper super Lie group instead the dimension of M is just equal to the number of generators T_A or, equivalently, of left-invariant 1-forms σ^A.) Suppose that on M we define a set of p-forms of various degree $\{\Theta^{A(p)}\}$ whose exterior derivative $\mathrm{d}\Theta^{A(p)}$ can still be expressed as a polynomial in $\Theta^{A(p)}$ with constant coefficients:

$$\mathrm{d}\Theta^{A(p)} + \sum_{n=1}^{N} \frac{1}{n} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} \Theta^{B_1(p_1)} \wedge \cdots \wedge \Theta^{B_n(p_n)} = 0 . \tag{2.9}$$

The number N is equal to $p_{\max} + 1$, where $p_{\max}$ is the highest degree in the set $\{\Theta^{A(p)}\}$.

Obviously, since all addends in eq. (2.9) have to be $(p+1)$-forms, the constants $C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)}$ are different from zero only if

$$p_1 + \cdots + p_n = p + 1 . \tag{2.10}$$

Moreover, they have the proper symmetry in the exchange of any two neighbouring indices:

$$C^{A(p)}_{B_1(p_1)B_2(p_2)\cdots B_i(p_i)B_{i+1}(p_{i+1})\cdots B_n(p_n)}$$

$$=(-1)^{B_iB_{i+1}+p_ip_{i+1}}C^{A(p)}_{B_1(p_1)B_2(p_2)\cdots B_{i+1}(p_{i+1})B_i(p_i)\cdots B_n(p_n)}\,. \tag{2.11}$$

We say that eq. (2.9) is a generalized Maurer-Cartan equation (GMCE) and that it describes a Cartan integrable system (CIS) if and only if the integrability condition $\mathrm{dd}\Theta^{A(p)}=0$ follows automatically from (2.9). Explicitly, the condition for (2.9) to be a CIS is the following:

$$\mathrm{dd}\Theta^{A(p)}=-\sum_{n=1}^{N}\sum_{m=1}^{N}C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)}C^{B_1(p_1)}_{D_1(q_1)\cdots D_m(q_m)}$$

$$\times\Theta^{D_1(q_1)\cdots}\wedge\cdots\wedge\Theta^{D_m(q_m)}\wedge\Theta^{B_2(p_2)}\wedge\cdots\wedge\Theta^{B_n(p_n)}=0\,. \tag{2.12}$$

Eq. (2.12) is the analogue of eq. (2.6) and therefore it is just the analogue of the Jacobi identities (2.2) of an ordinary Lie algebra.

Given a CIS all concepts advocated by the group manifold approach can be naturally extended. Let us go through their list.

(*i*) *Soft forms or CIS potentials.* A set $\{\Theta^{A(p)}\}$ satisfying the GMCE (2.9) is named a *left-invariant set*.

A new set $\{\Pi^{A(p)}\}$ which does not satisfy (2.9) will instead be a *soft-set*. $\Pi^{A(p)}$ may be viewed as the Yang-Mills potentials of the CIS, in the same way as μ^A are the Yang-Mills potentials of the ordinary supergroup described by the ordinary Maurer-Cartan equations (2.5)

(*ii*) *CIS curvatures, CIS Bianchi identities and CIS covariant derivatives.* Given a soft set $\Pi^{A(p)}$, its deviation from the GMCE (2.9) is named the *curvature set of* $\{\Pi^{A(p)}\}$:

$$R^{A(p+1)}=\mathrm{d}\Pi^{A(p)}+\sum_{n=1}^{N}\frac{1}{n}C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)}\Pi^{B_1(p_1)}\wedge\cdots\wedge\Pi^{B_n(p_n)}=0\,. \tag{2.13}$$

The integrability of the CIS, condition (2.12), yields a differential identity on the curvatures $R^{A(p+1)}$ which is worth the name of *Bianchi identity*:

$$\nabla R^{A(p+1)}=\mathrm{d}R^{A(p+1)}+\sum_{n=1}^{N}C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)}R^{B_1(p+1)}\wedge\Pi^{B_2(p_2)}\wedge\cdots\wedge\Pi^{B_n(p_n)}=0\,. \tag{2.14}$$

In complete analogy with what one does in Chevalley cohomology theory (see [9]) we say that the l.h.s. of eq. (2.14) defines the covariant derivative of an adjoint set.

Suppose $H^{A(p+1)}$ is a set of $(p+1)$-forms: the combination

$$\nabla H^{A(p+1)} = \mathrm{d} H^{A(p+1)} + \sum_{n=1}^{N} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} H^{B_1(p_1+1)} \wedge \Pi^{B_2(p_2)} \wedge \cdots \wedge \Pi^{B_n(p_n)} \tag{2.15}$$

will be named the *covariant adjoint derivative* of $H^{A(p+1)}$. With this definition, the Bianchi identity (2.14) just states that the covariant adjoint derivative of the curvature is zero as happens with ordinary supergroups. Let us now assume that we have a multiplet $\nu_{A(d-p-1)}$ of forms whose degree is the complement of the degree of $H^{A(p+1)}$ with respect to some fixed number d. We say that $\{\nu_{A(d-p-1)}\}$ is a *coadjoint set* of d-form if I, obtained multiplying $H^{A(p+1)}$ with $\nu_{A(d-p-1)}$ is an invariant:

$$I = H^{A(p+1)} \wedge \nu_{A(d-p-1)}. \tag{2.16}$$

Invariant just means the following: the covariant derivative of I coincides with its ordinary exterior derivative:

$$\begin{aligned}\nabla I &= \nabla H^{A(p+1)} \wedge \nu_{A(d-p-1)} + (-)^{p+1} H^{A(p+1)} \wedge \nabla \nu_{A(d-p-1)} \\ &= \mathrm{d} I = \mathrm{d} H^{A(p+1)} \wedge \nu_{A(d-p-1)} + (-1)^{p+1} H^{A(p+1)} \wedge \mathrm{d}\nu_{A(d-p-1)}.\end{aligned} \tag{2.17}$$

Eq. (2.17) provides the definition of *coadjoint covariant derivative*. Indeed, in order for (2.17) to be true, we must have

$$\begin{aligned}\nabla \nu_{A(d-p-1)} = \mathrm{d}\nu_{A(d-p-1)} - (-)^{p+1} \sum_{n=1}^{N} & C^{B_1(p_1)}_{A(p)B_2(p_2)\cdots B_n(p_n)} \\ & \times \Pi^{B_2(p_2)} \wedge \cdots \wedge \Pi^{B_n(p_n)} \wedge \nu_{B_1(d-p_1-1)},\end{aligned} \tag{2.18}$$

where

$$p_1 + 1 = p + p_2 + p_3 + \cdots + p_n.$$

(*iii*) *Contraction*. The notion of contraction of a generic polynomial Ω in the soft forms $\Pi^{A(p)}$ coincides with the concept of functional variation. Therefore, we set

$$\underline{\quad A(p)}\!\rfloor\, \Omega = \frac{\delta}{\delta \Pi^{A(p)}} \Omega. \tag{2.19}$$

(*iv*) *Geometric actions and cosmo-cocycle equation for the vacuum condition.* A (pure) geometric theory on a CIS manifold will be characterized by an action principle of the following type:

$$\mathcal{Q}=\int_{\mathrm{M}_d}\left\{\Lambda+R^{A(p+1)}\wedge\nu_{A(d-p-1)}+R^{A(p+1)}\wedge R^{B(q+1)}\wedge\nu_{AB(d-p-q-2)}+\cdots\right\}, \tag{2.20}$$

where $\mathrm{M}_d\subset\mathrm{M}$ is a floating hypersurface of dimension d and Λ, $\nu_{A(d-p-1)}$, $\nu_{AB(d-p-q-2)}$ are polynomials in $\Pi^{C(n)}$ of degree $d, d-p-1, d-p-q-2$, respectively.

The condition to be satisfied by (2.20) in order to admit the vacuum solution is the straightforward generalization of the cosmo-cocycle condition customary in the group manifold approach (see [9] or [6]). It reads

$$\underline{\ {}_{A(p)}}\rfloor\Lambda+\nabla\nu_{A(d-p-1)}=0, \qquad \text{at } R^{A(p+1)}=0. \tag{2.21}$$

In the sequel we shall construct $D=11$ supergravity as a geometric theory on an appropriate CIS manifold. Before coming to that we want to address another algebraic question of some relevance. Is a Cartan integrable system reducible to an ordinary (super)group?

This question arises naturally when we try to identify the manifold M on which the left-invariant forms $\Theta^{A(p)}$ or their soft analogues $\Pi^{A(p)}$ live.

Indeed, as we pointed out at the very beginning of our discussion, the dimension of M has not been fixed, so we do not know how many independent tangent vectors T_α there are on which to project our generalized Maurer-Cartan equation (2.9). A very natural set of questions to ask is, therefore, the following:

(i) Is there a manifold M of minimal dimension $\delta=\dim \mathrm{M}$ which supports the forms $\Theta^{A(p)}$?

(ii) Is there a basis of T(M) composed of left-invariant tangent vectors (left invariant means that the components of their commutator are constants $C^{\alpha}_{\cdot\beta\gamma}$)

$$\left[T_\alpha, T_\beta\right\}=C^{\gamma}_{\cdot\alpha\beta}T_\gamma, \tag{2.22}$$

and such that the value of $\Theta^{A(p)}$ on a set of p tangent vectors T_α is a constant,

$$\Theta^{A(p)}\left(T_{\alpha_1},\ldots,T_{\alpha_p}\right)=\frac{1}{p}K^{A(p)}_{\alpha_1,\alpha_2,\ldots,\alpha_p}? \tag{2.23}$$

If the answer to both questions is yes, the manifold M is an ordinary (super) Lie Group G and the GMCE (2.9) are just the shadow of the ordinary Lie algebra (2.22).

The most appropriate way to answer these questions is to go over to a dual description of the Lie algebra (2.22) in terms of left invariant 1-forms σ^α. The problem can be formalized as it follows. First we set

$$\Theta^{A(p)} = \frac{1}{p} K^{A(p)}_{\alpha_1,\ldots,\alpha_p} \sigma^{\alpha_1} \wedge \sigma^{\alpha_2} \wedge \cdots \wedge \sigma^{\alpha_p}, \tag{2.24}$$

where $K^{A(p)}_{\alpha_1,\ldots,\alpha_p}$ are constants. Next we put

$$\mathrm{d}\sigma^\alpha + \tfrac{1}{2} C^{\alpha}_{\cdot\beta\gamma} \sigma^\beta \wedge \sigma^\gamma = 0. \tag{2.25}$$

Then the constants $K^{A(p)}_{\alpha_1\cdots\alpha_p}$ and $C^{\alpha}_{\cdot\beta\gamma}$ have to satisfy two conditions:

(*A*) *Jacobi identities on* $C^{\alpha}_{\cdot\beta\gamma}$:

$$\mathrm{dd}\sigma^\alpha = -\tfrac{1}{2} C^{\alpha}_{\cdot\beta\gamma} C^{\beta}_{\cdot\delta\eta} \sigma^\delta \wedge \sigma^\eta \wedge \sigma^\gamma = 0. \tag{2.26}$$

(*B*) *Equivalence with eq.* (2.9), namely

$$\begin{aligned}
\mathrm{d}\Theta^{A(p)} &= K^{A(p)}_{\alpha_1\cdots\alpha_p} \mathrm{d}\sigma^{\alpha_1} \wedge \sigma^{\alpha_2} \wedge \cdots \wedge \sigma^{\alpha_p} \\
&= -\tfrac{1}{2} K^{A(p)}_{\alpha_1\cdots\alpha_p} C^{\alpha_1}_{\cdot\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \wedge \sigma^{\alpha_2} \wedge \cdots \wedge \sigma^{\alpha_p} \\
&= -\sum_{n=1} \frac{1}{n} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} \Theta^{B_1(p_1)} \wedge \Theta^{B_2(p_2)} \wedge \cdots \wedge \Theta^{B_n(p_n)} \\
&= -\sum_{n=1}^{N} \frac{1}{n} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} K^{B_1(p_1)}_{\overset{1}{\beta}_1\ldots\overset{1}{\beta}_{p_1}} K^{B_2(p_2)}_{\overset{2}{\beta}_1\ldots\overset{2}{\beta}_{p_2}} \\
&\quad \cdots K^{B_n(p_n)}_{\overset{n}{\beta}_1\ldots\overset{n}{\beta}_{p_n}} \sigma^{\overset{1}{\beta}_1} \wedge \cdots \wedge \sigma^{\overset{1}{\beta}_{p_1}} \wedge \cdots \wedge \sigma^{\overset{n}{\beta}_1} \wedge \cdots \wedge \sigma^{\overset{n}{\beta}_{p_n}}.
\end{aligned} \tag{2.27}$$

Any solution of these algebraic equations on the coefficients $C^{\alpha}_{\cdot\beta\gamma}$ and $K^{A(p)}_{\alpha_1\cdots\alpha_p}$ yields a supergroup interpretation of the CIS and reduces a theory on a CIS manifold to a theory on a ordinary supergroup manifold.

What is by no means guaranteed is the uniqueness of the solution of eqs. (2.26) and (2.27). For instance in the case of the $D=11$ supergravity CIS we shall find a dichotomic solution yielding two supergroups as possible substitutes of the CIS. This means that a Çartan integrable system is a compact way of describing a collection of (super)groups and a geometric theory on a CIS manifold is actually a class of group manifold theories which are physically equivalent.

3. Systematics of Fierz identities in $D=11$ and identification of a Cartan integrable system

The basic technical tool in the derivation of geometric supergravity theories both at the on-shell and off-shell level is provided by the group theoretical decomposition of gravitino 1-form wedge products popularly called Fierz identities. This group theoretical technique has been described in [7] and was already applied to the auxiliary field problem of $D=5$ supergravity in [7] and of $D=10$ super Yang-Mills theory in [10]. In this section we present the systematics of $D=11$ Fierz identities to be used both in the identification of the CIS and in the construction of the lagrangian. Most of the results of this section were already obtained by D'Adda and Regge in unpublished notes. They used different normalizations and conventions: however, an *a posteriori* comparison of our numbers revealed a perfect match providing a very important check.

We start by giving the dimensionality of the SO(1, 10) representations appearing in the symmetric product of two, three and four gravitino 1-forms ψ (ψ is a spin $\frac{1}{2}$ Majorana 1-form). The notations of table 1 are similar to the notations of table 1 of ref. [10] and are easily explained.

The eleven-dimensional Lorentz group SO(1, 10) has, like SO(1, 9), rank 5 and therefore its irreducible representations are labeled by 5 integer or half-integer numbers.

In the integer case we are dealing with a bosonic representation and the 5-numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ labeling it can be identified with the number of boxes in each row of a Young tableau. In this way the representation $(1)^2(0)^3$ corresponds, for

TABLE 1

Dimensions of SO(1, 10) irreps appearing in the symmetric products of 2, 3, 4 irrep $\left(\frac{1}{2}\right)^5$

Bose irreps		Fermi irreps	
type	dimension	type	dimension
$(0)^5$	1	$\left(\frac{1}{2}\right)^5$	32
$(1)(0)^4$	11	$\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^4$	320
$(1)^2(0)^3$	55	$\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right)^3$	1408
$(1)^3(0)^2$	165		
$(1)^4(0)$	330		
$(1)^5$	462	$\left(\frac{3}{2}\right)^5$	4224
$(2)(0)^4$	65		
$(2)(1)(0)^3$	429		
$(2)^2(0)^3$	1144		
$(2)(1)^4$	4290		
$(2)^2(1)^3$	17160		
$(2)^5$	32604		

instance, to the tableau $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$, namely to an antisymmetric tensor $T_{a_1a_2}$. Analogously $(2)^2(0)^3$ corresponds to the tableau $\begin{array}{|c|c|}\hline a_1 & a_3 \\ \hline a_2 & a_4 \\ \hline\end{array}$ that is to the tensor $T_{a_1a_2 \atop a_3a_4}$ while (1^5) is a skew-symmetric 5-index tensor $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$ $\sim T_{a_1\cdots a_5}$.

In the half-integer case the representation is of the Fermi type. The corresponding object is a spinor tensor having in its vectorial indices the symmetry of the Young tableau $\lambda_1-\frac{1}{2}, \lambda_2-\frac{1}{2}, \lambda_3-\frac{1}{2}, \lambda_4-\frac{1}{2}, \lambda_5-\frac{1}{2}$. Moreover, it is irreducible in the sense that whatever trace can be obtained by contracting it with Γ-matrices is zero.

For instance the irrep $(\frac{3}{2})(\frac{1}{2})^4$ is a spinor tensor with the symmetry $(1)(0)^4$ in its Bose indices, namely Ξ_a. The irreducibility means $\Gamma^a\Xi_a=0$. Analogously $(\frac{3}{2})^2(\frac{1}{3})^3$ is a spinor tensor with Bose indices of the type $(1)^2(0)^3$, namely $\Xi_{a_1a_2}$ (skew symmetric). The irreducibility condition is $\Gamma^{a_2}\Xi_{a_1a_2}=0$.

The use of numerology provides an easy tool to work out the representations appearing in each symmetric product. We find

$$\left\{(\tfrac{1}{2})^5\otimes(\tfrac{1}{2})^5\right\}_{\text{sym}}=(1)(0)^4\oplus(1)^2(0)^3\oplus(1)^5,$$

$$\left(\frac{32\times 33}{2}=528=\quad 11\quad+\quad 55\quad+462\right); \tag{3.1}$$

$$\left\{(\tfrac{1}{2})^5\otimes(\tfrac{1}{2})^5\otimes(\tfrac{1}{2})^5\right\}_{\text{sym}}=(\tfrac{1}{2})^5\oplus(\tfrac{3}{2})(\tfrac{1}{2})^4\otimes(\tfrac{3}{2})^2(\tfrac{1}{2})^3\otimes(\tfrac{3}{2}),$$

$$\left(\frac{32\times 33\times 34}{3\times 2}=32\;+\;320\;+\;1408\;+4224\right), \tag{3.2}$$

$$\left\{(\tfrac{1}{2})^5\otimes(\tfrac{1}{2})^5\otimes(\tfrac{1}{2})^5\otimes(\tfrac{1}{2})^5\right\}_{\text{sym}}=$$

$$(0)^5\oplus(1)^3(0)^2\oplus(1)^4(0)\oplus(1)^5\oplus(2)(0)^4\oplus(2)(1)(0)^3\oplus(2)^2(0)^3\oplus(2)^2(1)^3\oplus(2)^5,$$

$$\left(\frac{32\times 33\times 34\times 35}{4\times 3\times 2}=\right.$$

$$\left. 1\;+\;165\;+\;330\;+462+\;65\;+\;429\;+\;1144\;+\;17160\;+32604\right), \tag{3.3}$$

These decompositions are made explicit in the following way. Let ψ be the Majorana gravitino 1-form and $\bar{\psi}=\psi^\dagger\Gamma_0=\psi^TC$ be its bar conjugate. Then we can write the Fierz decompositions given in table 2, where $\Xi^{(32)}$, $\Xi^{(320)}_a(\Gamma^a\Xi^{(320)}_a=0)$, $\Xi^{(1408)}_{a_1a_2}(\Gamma^{a_2}\Xi^{(1408)}_{a_1a_2}=0)$, $\Xi_{a_1\cdots a_5}(\Gamma^{a_5}\Xi_{a_1\cdots a_4a_5}=0)$ are, respectively, the irreducible representations $(\frac{1}{2})^5$, $(\frac{3}{2})(\frac{1}{2})^4$, $(\frac{3}{2})^2(\frac{1}{2})^3$, $(\frac{3}{2})^5$ listed in table 1. Similarly, $X^{(1)}$, $X^{(65)a}{}_b$,

TABLE 2

Explicit Fierz decomposition
$\psi\wedge\bar\psi=\frac{1}{32}\left(\Gamma_a\bar\psi\wedge\Gamma^a\psi-\frac{1}{2}\Gamma_{a_1a_2}\bar\psi\Gamma^{a_1a_2}\wedge\psi+\frac{1}{5!}\Gamma_{a_1\cdots a_5}\bar\psi\Gamma^{a_1\cdots a_5}\wedge\psi\right)$
$\psi\wedge\bar\psi\wedge\Gamma_a\psi=\Xi^{(320)}_a+\frac{1}{11}\Gamma_a\Xi^{(32)}$
$\psi\wedge\bar\psi\Gamma_{a_1a_2}\psi=\Xi^{(1408)}_{a_1a_2}-\frac{2}{9}\Gamma_{[a_1}\Xi^{(320)}_{a_2]}+\frac{1}{11}\Gamma_{a_1a_2}\Xi^{(32)}$
$\psi\wedge\bar\psi\wedge\Gamma_{a_1\cdots a_5}\psi=\Xi^{(4224)}_{a_1\cdots a_5}+2\Gamma_{[a_1a_2a_3}\Xi^{(1408)}_{a_4a_5]}$
$+\frac{5}{9}\Gamma_{[a_1\cdots a_4}\Xi^{(320)}_{a_5]}-\frac{1}{77}\Gamma_{a_1\cdots a_5}\Xi^{(32)}$
$\bar\psi\wedge\Gamma_{a_1}\psi\wedge\bar\psi\wedge\Gamma_{a_2}\psi=X^{(65)}_{\substack{a_1\\a_2}}+\frac{1}{11}\delta_{a_1a_2}X^{(1)}$
$\bar\psi\wedge\Gamma_{a_1a_2}\psi\wedge\bar\psi\wedge\Gamma_{a_3}\psi=X^{(429)}_{\substack{a_1a_2\\a_3}}+X^{(165)}_{a_1a_2a_3}$
$\bar\psi\wedge\Gamma_{a_1a_2}\psi\wedge\bar\psi\wedge\Gamma_{a_3a_4}\psi=X^{(1144)}_{\substack{a_1a_2\\a_3a_4}}+X^{(330)}_{a_1a_2a_3a_4}$
$+\frac{4}{9}\delta_{\substack{[a_1\\[a_3}}X^{(65)}_{\substack{a_2]\\a_4]}}-\frac{2}{11}\delta^{a_1a_2}_{a_3a_4}X^{(1)}$
$\bar\psi\wedge\Gamma_{a_1\cdots a_5}\psi\wedge\bar\psi\wedge\Gamma_{a_6}\psi=\varepsilon_{a_1\cdots a_6b_1\cdots b_5}X^{(462)}_{b_1\cdots b_5}+X^{(4290)}_{\substack{a_1\cdots a_5\\a_6}}$
$+\frac{15}{7}\delta_{a_6[a_1}X^{(330)}_{a_2\cdots a_5]}.$
$\bar\psi\wedge\Gamma_{a_1\cdots a_5}\psi\wedge\bar\psi\wedge\Gamma_{a_6a_7}\psi=\frac{1}{56}i\varepsilon_{a_1\cdots a_7b_1\cdots b_4}X^{(330)}_{b_1\cdots b_4}$
$-\frac{1}{300}i\varepsilon_{b_1\cdots b_5a_1\cdots a_5[a_6}X^{(4290)}_{\substack{b_1\cdots b_5\\a_7]}}+X^{(17160)}_{\substack{a_1\cdots a_5\\a_6a_7}}$
$-\frac{180}{21}\delta^{a_6a_7}_{[a_1a_2}X^{(165)}_{a_3a_4a_5]}-i1200\,\delta^{[a_6}_{[a_1}\tilde X^{(462)}_{a_2\cdots a_5]a_7]}$

$X^{(165)}_{a_1a_2a_3}$, $X^{(330)}_{a_1\cdots a_4}$, $X^{(462)}_{a_1\cdots a_5}$, $X^{(429)}_{\substack{a_1a_2\\a_3}}$, $X^{(1144)}_{\substack{a_1a_2\\a_3a_4}}$, $X^{(4290)}_{\substack{b_1\cdots b_5\\a_2}}$, $X^{(17160)}_{\substack{a_1\cdots a_5\\b_1b_2}}$ are, respectively, the bosonic irreducible representations $(0)^5$, $(2)(0)^4$, $(1^3)(0)^2$, $(1^4)(0)$, $(1)^5$, $(2)(1)(0)^3$, $(2)^2(0)^3$, $(2)(1)^4$, $(2)^2(1)^3$, also listed in table 1. Moreover, we have

$$\tilde X^{(462)}_{a_1\cdots a_6}=\varepsilon_{a_1\cdots a_6b_1\cdots b_5}X^{(462)}_{b_1\cdots b_5}. \tag{3.4}$$

As we have explained in [7] the decomposition of table 2 is a substitute for all Fierz identities which correspond to the appearance of the same irreps in several different products of fermionic currents. The irreps Ξ and X form a complete and orthonormal basis for the decomposition of, respectively, 3-ψ and 4-ψ terms.

With these tools we are now ready to address the question about the Cartan integrable system suitable for $D=11$ supergravity.

We first narrow down our hunting ground by taking into account the following remarks.

(i) Since supergravity contains ordinary gravity plus the Rarita-Schwinger field, our CIS must be an extension of the following ordinary Maurer-Cartan equations:

$$\mathrm{d}\omega^{ab}-\omega^{a}_{\cdot c}\wedge\omega^{cb}=0\,, \tag{3.5a}$$

$$\mathrm{d}V^{a}-\omega^{ab}\wedge V_{b}-\tfrac{1}{2}i\bar{\psi}\wedge\Gamma^{a}\psi=0\,, \tag{3.5b}$$

$$\mathrm{d}\psi-\tfrac{1}{4}\omega^{ab}\wedge\Gamma_{ab}\psi=0\,, \tag{3.5c}$$

which correspond to the super Lie algebra of the graded Poincaré group in eleven dimensions. The indices a, b, c run from 0 to 10 and the standard minkowskian metric

$$\eta_{ab}=\begin{pmatrix} 1 & \cdots & \cdots & 0\\ 0 & -1 & & \\ \vdots & & & \\ 0 & & & -1\end{pmatrix} \tag{3.6}$$

is used in the raising and lowering operations.

The skew-symmetric $\omega^{ab}=-\omega^{ba}$ is the Lorentz connection 1-form, V^{a} is the elfbein 1-form and ψ is the Majorana gravitino 1-form. The conventions adopted for Γ-matrices are listed in the appendix.

(ii) Since in $D=11$ there is no internal symmetry group whose indices can be used and since we admit only massless particles of spin smaller than 2 the only other Bose fields which might enter the lagrangian are skew-symmetric tensors of the type $A_{\mu_1\mu_2\cdots\mu_p}$. These latter are nothing other than p-forms.

(iii) If we assume that supersymmetry is linearly realized, the transformation rule of $A_{\mu_1\cdots\mu_p}$ must be of the following type:

$$\delta A_{\mu_1\cdots\mu_p}=\mathrm{const}\times\bar{\varepsilon}\Gamma_{[\mu_1\cdots\mu_{p-1}}\psi_{\mu_p]}\,. \tag{3.7}$$

Eq. (3.7) means that, in the vacuum, which is what matters for the derivation of

GMCE, the exterior derivative of $A^{(p)}$ has to be the following:

$$\mathrm{d}A^{(p)} = \alpha_p \bar{\psi} \wedge \Gamma^{a_1 \cdots a_{p-1}} \psi \wedge V_{a_1} \wedge \cdots \wedge V_{a_{p-1}}, \tag{3.8}$$

where α_p is some non-zero constant. Since the only non-vanishing currents are those corresponding to symmetric Γ-matrices, namely

$$\bar{\psi} \wedge \Gamma^a \psi, \qquad \bar{\psi} \wedge \Gamma^{a_1 a_2} \psi, \qquad \bar{\psi} \wedge \Gamma^{a_1 \cdots a_5} \psi, \tag{3.9}$$

and their duals

$$\bar{\psi} \wedge \Gamma^{a_1 \cdots a_{10}} \psi, \qquad \bar{\psi} \wedge \Gamma^{a_1 \cdots a_9} \psi, \qquad \bar{\psi} \wedge \Gamma^{a_1 \cdots a_6} \psi, \tag{3.10}$$

we conclude that the only *a priori* viable forms are $A^{(2)}$, $A^{(3)}$, $A^{(6)}$, $A^{(7)}$, $A^{(10)}$ and $A^{(11)}$. The Cartan system obtained by the addition of eq. (3.8) to eqs. (3.5) must, however, be integrable; namely, we must have

$$\begin{aligned} \mathrm{dd}A^{(p)} &= \alpha_p \mathscr{D}\left(\bar{\psi} \wedge \Gamma^{a_1 \cdots a_{p-1}} \psi \wedge V_{a_1} \wedge \cdots \wedge V_{a_{p-1}} \right) \\ &= (p-1) \alpha_p \frac{i}{2} \bar{\psi} \wedge \Gamma^{a_1 \cdots a_{p-1}} \psi \wedge \bar{\psi} \wedge \Gamma_{a_1} \psi \wedge V_{a_2} \wedge \cdots \wedge V_{a_{p-1}} = 0 . \end{aligned} \tag{3.11}$$

Whether eq. (3.11) holds depends on the structure of the Fierz identities listed in table 2. Indeed, in order for (3.11) to be true we must have

$$\bar{\psi} \wedge \Gamma^{m a_1 \cdots a_{p-2}} \psi \wedge \bar{\psi} \wedge \Gamma_m \psi = 0, \tag{3.12}$$

which happens only if

$$\begin{aligned} &p-2=1, \qquad p-2=10, \\ &p-2=2, \qquad p-2=9. \end{aligned} \tag{3.13}$$

Conditions (3.13) are easily understood recalling eq. (3.3) which states that the only antisymmetric tensors absent in the decomposition of $\{(\frac{1}{2})^5 \otimes (\frac{1}{2})^5 \otimes (\frac{1}{2})^5 \otimes (\frac{1}{2})^5\}_{\mathrm{sym}}$ are $(1)(0)^4$, $(1)^2(0)^3$ and obviously their duals $(1)^{10}(0)$, $(1)^9(0)^2$.

Therefore, the viable p-forms which can be embedded together with ω^{ab}, V^a, ψ in a CIS are those among $p = 2, 3, 6, 7, 10, 11$ which also satisfy eq. (3.13), namely

$$p = 3, \qquad p = 11. \tag{3.14}$$

Now since $A^{(11)}$ is a form of maximum degree, its curl (= exterior derivative) cannot enter the lagrangian of $D = 11$. Hence it is to be dismissed. Therefore, we conclude that the CIS corresponding to a linear representation of supersymmetry in eleven dimensions later to be called C_1 is described by the following generalized curvatures:
Cartan integrable system C_1

$$\mathfrak{R}^{ab} = \mathrm{d}\omega^{ab} - \omega^{ac} \wedge \omega_c^{\cdot b}, \tag{3.15a}$$

$$R^a = \mathfrak{D}V^a - \tfrac{1}{2}i\bar{\psi} \wedge \Gamma^a\psi, \tag{3.15b}$$

$$\rho = \mathfrak{D}\psi, \tag{3.15c}$$

$$R^\square = \mathrm{d}A - \tfrac{1}{2}\bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b. \tag{3.15d}$$

The GMCE obtains when ω^{ab}, V^a, ψ, A are left invariant and the curvatures are set to zero. In the soft-case, when the curvatures are different from zero, the integrability of the system shows up as Bianchi identities:
CIS Bianchi of C_1

$$\nabla R^{ab} = \mathfrak{D}R^{ab} = 0, \tag{3.16a}$$

$$\nabla R^a = \mathfrak{D}R^a + R^{ab} \wedge V_b - i\bar{\psi} \wedge \Gamma^a\rho = 0, \tag{3.16b}$$

$$\nabla\rho = \mathfrak{D}\rho + \tfrac{1}{4}\Gamma_{ab}\psi \wedge R^{ab} = 0, \tag{3.16c}$$

$$\nabla R^\square = \mathrm{d}R^\square - \bar{\psi} \wedge \Gamma^{a_1a_2}\rho \wedge V_{a_1} \wedge V_{a_2} + \bar{\psi} \wedge \Gamma^{a_1a_2}\psi \wedge R_{a_1} \wedge V_{a_2} = 0. \tag{3.16d}$$

If $\{\nu_{ab}, \nu_a, n, \nu_\square\}$ is a coadjoint set where ν_{ab}, ν_a, n are of degree $(d-2)$ and $\nu_\square$ is of degree $d-4$, and we write the invariant

$$I = R^{ab} \wedge \nu_{ab} + R^a \wedge \nu_a + \bar{\rho} \wedge n + R^\square \wedge \nu_\square, \tag{3.17}$$

the procedure outlined in sect. 2 (eq. (2.16) and following ones) yields the definition of the coadjoint covariant derivative:
Coadjoint covariant derivative of C_1

$$\nabla\nu_{ab} = \mathfrak{D}\nu_{ab} + V_{[a} \wedge \nu_{b]} + \tfrac{1}{4}\bar{\psi} \wedge \Gamma_{ab}n, \tag{3.18a}$$

$$\nabla\nu_a = \mathfrak{D}\nu_a - \bar{\psi} \wedge \Gamma_{ab}\psi \wedge V^b \wedge \nu_\square, \tag{3.18b}$$

$$\nabla n = \mathfrak{D}n - \Gamma_{a_1a_2}\psi \wedge V^{a_1} \wedge V^{a_2} \wedge \nu_\square - i\Gamma_a\psi \wedge \nu^a, \tag{3.18c}$$

$$\nabla\nu_\square = \mathrm{d}\nu_\square. \tag{3.18d}$$

Being through with these preliminaries, we could now start turning the crank and constructing our geometric lagrangian based on C_1. We wish, however, to anticipate a problem we are going to have. It concerns the propagation of the 3-form A, namely the $A_{\mu\nu\rho}$ field of the Cremmer-Julia-Scherk formulation [1]. In fact, since $A_{\mu\nu\rho}$ is a physical particle, it demands a kinetic term of the type

$$*R^{\square} \wedge R^{\square} \tag{3.19}$$

involving the notion of Hodge duality on the space-time submanifold. As is well known, Hodge dualization is a meaningless operation in the geometric group manifold approach and terms like (3.19) have to emerge in the second-order lagrangian after the elimination of some non-propagating fields appearing in the first-order one. So far only two mechanisms are known to get this result. One was found in $D=5$ supergravity [11, 12] and also in the coupling of a scalar field to gravity [16]. In $D=5$ supergravity it works in the following way.

The torsion equation, obtained through the ω^{ab} variation yields

$$\varepsilon_{abc_1c_2c_3} R^{c_1} \wedge V^{c_2} \wedge V^{c_3} + \eta V_a \wedge V_b \wedge R^{\otimes} = 0\,, \tag{3.20}$$

where $\eta = \pm 1$, R^{c_1} is the supertorsion and

$$R^{\otimes} = \mathrm{d}B^{\otimes} - \tfrac{1}{2} i \bar{\xi} \wedge \xi \tag{3.21}$$

is the curvature associated to the Maxwell 1-form $B = B_\mu \mathrm{d}x^\mu$. Eq. (3.20) implies that the supertorsion R^a has space-time components proportional to the curl of B_μ. Indeed the solution of (3.20) is

$$R^{\otimes} = F_{ab} V^a \wedge V^b\,, \tag{3.22a}$$

$$R^a = -\tfrac{1}{4}\eta \varepsilon^{abcdf} F_{bc} V_d \wedge V_f\,. \tag{3.22b}$$

Inserting (3.22) back into the first-order lagrangian, one realizes that the geometric, Hodge-dual-free, term,

$$R^{\otimes} \wedge R^a \wedge V_a\,, \tag{3.23}$$

becomes the kinetic term,

$$F^{rs} F_{rs} \varepsilon_{a_1 \cdots a_5} V^{a_1} \wedge \cdots \wedge V^{a_5}\,, \tag{3.24}$$

of the B_μ field. Unfortunately, this beautiful mechanism is not accessible to the $A_{\mu\nu\rho}$ field of $D=11$ supergravity because in $D=11$ the analogue of eq. (3.20) would be

$$\varepsilon_{abc_1 \cdots c_9} R^{c_1} \wedge V^{c_2} \wedge \cdots \wedge V^{c_9} + \eta V_a \wedge V_b \wedge R^{(8)} = 0\,, \tag{3.25}$$

where $R^{(8)}$ is the curl of a 7-form $A_{\mu_1\cdots\mu_7}$ and not of a 3-form $A_{\mu_1\mu_2\mu_3}$ or of a 6-form $A_{\mu_1\cdots\mu_6}$, interpretable as the dual potential of the former.

The second mechanism for the geometric generation of the dual was introduced in $N=2$ and $N=3$ supergravity by one of us [13]. It consists of the addition of the 0-form F_{ab} as an independent dynamical field and it corresponds to a first-order formulation of the Maxwell lagrangian. The analogue of this mechanism in $D=11$ supergravity would be the addition of a 0-form $F_{a_1\cdots a_4}$. In the sequel we shall be forced to introduce this trick, which, however, results in an impure character of the geometric lagrangian. In fact, $F_{a_1\cdots a_4}$ is not directly interpretable as the 1-form potential of any group generator.

Because of that, before resorting to this mechanism we shall explore another possibility suggested by conformal supergravity [15]. In that theory one has a dilation field D and an axion field A. The equation of motion of the conformino, namely the gauge field of the S-supersymmetry, implies that curl $R(D)$ is the dual of curl $R(A)$. This relation inserted back into the lagrangian transforms the geometric term

$$R(A)\wedge R(D)$$

into the kinetic term of the axion. A similar thing might happen also in $D=11$ supergravity. If, besides A we also had a 6-form B, then we might hope that the gravitino equation forces curl B to be the dual of curl A, transforming, in this way, the geometric term

$$R(A)\wedge R(B)$$

into the kinetic term of A.

This conjecture is to be taken into serious consideration because it is also supported by a remarkable algebraic fact rooted in Fierz identities: the CIS C_1 can be extended in a non-trivial way, precisely by the introduction of a 6-form B.

Indeed if we add the following equation to eqs. (3.15) we still get an integrable system, hereafter named C_2:

$$\begin{aligned} \mathrm{d}B-\tfrac{1}{2}i\bar{\psi}\wedge\Gamma^{a_1\cdots a_5}\psi\wedge V_{a_1}\wedge\cdots\wedge V_{a_5} \\ -\tfrac{15}{2}\bar{\psi}\wedge\Gamma^{a_1a_2}\psi\wedge V_{a_1}\wedge V_{a_2}\wedge A=0\,. \end{aligned} \tag{3.26}$$

The Maurer-Cartan equation (3.26) is integrable because

$$\begin{aligned} \mathrm{dd}B= & -\tfrac{5}{4}\bar{\psi}\wedge\Gamma^{a_1\cdots a_4m}\psi\wedge V_{a_1}\wedge\cdots\wedge V_{a_4}\wedge\bar{\psi}\Gamma_m\wedge\psi \\ & +15\bar{\psi}\wedge\Gamma^{am}\psi\wedge V_a\wedge\bar{\psi}\wedge\Gamma_m\psi\wedge A \\ & +\tfrac{15}{4}\bar{\psi}\wedge\Gamma_{a_1a_2}\psi\wedge\bar{\psi}\wedge\Gamma_{a_3a_4}\psi\wedge V^{a_1}\wedge\cdots\wedge V^{a_4}, \end{aligned} \tag{3.27}$$

and

$$\bar{\psi} \wedge \Gamma^{a_1 \cdots a_4 m} \psi \wedge \bar{\psi} \wedge \Gamma_m \psi = 3 \bar{\psi} \wedge \Gamma^{[a_1 a_2} \psi \wedge \bar{\psi} \wedge \Gamma^{a_3 a_4]} \psi , \tag{3.28a}$$

$$\bar{\psi} \wedge \Gamma^{am} \psi \wedge \bar{\psi} \wedge \Gamma_m \psi = 0 , \tag{3.28b}$$

as may be checked by looking at table 2.

The Bianchi identities and coadjoint covariant derivative of the C_2 CIS are, respectively:

C_2 Bianchi identities

$$\nabla R^{ab} = \mathcal{D} R^{ab} = 0 , \tag{3.29a}$$

$$\nabla R^{a} = \mathcal{D} R^{a} + R^{ab} \wedge V_b - i \bar{\psi} \wedge \Gamma^a \rho = 0 , \tag{3.29b}$$

$$\nabla \rho = \mathcal{D} \rho + \tfrac{1}{4} \Gamma_{ab} \psi \wedge R^{ab} = 0 , \tag{3.29c}$$

$$\nabla R^{\square} = \mathrm{d} R^{\square} - \bar{\psi} \wedge \Gamma^{a_1 a_2} \psi \wedge V_{a_1} \wedge V_{a_2} + \bar{\psi} \wedge \Gamma^{a_1 a_2} \psi \wedge R_{a_1} \wedge V_{a_2} = 0 , \tag{3.29d}$$

$$\begin{aligned} \nabla R^{\otimes} = \mathrm{d} R^{\otimes} & - i \bar{\psi} \wedge \Gamma^{a_1 \cdots a_5} \rho \wedge V_{a_1} \wedge \cdots \wedge V_{a_5} \\ & + \tfrac{5}{2} i \bar{\psi} \wedge \Gamma^{a_1 \cdots a_5} \psi \wedge R_{a_1} \wedge V_{a_2} \wedge \cdots \wedge V_{a_5} \\ & - 15 \bar{\psi} \wedge \Gamma^{a_1 a_2} \rho \wedge A \wedge V_{a_1} \wedge V_{a_2} + \tfrac{15}{2} \bar{\psi} \wedge \Gamma^{a_1 a_2} \psi \wedge R^{\square} \wedge V_{a_1} \wedge V_{a_2} \\ & - 15 \bar{\psi} \wedge \Gamma^{a_1 a_2} \psi \wedge A \wedge R_{a_1} \wedge V_{a_2} = 0 . \end{aligned} \tag{3.29e}$$

C_2 coadjoint covariant derivative

$$\nabla \nu_{ab} = \mathcal{D} \nu_{ab} + V_{[a} \wedge \nu_{b]} + \tfrac{1}{4} \bar{\psi} \wedge \Gamma_{ab} \eta , \tag{3.30a}$$

$$\begin{aligned} \nabla \nu_{a} = \mathcal{D} \nu_{a} & - \bar{\psi} \wedge \Gamma_{ab} \psi \wedge V^b \wedge V_{\square} - \tfrac{5}{2} i \bar{\psi} \wedge \Gamma_{ab_1 \cdots b_4} \psi \wedge V^{b_1} \wedge \cdots \wedge V^{b_4} \wedge \nu_{\otimes} \\ & - 15 \bar{\psi} \wedge \Gamma_{ab} \psi \wedge V^b \wedge A \wedge \nu_{\otimes} , \end{aligned} \tag{3.30b}$$

$$\begin{aligned} \nabla n = \mathcal{D} n & - i \Gamma^a \psi \wedge \nu_a - \Gamma^{a_1 a_2} \psi \wedge V_{a_1} \wedge V_{a_2} \wedge \nu_{\square} \\ & - i \Gamma^{a_1 \cdots a_5} \psi \wedge V_{a_1} \wedge \cdots \wedge V_{a_5} \wedge \nu_{\otimes} - 15 \Gamma^{a_1 a_2} \psi \wedge V_{a_1} \wedge V_{a_2} \wedge A \wedge \nu_{\otimes} , \end{aligned} \tag{3.30c}$$

$$\nabla \nu_{\otimes} = \mathrm{d} \nu_{\otimes} , \tag{3.30d}$$

$$\nabla \nu_{\square} = \mathrm{d} \nu_{\square} - \tfrac{15}{2} \bar{\psi} \wedge \Gamma^{a_1 a_2} \psi \wedge V_{a_1} \wedge V_{a_2} \wedge \nu_{\otimes} , \tag{3.30e}$$

where R^{ab}, R^a, $R^{\square}$, ρ are given by eqs. (3.15) and

$$R^{\otimes} = \mathrm{d}B - \tfrac{1}{2}i\bar{\psi}\wedge\Gamma^{a_1\cdots a_5}\psi\wedge V_{a_1}\wedge\cdots\wedge V_{a_5}$$

$$-\tfrac{15}{2}\bar{\psi}\wedge\Gamma^{a_1a_2}\psi\wedge V_{a_1}\wedge V_{a_2}\wedge A \tag{3.31}$$

is the curvature associated to the 6-form B; $\nu_{\otimes}$ is a $(d-7)$-form, $\nu_{\square}$ is a $(d-4)$-form and ν_{ab}, ν_a, n are $(d-2)$-forms.

In the following sections we construct the geometric theory of the C_2 CIS manifold and we show how the requirement of gauge invariance under the transformation

$$\delta A = \mathrm{d}\varphi,$$

necessary for rheonomy, kills all the B-dependent terms reducing the theory to the C_1 CIS manifold. This result is in full agreement with the results of ref. [3] and might be related to the fact that eq. (3.26) introduces a non-linear representation of supersymmetry. In any case it is a confirmation of the fact that a magnetic dual formulation of $D=11$ supergravity does not exist and forces us to resort to the 0-form trick.

4. Construction of the geometric lagrangian

In this section, following the scheme outlined in eq. (2.20), we construct a geometric action associated to the CIS C_2. It will be of the following type:

$$\mathcal{A} = \int_{\mathrm{M}_{11}} \left\{\Lambda + R^{ab}\wedge\nu_{ab} + R^a\wedge\nu_a + \bar{\rho}\wedge n + R^{\square}\wedge\nu_{\square} + R^{\otimes}\wedge\nu_{\otimes} + R^A\wedge R^B\wedge\nu_{AB}\right\}, \tag{4.1}$$

where M_{11} is an eleven-dimensional floating surface in the C_2 CIS manifold and accordingly, all addends in the action (4.1) are 11-forms.

We shall make use of the following three building principles:

(*a*) *The action is locally Lorentz invariant.* This means that $\Lambda, \nu_{ab}, \nu_a, n, \nu_{\square}, \nu_{\otimes}, \nu_{AB}$ are polynomials in V^a, ψ, A, B the spin-connection ω^{ab} being excluded. Moreover, everything is an SO(1,9) good tensor.

(*b*) *The vacuum* ($R^{ab}=R^a=R^{\otimes}=R^{\square}=\rho=0$) *is a solution.* This condition is fulfilled if the following cosmo-cocycle conditions are satisfied by the $\{\nu_{ab}, \nu_a, n, \nu_{\square}, \nu_{\otimes}\}$ multiplet:

$$\nabla\nu_{ab} = 0 \tag{4.2a}$$

$$\underline{a}\rfloor\Lambda + \nabla\nu_a = 0 \tag{4.2b}$$

$$\psi \rfloor \Lambda + \nabla n = 0 \qquad \text{at } R^{ab} = R^a = R^{\square} = R^{\otimes} = \rho = 0\,. \tag{4.2c}$$

$$\square \rfloor \Lambda + \nabla \nu_{\square} = 0 \tag{4.2d}$$

$$\otimes \rfloor \Lambda + \nabla \nu_{\otimes} = 0 \tag{4.2e}$$

(*c*) *The equations of motion are invariant under the scale transformation which leaves the GMCE invariant.* The last requirement needs further explanations. Let us first note that the definitions (3.15) and (3.31) of the C_2 curvatures are invariant under the following scale transformation:

$$\begin{aligned}
R^{ab} &\to R'^{ab} = R^{ab}\,, & \omega^{ab} &\to \omega'^{ab} = \omega^{ab}\,,\\
R^{a} &\to R'^{a} = eR^{a}\,, & V^{a} &\to V'^{a} = eV^{a}\,,\\
\rho &\to \rho' = \sqrt{e}\,\rho\,, & \psi &\to \psi' = \sqrt{e}\,\psi\,,\\
R^{\square} &\to R'^{\square} = e^3 R^{\square}\,, & A &\to A' = e^3 A\,,\\
R^{\otimes} &\to R'^{\otimes} = e^6 R^{\otimes}\,, & B &\to B' = e^6 B\,,
\end{aligned} \tag{4.3}$$

where e is a real parameter. Since the equation of motions of the theory are relations among the curvatures and the potentials, in order to be consistent, they must not depend on the specific choice of e.

Indeed every value of e singles out an element in an equivalence class of isomorphic CIS. The equations of motion of the dynamical theory should depend only on the equivalence class and not on the specific element in the class. Otherwise, it is almost evident that the theory will be trivial, admitting, at most, the vacuum solution. In fact, if the equations of motion depend on e, they will provide relations among the curvature components also depending on e.

Suppose that, this notwithstanding, some of the curvature components are different from zero $R^{A}{}_{F_1\cdots F_p} = f^{A}_{F_1\cdots F_p}(e)$. Giving a special value to e we could, nonetheless, put them to zero yet working with the same CIS as before. The only solution to this paradox is that $f^{A}_{F_1\cdots F_p} = 0$.

This scale criterion is very powerful and easily implemented: it is just sufficient that, under the transformation (4.3), all terms in the action (4.1) scale with the same power of e. Since the Einstein term

$$R^{ab} \wedge V^{c_1} \wedge \cdots \wedge V^{c_9} \varepsilon_{abc_1\cdots c_9} \tag{4.4}$$

has to be there and it has scale dimension e^9 this fixes the scale of all other terms.

The scale criterion was not clearly stated in previous work on the group manifold approach, but it can be checked that in existing theories like $D=4$ and $D=5$ supergravity it just kills those terms which have to be suppressed in order for the theory to be non-rigid. (For example in $D=4$ the requirement of Lorentz and parity invariance plus the vacuum condition yields the action

$$\mathcal{Q}=\int_{M^4}\{R^{ab}\wedge V^c\wedge V^d\varepsilon_{abcd}+4\bar{\psi}\wedge\gamma_5\gamma_a\rho\wedge V^a+aR^{ab}_a\bar{\psi}\wedge\gamma_5\Sigma_{ab}\psi\}\,, \quad (4.5)$$

which is trivial unless $a=0$ (for a discussion of this point see ref. [9], p. 26). Now it happens that the last term in (4.5) has scale dimension e while all the others have scale dimension e^2: hence it must be suppressed.)

The most general form of the polynomials Λ, ν_{ab}, ν_a, n, $\nu_\square$, $\nu_\otimes$, ν_{AB} which fulfills criteria (a) and (c) is the following:

$$\Lambda=a\bar{\psi}\wedge\Gamma^{a_1a_2}\psi\wedge\bar{\psi}\wedge\Gamma^{a_3a_4}\psi\wedge V^{a_5}\wedge\cdots\wedge V^{a_{11}}\varepsilon_{a_1\cdots a_{11}}$$

$$+b\bar{\psi}\wedge\Gamma^{a_1a_2}\psi\wedge\bar{\psi}\wedge\Gamma^{a_3a_4}\psi\wedge V_{a_1}\wedge\cdots\wedge V_{a_4}\wedge A\,, \quad (4.6a)$$

$$\nu_{ab}=-\tfrac{1}{9}\varepsilon_{abc_1\cdots c_9}V^{c_1}\wedge\cdots\wedge V^{c_9}\,, \quad (4.6b)$$

$$\nu_a=i\beta_1V_a\wedge\bar{\psi}\wedge\Gamma^{c_1\cdots c_5}\psi\wedge V^{c_6}\wedge\cdots\wedge V^{c_{11}}\varepsilon_{c_1\cdots c_{11}}$$

$$+\beta_2\bar{\psi}\wedge\Gamma^{ab}\psi_a\wedge V_b\wedge B^{\otimes}+i\beta_3\bar{\psi}\wedge\Gamma_{ac_1\cdots c_4}\psi\wedge V^{c_1}\wedge\cdots\wedge V^{c_4}\wedge A\,, \quad (4.6c)$$

$$n=h_1\Gamma_{c_1\cdots c_8}\psi\wedge V^{c_1}\wedge\cdots\wedge V^{c_8}+h_2\Gamma_{ab}\psi\wedge V^a\wedge V^b\wedge B^{\otimes}$$

$$+ih_3\Gamma_{c_1\cdots c_5}\psi\wedge V^{c_1}\wedge\cdots\wedge V^{c_5}\wedge A\,, \quad (4.6d)$$

$$\nu_\square=ik_1\bar{\psi}\wedge\Gamma_{a_1\cdots a_5}\psi\wedge V^{a_1}\wedge\cdots\wedge V^{a_5}+k_2\bar{\psi}\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b\wedge A\,, \quad (4.6e)$$

$$\nu_\otimes=k_3\bar{\psi}\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b\,, \quad (4.6f)$$

$$R^A\wedge R^B\wedge\nu_{AB}=\gamma_1R^\square\wedge R^\square\wedge A+\gamma_2R^\square\wedge R^\otimes\,, \quad (4.6g)$$

where a, b, β_1, β_2, β_3, h_1, h_2, h_3, k', k'', k, γ_1, γ_2 are numerical constants. The first eleven, namely all except γ_1 and γ_2 are determined by the vacuum conditions (4.2).

Implementing eqs. (4.2), after extremely long but straightforward manipulations which make essential use of the Fierz-decomposition of table 2 we arrive at the following system of 16 algebraic equations:

$$
\begin{aligned}
&(\mathrm{EQ1}) && h_1 = 2\,, \\
&(\mathrm{EQ2}) && \beta_1 - \frac{14}{5!} h_1 = 0\,, \\
&(\mathrm{EQ3}) && \beta_2 - h_2 = 0\,, \\
&(\mathrm{EQ4}) && \beta_3 - \tfrac{5}{3} h_3 = 0\,, \\
&(\mathrm{EQ5}) && k_2 - 15\,(k_3 + k_1) + 2b = 0\,, \\
&(\mathrm{EQ6}) && 6\beta_3 + 4b - \tfrac{15}{2}\beta_2 - k_2 - 15k_3 = 0\,, \\
&(\mathrm{EQ7}) && 7a - \tfrac{15}{14}\beta_1 + \tfrac{1}{56}(k_1 + \tfrac{1}{2}\beta_2) + \tfrac{1}{56}(\tfrac{5}{2}k_3 - \beta_3) = 0\,, \\
&(\mathrm{EQ8}) && \beta_3 - 5k_3 + 5(\tfrac{1}{2}\beta_2 + k_1) + 6!\tfrac{5}{2}\beta_1 = 0\,, \\
&(\mathrm{EQ9}) && -\tfrac{1}{2}\beta_1 - \tfrac{5}{3600}(\tfrac{1}{2}\beta_2 + k_1) + \tfrac{1}{3600}(5k_3 - \beta_3) = 0\,, \\
&(\mathrm{EQ10}) && \beta_2 = h_2\,, \\
&(\mathrm{EQ11}) && 6\beta_3 + (4b - \tfrac{15}{2}h_2 - 15k_3 - k_2) = 0\,, \\
&(\mathrm{EQ12}) && 12\beta_3 - 45h_3 + (4b - \tfrac{15}{2}h_2 - 15k_3 - k_2) = 0\,, \\
&(\mathrm{EQ13}) && -\beta_3 + \tfrac{35}{2}h_3 + (4b - \tfrac{15}{2}h_2 - 15k_3 - k_2) = 0\,, \\
&(\mathrm{EQ14}) && -12 \cdot 5!(\tfrac{1}{2}h_3 - k_3) + 24 \cdot 5!(\tfrac{1}{2}h_2 + k_1) + 2 \cdot 6!5!3\beta_1 + 7!4!4a = 0\,, \\
&(\mathrm{EQ15}) && -28h_1 - \tfrac{5}{18}h_2 - \tfrac{5}{9}k_1 - \tfrac{1}{9}h_3 + \tfrac{2}{9}k + 80\beta_1 - \tfrac{112}{3}a = 0\,, \\
&(\mathrm{EQ16}) && -112h_1 - \tfrac{1}{2}h_2 - k_1 - \tfrac{7}{2}h_3 + 7k_3 - 120\beta_1 + 672a = 0\,.
\end{aligned}
\tag{4.7}
$$

These 16 equations are subdivided in the following way. (EQ1)–(EQ4) come from condition (4.2a) and correspond, respectively, to the annihilation of the following terms:

$$\text{(EQ1)} \qquad \varepsilon_{abc_1\cdots c_9}\bar{\psi}\wedge\Gamma^{c_1}\psi\wedge V^{c_2}\wedge\cdots\wedge V^{c_9},$$

$$\text{(EQ2)} \qquad V_a\wedge V_b\wedge\bar{\psi}\wedge\Gamma^{c_1\cdots c_5}\psi\wedge V^{c_6}\wedge\cdots\wedge V^{c_{11}}\varepsilon_{c_1\cdots c_{11}},$$

$$\text{(EQ3)} \qquad V_{[a}\wedge\bar{\psi}\wedge\Gamma_{b]m}\psi\wedge V^m\wedge B,$$

$$\text{(EQ4)} \qquad V_{[a}\wedge\bar{\psi}\wedge\Gamma_{b]c_1\cdots c_4}\psi\wedge V^{c_1}\wedge\cdots\wedge V^{c_4}\wedge A. \tag{4.8}$$

(EQ5) comes from condition (4.2d) and corresponds to the annihilation of the term:

$$\text{(EQ5)} \qquad X^{(330)}_{a_1\cdots a_4}\wedge V^{a_1}\wedge\cdots\wedge V^{a_4}. \tag{4.9}$$

(EQ6)–(EQ9) come from condition (4.2b) and correspond, respectively, to the annihilation of:

$$\text{(EQ6)} \qquad X^{(330)}_{ac_1c_2c_3}\wedge V^{c_1}\wedge V^{c_2}\wedge V^{c_3}\wedge A, \tag{4.10a}$$

$$\text{(EQ7)} \qquad X^{(330)}_{f_1\cdots f_4}\wedge V_{f_5}\wedge\cdots\wedge V_{f_{10}}\varepsilon^{af_1\cdots f_{10}}, \tag{4.10b}$$

$$\text{(EQ8)} \qquad X^{(462)}_{f_1\cdots f_5}\wedge V^{f_1}\wedge\cdots\wedge V^{f_5}\wedge V^a, \tag{4.10c}$$

$$\text{(EQ9)} \qquad X^{(4920)}_{\substack{f_1\cdots f_5\\ a}}\wedge V_{f_6}\wedge\cdots\wedge V_{f_{11}}\varepsilon^{f_1\cdots f_{11}}, \tag{4.10d}$$

where $X^{(330)}_{f_1\cdots f_4}$, $X^{(462)}_{f_1\cdots f_5}$, $X^{(4290)}_{\substack{f_1\cdots f_5\\ a}}$ are the irreducible representations appearing in the decomposition of table 2. Finally (EQ10)–(EQ16) come from condition (4.2c). They correspond to the annihilation of the following terms:

$$\text{(EQ10)} \qquad \Xi^{(320)}_a\wedge V^a\wedge B^{\otimes}, \qquad \Gamma_a\Xi^{(32)}\wedge V^a\wedge B^{\otimes}, \tag{4.11a}$$

$$\text{(EQ11)} \qquad \Gamma_{[a_1a_2}\Xi^{(1408)}_{a_3a_4]}\wedge V^{a_1}\wedge\cdots\wedge V^{a_4}\wedge A, \tag{4.11b}$$

$$\text{(EQ12)} \qquad \Gamma_{[a_1a_2a_3}\Xi^{(320)}_{a_4]}\wedge V^{a_1}\wedge\cdots\wedge V^{a_4}\wedge A, \tag{4.11c}$$

$$\text{(EQ13)} \qquad \Gamma_{a_1\cdots a_4}\Xi^{(32)}\wedge V^{a_1}\wedge\cdots\wedge V^{a_4}\wedge A, \tag{4.11d}$$

$$\text{(EQ14)} \qquad \Gamma_{[a_1\cdots a_5}\Xi^{(1408)}_{a_6a_7]}\wedge V^{a_1}\wedge\cdots\wedge V^{a_7}, \tag{4.11e}$$

$$\text{(EQ15)} \qquad \Gamma_{a_1\cdots a_6}\Xi^{(320)}_{a_7}\wedge V^{a_1}\wedge\cdots\wedge V^{a_7}, \tag{4.11f}$$

$$\text{(EQ16)} \qquad \Gamma_{a_1\cdots a_7}\Xi^{(32)}\wedge V^{a_1}\wedge\cdots\wedge V^{a_7}. \tag{4.11g}$$

The linear system (4.7) contains more equations than unknowns: many equations, however, are linearly dependent and because of that the system is solvable. Actually a little bit of inspection reveals that, after use of (EQ1)–(EQ4) the remaining equations depend only on the variables

$$a, b, \beta_1, (\tfrac{1}{2}\beta_2 + k_1), (\beta_3 - 5k_3), k_2 .$$

With respect to these unknowns the system has a 1-parameter family of solutions. The reason why 4 variables patch together in 2 fixed combinations is that there are just two Lorentz invariant 10-forms whose scaling degree is e^9, namely

$$\Phi_1 = i\bar{\psi} \wedge \Gamma^{a_1 \cdots a_5}\psi \wedge V_{a_1} \wedge \cdots \wedge V_{a_5} \wedge A , \tag{4.12a}$$

$$\Phi_2 = \bar{\psi} \wedge \Gamma^{ab}\psi \wedge V_a \wedge V_b \wedge B^{\otimes} . \tag{4.12b}$$

Why is this relevant to the solution of system (4.7)? The argument is the following. Assume that we have a solution $\{\nu_{ab}, \nu_a, \nu_\Box, \nu_\otimes, n, \Lambda\}$ of the cosmo-cocycle condition (4.2); we can form the following two linear lagrangians:

$$\mathcal{L}_{\text{linear}} = R^{ab} \wedge \nu_{ab} + R^a \wedge \nu_a + R^\Box \wedge \nu_\Box + R^\otimes \wedge \nu_\otimes + \bar{\rho} \wedge n + \Lambda , \tag{4.13a}$$

$$\mathcal{L}'_{\text{linear}} = \mathcal{L}_{\text{linear}} + \alpha_1 \, d\phi_1 + \alpha_2 \, d\phi_2 . \tag{4.13b}$$

Obviously (4.13a) and (4.13b) are physically equivalent because they differ by the total divergence

$$\alpha_1 \, d\phi_1 + \alpha_2 \, d\phi_2 . \tag{4.14}$$

On the other hand, explicitly computing the derivatives $d\phi_1$ and $d\phi_2$ we see that:

$$\mathcal{L}'_{\text{linear}} = R^{ab} \wedge \nu'_{ab} + R^a \wedge \nu'_a + R^\Box \wedge \nu'_\Box + R^\otimes \wedge \nu'_\otimes + \bar{\rho} \wedge n' + \Lambda' , \tag{4.15}$$

where $\{\nu'_{ab}, \nu'_a, \nu'_\Box, \nu'_\otimes, n', \Lambda\}$ is a new solution of the cosmo-cocycle condition (4.2) which is related to the previous one by the following transformation:

$$\begin{aligned}
&a' = a + \tfrac{1}{112}(\alpha_1 - \alpha_2), \qquad b' = b - \tfrac{15}{2}\alpha_1 , \\
&\beta'_1 = \beta_1 , \qquad \beta'_2 = \beta_2 + 2\alpha_2 , \qquad \beta'_3 = \beta_3 + 5\alpha_1 , \\
&k'_1 = k_1 - \alpha_1 , \qquad k'_2 = k_2 , \qquad k'_3 = k_3 + \alpha_2 , \\
&h'_1 = h_1 , \qquad h'_2 = h_2 + 2\alpha_2 , \qquad h'_3 = h_3 + 2\alpha_1 .
\end{aligned} \tag{4.16}$$

It follows that the linear system (4.7) must be invariant under the transformation

(4.16) and this explains why, after implementation of (EQ1)–(EQ4), the effective variables are 6 instead of 8. Since $d\phi_1$ and $d\phi_2$ represent total divergences, without any loss of generality, we can use them to set

$$\beta_2 = 0\,, \qquad \beta_3 = 0\,. \tag{4.17}$$

With this choice the solution of system (4.7) is

$$\begin{aligned} &a = \tfrac{1}{4}\left(1 - \tfrac{1}{28}k\right), \qquad b = -15\left(14 - \tfrac{1}{2}k\right), \\ &\beta_1 = \tfrac{7}{30}, \qquad \beta_2 = \beta_3 = 0\,, \\ &k_1 = -84 + k\,, \qquad k_2 = -840\left(1 - \tfrac{1}{56}k\right), \\ &h_1 = 2\,, \qquad h_2 = h_3 = 0\,, \end{aligned} \tag{4.18}$$

corresponding to the following action:

$$\mathcal{A}(\gamma_1, \gamma_2, k) = \int \mathcal{L}_0(\gamma_1, \gamma_2, k)\,, \tag{4.19a}$$

$$\begin{aligned} \mathcal{L}_0(\gamma_1, \gamma_2, k) = &-\tfrac{1}{9} R^{a_1 a_2} \wedge V^{a_3} \wedge \cdots \wedge V^{a_{11}} \varepsilon_{a_1 \cdots a_{11}} \\ &+ \tfrac{7}{30} i R^a \wedge V_a \wedge \bar{\psi} \wedge \Gamma^{b_1 \cdots b_5} \psi \wedge V^{b_6} \wedge \cdots \wedge V^{b_{11}} \varepsilon_{b_1 \cdots b_{11}} \\ &+ k R^{\otimes} \wedge \bar{\psi} \wedge \Gamma_{ab} \psi \wedge V^a \wedge V^b \\ &+ i(k - 84) R^{\square} \wedge \bar{\psi} \wedge \Gamma_{a_1 \cdots a_5} \psi \wedge V^{a_1} \wedge \cdots \wedge V^{a_5} \\ &+ 840\left(\tfrac{1}{56}k - 1\right) R^{\square} \wedge \bar{\psi} \wedge \Gamma_{ab} \psi \wedge V^a \wedge V^b \wedge A \\ &+ 2\bar{\rho} \wedge \Gamma_{c_1 \cdots c_8} \psi \wedge V^{c_1} \wedge \cdots \wedge V^{c_8} \\ &+ \tfrac{1}{4}\left(1 - \tfrac{1}{28}k\right) \bar{\psi} \wedge \Gamma^{a_1 a_2} \psi \wedge \bar{\psi} \wedge \Gamma^{a_3 a_4} \psi \wedge V^{a_5} \wedge \cdots \wedge V^{a_{11}} \varepsilon_{a_1 \cdots a_{11}} \\ &+ 15\left(\tfrac{1}{2}k - 14\right) \bar{\psi} \wedge \Gamma^{a_1 a_2} \psi \wedge \bar{\psi} \wedge \Gamma^{a_3 a_4} \psi \wedge V_{a_1} \wedge \cdots \wedge V_{a_4} \wedge A \\ &+ \gamma_1 R^{\square} \wedge R^{\square} \wedge A + \gamma_2 R^{\square} \wedge R^{\otimes}\,, \end{aligned} \tag{4.19b}$$

which still depends on the 3-parameters k, γ_1, γ_2. They are now fixed by the requirement of gauge invariance of the action (4.19) under the transformation

$$A \to A + \delta A\,, \qquad \delta A = d\varphi\,, \tag{4.20}$$

where φ is an arbitrary 2-form. The motivations of this requirement are the following:

(a) Analogy with $D=5$ supergravity where the gauge invariance under

$$\delta B = \mathrm{d}\varphi \tag{4.21}$$

fixes the coefficients of the quadratic terms in such a way as to guarantee non-triviality of the theory [12].

(b) Analogy with the Cremmer-Julia-Scherk formulation where

$$\delta A_{\nu\nu\rho} = \partial_{[\mu}\varepsilon_{\nu\rho]} \tag{4.22}$$

is indeed an invariance of the action

(c) Actual inspection of the equations of motion which reveals the following: if the terms with a bare A do not cancel identically in all equations, the only possible solution is the vacuum ($R^{ab} = R^a = R^\square = R^\otimes = \rho = 0$).

Performing the explicit variation of $\mathcal{L}_0$ we obtain

$$\begin{aligned}
\delta A \Rightarrow \mathrm{d}\varphi \Rightarrow \delta\mathcal{L}_0 = &-840\left(1-\tfrac{1}{56}k\right)R^\square\wedge\bar\psi\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b\wedge\mathrm{d}\varphi \\
&+\gamma_1 R^\square\wedge R^\square\wedge\mathrm{d}\varphi + 15\left(\tfrac{1}{2}k-14\right)\bar\psi\wedge\Gamma_{a_1a_2}\psi\wedge\bar\psi\wedge\Gamma_{a_3a_4}\psi\wedge V^{a_1}\wedge\cdots\wedge V^{a_4}\wedge\mathrm{d}\varphi \\
&-\tfrac{15}{2}\gamma_2 R^\square\wedge\bar\psi\wedge\Gamma^{a_1a_2}\psi\wedge V_{a_1}\wedge V_{a_2}\wedge\mathrm{d}\varphi .
\end{aligned} \tag{4.23}$$

An integration by part shows that $\delta\mathcal{L}_0$ is a total divergence only if the following conditions are satisfied:

$$\gamma_1 = -840, \qquad \gamma_2 = 2k. \tag{4.24}$$

When (4.24) holds, the k-dependent terms of the lagrangian sum up to a total divergence:

$$\begin{aligned}
k\Big\{&\tfrac{840}{56}R^\square\wedge\bar\psi\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b\wedge A + iR^\square\wedge\bar\psi\wedge\Gamma_{a_1\cdots a_5}\psi\wedge V^{a_1}\wedge\cdots\wedge V^{a_5} \\
&+R^\otimes\wedge\bar\psi\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b + 2R^\otimes\wedge R^\square \\
&-\tfrac{1}{112}\bar\psi\wedge\Gamma^{a_1a_2}\psi\wedge\bar\psi\wedge\Gamma^{a_3a_4}\psi\wedge V^{a_5}\wedge\cdots\wedge V^{a_{11}}\varepsilon_{a_1\cdots a_{11}} \\
&+\tfrac{15}{2}\bar\psi\wedge\Gamma^{a_1a_2}\psi\wedge\bar\psi\wedge\Gamma^{a_3a_4}\psi\wedge V_{a_1}\wedge\cdots\wedge V_{a_4}\wedge A\Big\} \\
&= 2k\,\mathrm{d}(A\wedge\mathrm{d}B)
\end{aligned} \tag{4.25}$$

and, therefore, may be dropped.

However, the k-dependent terms are also the only ones containing the 6-form B. Hence the A-gauge invariant lagrangian $\mathcal{L}_0(\gamma_1=-840, \gamma_2=2k, k)$ does not contain B and it is based on the C_1 CIS described by eqs. (3.15). Although we started with a larger CIS we end up with the minimal one containing only the 3-form A. This is a confirmation of the component approach result of ref. [3] ruling out the $B_{\mu_1\cdots\mu_6}$ formulation.

The sad point is that our hopes for a spontaneous generation of the Maxwell kinetic term die simultaneously with B. The lagrangian $\mathcal{L}_0(\gamma_1=-840, \gamma_2=2k, k)$ lacks the Maxwell lagrangian of A and is, therefore, bound to yield only the vacuum solution. Indeed, setting $k=0, \gamma_1=-840, \gamma_2=0$ and performing the variation of (4.19) in the A 3-form, we obtain the following equation of motion:

$$15R^{\square}\wedge R^{\square}+15R^{\square}\wedge\bar{\psi}\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b+i\bar{\psi}\wedge\Gamma_{a_1\cdots a_5}\rho\wedge V^{a_1}\wedge\cdots\wedge V^{a_5}=0. \tag{4.26}$$

Projected onto 8 elfbeins, eq. (4.26) yields

$$R^{\square}_{a_1\cdots a_4}R^{\square}_{b_1\cdots b_4}\varepsilon^{a_1\cdots a_4b_1\cdots b_4c_1c_2c_3}=0, \tag{4.27}$$

which instead of being the Maxwell equation for the space-time components $R^{\square}_{a_1\cdots a_4}$ of $R^{\square}$ is an algebraic constraint on the latter implying $R^{\square}_{a_1\cdots a_4}=0$.

Since all other possibilities have been explored we have now no other option than resorting to the 0-form trick. In complete analogy with the procedure adopted for $N=2$, $D=4$ supergravity [13] we introduce the following action:

$$\mathcal{A}=\int\mathcal{L}(m,n), \tag{4.28a}$$

$$\mathcal{L}(m,n)=\mathcal{L}_0(\gamma_1=-840, \gamma_2=0, k=0)+\mathcal{L}'(m,n), \tag{4.28b}$$

where $\mathcal{L}_0(\gamma_1, \gamma_2, k)$ is given by eq. (4.19b) and $\mathcal{L}'(m,n)$ is given below:

$$\begin{aligned}\mathcal{L}'(m,n)=&\,mF^{a_1\cdots a_4}R^{\square}\wedge V^{a_5}\wedge\cdots\wedge V^{a_{11}}\varepsilon_{a_1\cdots a_{11}}\\&+nF_{a_1\cdots a_4}F^{a_1\cdots a_4}V^{c_1}\wedge\cdots\wedge V^{c_{11}}\varepsilon_{c_1\cdots c_{11}},\end{aligned} \tag{4.29}$$

$F^{a_1\cdots a_4}$ being a 4-index skew-symmetric 0-form and m, n two numerical parameters. Eq. (4.29) corresponds to the 1st-order formulation of the Maxwell lagrangian.

In the next section we show that, provided m and n take specific values, the lagrangian (4.28b) is non-trivial and describes a rheonomic theory.

5. Equations of motion: non-triviality and rheonomy

The equations of motion of the theory (4.28) are the following ones.
Torsion equation (variation in ω^{ab}):

$$\varepsilon_{abc_1\cdots c_9} R^{c_1}\wedge V^{c_2}\wedge\cdots\wedge V^{c_9}=0. \tag{5.1a}$$

First Maxwell equation ($F_{a_1\cdots a_4}$ variation):

$$mR^{\square}\wedge V^{a_5}\wedge\cdots\wedge V^{a_{11}}\varepsilon_{a_1\cdots a_4a_5\cdots a_{11}}+2nF_{a_1\cdots a_4}V^{c_1}\wedge\cdots\wedge V^{c_{11}}\varepsilon_{c_1\cdots c_{11}}=0. \tag{5.1b}$$

Second Maxwell equation (variation in A):

$$\begin{aligned}&168i\bar\psi\wedge\Gamma_{a_1\cdots a_5}\rho\wedge V^{a_1}\wedge\cdots\wedge V^{a_5}-2520\bar\psi\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b\wedge R^{\square}\\&-2520R^{\square}\wedge R^{\square}+m\mathscr{D}F_{a_1\cdots a_4}\wedge V_{a_5}\wedge\cdots\wedge V_{a_{11}}\varepsilon^{a_1\cdots a_{11}}\\&+\tfrac{7}{2}miF_{a_1\cdots a_4}\bar\psi\wedge\Gamma_{a_6}\psi\wedge V_{a_7}\wedge\cdots\wedge V_{a_{11}}\varepsilon^{a_1\cdots a_{11}}=0.\end{aligned} \tag{5.1c}$$

Gravitino equation (variation in $\bar\psi$):

$$\begin{aligned}&4\Gamma_{a_1\cdots a_8}\rho\wedge V^{a_1}\wedge\cdots\wedge V^{a_8}-168i\Gamma_{a_1\cdots a_5}\psi\wedge V^{a_1}\wedge\cdots\wedge V^{a_5}\wedge R^{\square}\\&-m\Gamma_{ab}\psi\wedge V^a\wedge V^b\wedge F_{c_1\cdots c_4}V_{c_5}\wedge\cdots\wedge V_{c_{11}}\varepsilon^{c_1\cdots c_{11}}=0.\end{aligned} \tag{5.1d}$$

Einstein equation (variation in V_r):

$$\begin{aligned}&-R^{a_1a_2}\wedge V^{a_3}\wedge\cdots\wedge V^{a_{10}}\varepsilon_{a_1\cdots a_{10}r}\\&\quad+\tfrac{7}{15}iR_r\wedge\bar\psi\wedge\Gamma_{b_1\cdots b_5}\psi\wedge V_{b_6}\wedge\cdots\wedge V_{b_{11}}\varepsilon^{b_1\cdots b_{11}}\\&\quad+\tfrac{7}{5}iR_a\wedge V^a\wedge\bar\psi\wedge\Gamma_{b_1\cdots b_5}\psi\wedge V_{b_6}\wedge\cdots\wedge V_{b_{10}}\varepsilon^{b_1\cdots b_{10}r}\\&\quad+\tfrac{7}{15}iV_r\wedge\bar\psi\wedge\Gamma_{b_1\cdots b_5}\rho\wedge V_{b_6}\wedge\cdots\wedge V_{b_{11}}\varepsilon^{b_1\cdots b_{11}}\\&\quad-\tfrac{7}{5}i\bar\psi\wedge\Gamma_{b_1\cdots b_5}\wedge V_r\wedge R_{b_6}\wedge V_{b_7}\wedge\cdots\wedge V_{b_{11}}\varepsilon^{b_1\cdots b_{11}}\\&\quad-420iR^{\square}\wedge\bar\psi\wedge\Gamma^{a_1\cdots a_4r}\psi\wedge V_{a_1}\wedge\cdots\wedge V_{a_4}\\&\quad+16\bar\rho\wedge\Gamma_{c_1\cdots c_7r}\psi\wedge V^{c_1}\wedge\cdots\wedge V^{c_7}\\&\quad+11nF_{a_1\cdots a_4}F^{a_1\cdots a_4}\wedge V^{c_1}\wedge\cdots\wedge V^{c_{10}}\varepsilon_{c_1\cdots c_{10}r}\\&\quad+7mF_{a_1\cdots a_4}V_{a_5}\wedge\cdots\wedge V_{a_{10}}\wedge R^{\square}\varepsilon^{a_1\cdots a_{10}r}\\&\quad-mF_{a_1\cdots a_4}\bar\psi\wedge\Gamma_{rb}\psi\wedge V^b\wedge V_{a_5}\wedge\cdots\wedge V_{a_{11}}\varepsilon^{a_1\cdots a_{11}}=0.\end{aligned} \tag{5.1e}$$

Considering first eq. (5.1a) we immediately obtain

$$R^a = 0. \tag{5.2}$$

Therefore the supertorsion vanishes on-shell just as in $N=1$ and $N=2$ 4-dimensional supergravities. Eq. (5.2) can be solved for the connection $\omega^{ab}{}_{\mu}$ as a functional of $V^a_{\cdot\mu}$ and ψ_μ. Explicitly:

$$\omega^{ab}{}_{\mu} = \mathring{\omega}^{ab}{}_{\mu} - \tfrac{1}{4} i \left(\bar{\psi}_\mu \Gamma_\lambda \psi_\nu + \bar{\psi}_\lambda \Gamma_\nu \psi_\mu + \bar{\psi}_\lambda \Gamma_\mu \psi_\nu - [\lambda \to \nu] \right) V^{\lambda|a} V^{\nu|b}, \tag{5.3}$$

where $\mathring{\omega}^{ab}{}_{\mu}$ is the usual connection satisfying the space-time torsionless condition

$$\partial_{[\mu} V^a_{\nu]} - \omega^{ab}_{[\mu} V^b_{\nu]} = 0. \tag{5.4}$$

Considering next eq. (5.1b) we obtain

$$R^{\square} = -\frac{2n11!}{7!4!m} F_{a_1\cdots a_4} V^{a_1} \wedge \cdots \wedge V^{a_4}. \tag{5.5}$$

Therefore, if we set

$$n = -\frac{m7!4!}{11!} = -\frac{m}{660}, \tag{5.6}$$

$F_{a_1\cdots a_4}$ can be identified with the space-time components of the curvature $R^{\square}$:

$$F_{a_1\cdots a_4} = R^{\square}_{a_1\cdots a_4}, \tag{5.7}$$

which, because of eq. (5.5), has no outer spinorial components:

$$R^{\square} = F_{a_1\cdots a_4} V^{a_1} \wedge \cdots \wedge V^{a_4}. \tag{5.8}$$

The choice (5.6) amounts to nothing else than a field redefinition of $F_{a_1\cdots a_4}$.

Using now eqs. (5.2) and (5.8) in eqs. (5.1c) and (5.1d) we obtain the following result:

If the parameter m takes the value:

$$m = 2, \tag{5.9}$$

then the gravitino equation is consistent with the second Maxwell equation and we have the solution

$$\rho = \rho_{ab} V^a \wedge V^b - \tfrac{1}{3}\left(i\Gamma^{a_1 a_2 a_3}\psi \wedge V^{a_4} F_{a_1\cdots a_4} + \tfrac{1}{8}\Gamma^{a_1\cdots a_5}\psi \wedge V_{a_5} F_{a_1\cdots a_4} \right), \tag{5.10}$$

TABLE 3
Summary of $D=11$ supergravity

Cartan integrable system

$$R^{ab}=\mathrm{d}\omega^{ab}-\omega^{ac}\wedge\omega_c^{\cdot b}$$

$$R^{a}=\mathcal{D}V^{a}-\tfrac{1}{2}i\bar{\psi}\wedge\Gamma^{a}\psi$$

$$\rho=\mathcal{D}\psi$$

$$R^{\square}=\mathrm{d}A-\tfrac{1}{2}\bar{\psi}\wedge\Gamma^{ab}\psi\wedge V_a\wedge V_b$$

Geometric action

$$\begin{aligned}\mathcal{A}=\int_{\mathrm{M}_{11}}\Big\{&-\tfrac{1}{9}R^{a_1a_2}\wedge V^{a_3}\wedge\cdots\wedge V^{a_{11}}\varepsilon_{a_1\cdots a_{11}}\\
&+\tfrac{7}{30}iR^{a}\wedge V_a\wedge\bar{\psi}\wedge\Gamma^{b_1\cdots b_5}\psi\wedge V^{b_6}\wedge\cdots\wedge V^{b_{11}}\varepsilon_{b_1\cdots b_{11}}\\
&+2\bar{\rho}\wedge\Gamma_{c_1\cdots c_8}\psi\wedge V^{c_1}\wedge\cdots\wedge V^{c_8}\\
&-84R^{\square}\wedge\Big(i\bar{\psi}\wedge\Gamma_{a_1\cdots a_5}\psi^{a}\wedge V^{a_1}\wedge\cdots\wedge V^{a_5}-10A\wedge\bar{\psi}\wedge\Gamma_{ab}\psi\wedge V^{a}\wedge V^{b}\Big)\\
&+\tfrac{1}{4}\bar{\psi}\wedge\Gamma^{a_1a_2}\psi\wedge\bar{\psi}\wedge\Gamma^{a_3a_4}\psi\wedge V^{a_5}\wedge\cdots\wedge V^{a_{11}}\varepsilon_{a_1\cdots a_{11}}\\
&-210\bar{\psi}\wedge\Gamma^{a_1a_2}\psi\wedge\bar{\psi}\wedge\Gamma^{a_3a_4}\psi\wedge V_{a_1}\wedge\cdots\wedge V_{a_4}\wedge A\\
&-840R^{\square}\wedge R^{\square}\wedge A-\tfrac{1}{330}F_{a_1\cdots a_4}F^{a_1\cdots a_4}V^{c_1}\wedge\cdots\wedge V^{c_{11}}\varepsilon_{c_1\cdots c_{11}}\\
&+2F_{a_1\cdots a_4}R^{\square}\wedge V_{a_5}\wedge\cdots\wedge V_{a_{11}}\varepsilon^{a_1\cdots a_{11}}\Big\}\end{aligned}$$

On-shell solution for the curvatures

$$R^{a}=0$$

$$R^{\square}=F_{a_1\cdots a_4}V^{a_1}\wedge\cdots\wedge V^{a_4}$$

$$\rho=\rho_{ab}V^{a}\wedge V^{b}-\tfrac{1}{3}\Big(i\Gamma^{a_1a_2a_3}\psi\wedge V^{a_4}+\tfrac{1}{8}\Gamma^{a_1\cdots a_4m}\psi\wedge V_m\Big)F_{a_1\cdots a_4}$$

$$\begin{aligned}R^{ab}=R^{ab}_{\cdot\cdot mn}V^{m}\wedge V^{n}+i\bar{\rho}_{mn}\Big(&\tfrac{1}{2}\Gamma^{abcmn}-\tfrac{2}{9}\Gamma^{mn[a}\delta^{b]c}\\
&+2\Gamma^{ab[m}\delta^{n]c}\Big)\psi\wedge V_c-\tfrac{7}{9}\bar{\psi}\wedge\Gamma_{mn}\psi F^{mnab}+\tfrac{55}{216}\bar{\psi}\wedge\Gamma^{abc_1\cdots c_4}\psi\wedge F_{c_1\cdots c_4}.\end{aligned}$$

Propagation equations

(i) $\Gamma^{abc}\rho_{bc}=0$

(ii) $\mathcal{D}_m F^{mc_1c_2c_3}-\frac{1}{2\cdot 4!\cdot 7!}\varepsilon^{c_1c_2c_3a_1\cdots a_8}F_{a_1\cdots a_4}F_{a_5\cdots a_8}=0$

(iii) $R^{am}_{\cdot bm}-\frac{1}{2}\delta^{a}_{b}R^{mn}_{\cdot mn}-3F^{ac_1c_2c_3}F_{bc_1c_2c_3}+\frac{3}{8}\delta^{a}_{b}F^{c_1\cdots c_4}F_{c_1\cdots c_4}=0$

where ρ_{ab} and $F_{a_1\cdots a_4}$ satisfy the following propagation equations:

$$\Gamma^{abc}\rho_{bc}=0\,, \tag{5.11a}$$

$$\mathfrak{D}_m F^{mc_1c_2c_3}-\frac{1}{2\cdot 4!\cdot 7!}\varepsilon^{c_1c_2c_3a_1\cdots a_8}F_{a_1\cdots a_4}F_{a_5\cdots a_8}=0\,. \tag{5.11b}$$

On the other hand, if $m\neq 2$ we obtain $F_{a_1\cdots a_4}=0$ and the only solution is $R^{ab}=R^a=\rho=F_{a_1\cdots a_4}=0$.

The various projections of the Einstein equation (5.1e) do not pose any further threat and, besides yielding the graviton propagation equation

$$R^{am}_{\cdot bm}-\tfrac{1}{2}\delta^a_b R^{mn}_{\cdot mn}-3F^{ac_1c_2c_3}F_{bc_1c_2c_3}+\tfrac{3}{8}\delta^a_b F^{c_1\cdots c_4}F_{c_1\cdots c_4}=0\,, \tag{5.12}$$

they give rheonomic conditions which express the outer components $R^{ab}_{\cdot\alpha m}$ and $R^{ab}_{\cdot\alpha\beta}$ in terms of the inner ones ρ_{ab} and $F_{a_1\cdots a_4}$ (see table 3).

Therefore when $m=2$ and $n=-\frac{1}{330}$ the theory described by action (4.28) becomes non-trivial and rheonomic: it goes without saying that upon transition to space-time second-order formalism it coincides with the Cremmer-Julia-Scherk theory [1].

We think it proper to conclude this section with a summary of the final result. It is given in table 3.

6. Supergroup interpretation of the $D=11$ Cartan integrable system

In sect. 2 we have discussed the possible equivalence of a CIS with an ordinary supergroup.

Everything boils down to solving the system of algebraic equations (2.26) and (2.27) relating the supergroup structure constants $C^{\alpha}_{\cdot\beta\gamma}$ with the components $K^{A(p)}_{\alpha_1\cdots\alpha_p}$ of the CIS forms $\Theta^{A(p)}$. In the present section we solve this problem for the specific CIS of $D=11$ super gravity, defined by eqs. (3.15) and recalled in table 3.

Since V^a, ω^{ab}, ψ are already 1-forms and eqs. (3.15) already define a supergroup, all we have to do is to find a suitable decomposition of the 3-form A in a basis of 1-forms.

Using a little bit of ingenuity we started with an ansatz, thus reducing eqs. (2.26) and (2.27) system of ordinary quadratic equations on a set of numerical parameters.

Our ansatz is the following:

$$\begin{aligned}A={}&B^{ab}\wedge V_a\wedge V_b+\alpha_1 B_{a_1a_2}\wedge B^{a_2}_{\cdot a_3}\wedge B^{a_3a_1}\\&+\alpha_2 B_{b_1a_1\cdots a_4}\wedge B^{b_1}_{\cdot b_2}\wedge B^{b_2a_1\cdots a_4}+\alpha_3\varepsilon_{a_1\cdots a_5b_1\cdots b_5m}B^{a_1\cdots a_5}\wedge B^{b_1\cdots b_5}\wedge V^m\\&+\alpha_4\varepsilon_{m_1\cdots m_6n_1\cdots n_5}B^{m_1m_2m_3p_1p_2}\wedge B^{m_4m_5m_6p_1p_2}\wedge B^{n_1\cdots n_5}\\&+i\beta_1\bar\psi\wedge\Gamma^a\eta\wedge V_a+\beta_2\bar\psi\wedge\Gamma^{ab}\eta\wedge B_{ab}+i\beta_3\bar\psi\wedge\Gamma^{a_1\cdots a_5}\eta B_{a_1\cdots a_5}\,,\end{aligned} \tag{6.1}$$

where B^{ab}, $B^{a_1\cdots a_5}$ are two new skew-symmetric 1-forms, η is a new spinorial 1-form and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3$ are parameters. The structure of the supergroup is described by curvatures of the following type:

$$R^{ab} = \mathrm{d}\omega^{ab} - \omega^{ac} \wedge \omega_c^{\cdot b}, \tag{6.2a}$$

$$R^a = \mathcal{D}V^a - \tfrac{1}{2}i\bar{\psi} \wedge \Gamma^a\psi, \tag{6.2b}$$

$$\rho = \mathcal{D}\psi, \tag{6.2c}$$

$$\overset{\square}{R}{}^{a_1a_2} = \mathcal{D}B^{a_1a_2} - \tfrac{1}{2}\bar{\psi} \wedge \Gamma^{a_1a_2}\psi, \tag{6.2d}$$

$$\overset{\square}{R}{}^{a_1\cdots a_5} = \mathcal{D}B^{a_1\cdots a_5} - \tfrac{1}{2}i\bar{\psi} \wedge \Gamma^{a_1\cdots a_5}\psi, \tag{6.2e}$$

$$\begin{aligned}\sigma = {} & \mathcal{D}\eta + i\delta\Gamma^a\psi \wedge V_a + \gamma_1\Gamma_{ab} \wedge B^{ab} \\ & + i\gamma_2\Gamma_{a_1\cdots a_5}\psi \wedge B^{a_1\cdots a_5}.\end{aligned} \tag{6.2f}$$

When we set $R^{ab} = R^a = \rho = \overset{\square}{R}{}^{a_1a_2} = R^{a_1\cdots a_5} = \sigma = 0$, we obtain the Maurer-Cartan equations which are viable only if they satisfy the integrability condition $\mathrm{dd} = 0$ (Jacobi identities). In our case the integrability of eqs. (6.2a)–(6.2e) is self-evident: all we have to do is to check the integrability of eq. (6.2f). At zero curvatures we obtain:

$$\begin{aligned}\mathcal{D}\mathcal{D}\eta = 0 = {} & \tfrac{1}{2}\delta\Gamma^a\psi \wedge \bar{\psi} \wedge \Gamma_a\psi - \tfrac{1}{2}\gamma_1\Gamma^{ab}\psi \wedge \bar{\psi} \wedge \Gamma_{ab}\psi \\ & + \tfrac{1}{2}\gamma_2\Gamma^{a_1\cdots a_5}\psi \wedge \bar{\psi} \wedge \Gamma_{a_1\cdots a_5}\psi.\end{aligned} \tag{6.3}$$

Using the Fierz decomposition of table 2 we see that eq. (6.3) is true only if

$$\delta + 10\gamma_1 - 720\gamma_2 = 0. \tag{6.4}$$

Eq. (6.4) is the specific form taken in our case by condition (2.26). The explicit form of eq. (2.27) is now worked out in the following way. We take the ansatz (6.1) and we compute $\mathrm{d}A$ at zero curvatures: $R^{ab} = R^a = \rho = \overset{\square}{R}{}^{a_1a_2} = R^{a_1\cdots a_5} = \sigma = 0$. Imposing that the result be equal to $\frac{1}{2}\bar{\psi}\wedge\Gamma^{ab}\psi\wedge V_a\wedge V_b$, we get

$$
\begin{aligned}
\mathrm{d}A = {} & \tfrac{1}{2}\bar{\psi}\wedge\Gamma^{ab}\psi\wedge V_a\wedge V_b - iB^{ab}\wedge\bar{\psi}\wedge\Gamma_a\psi\wedge V_b + \tfrac{3}{2}\alpha_1\bar{\psi}\wedge\Gamma^{ab}\psi\wedge B^{b}_{\cdot m}\wedge B^{ma} \\
& + i\alpha_2\bar{\psi}\wedge\Gamma^{a_1\cdots a_4 m}\psi\wedge B_m^{\cdot n}\wedge B_{na_1\cdots a_4} \\
& - \tfrac{1}{2}\alpha_2\bar{\psi}\wedge\Gamma_{b_1b_2}\psi\wedge\bar{\psi}\wedge B^{a_1\cdots a_4 b_1}\wedge B_{a_1\cdots a_4}{}^{\cdot b_2} \\
& + i\alpha_3\varepsilon_{a_1\cdots a_5 b_1\cdots b_5 m}\bar{\psi}\wedge\Gamma^{a_1\cdots a_5}\psi\wedge B^{b_1\cdots b_5}\wedge V^m \\
& + \tfrac{1}{2}i\alpha_3\varepsilon_{a_1\cdots a_5 b_1\cdots b_5 m}B^{a_1\cdots a_5}\wedge B^{b_1\cdots b_5}\wedge\bar{\psi}\wedge\Gamma^m\psi \\
& + i\alpha_4\varepsilon_{a_1\cdots a_6 b_1\cdots b_5}\bar{\psi}\wedge\Gamma^{a_1a_2a_3p_1p_2}\psi\wedge B^{p_1p_2a_4a_5a_6}\wedge B^{b_1\cdots b_5} \\
& + \tfrac{1}{2}i\alpha_4\bar{\psi}\wedge\Gamma^{a_1\cdots a_5}\psi\wedge\varepsilon_{a_1\cdots a_5 b_1\cdots b_6}B^{b_1b_2b_3p_1p_2}\wedge B^{b_4b_5b_6}{}_{\cdot p_1p_2} \\
& - i\beta_1\bar{\psi}\wedge\Gamma^a\Big(i\delta\Gamma^m\psi\wedge V_m + i\gamma_1\Gamma^{mn}\psi\wedge B_{mn} + i\gamma_2\Gamma^{m_1\cdots m_5}\psi\wedge B_{m_1\cdots m_5}\Big)\wedge V_a \\
& + \tfrac{1}{2}i^2\beta_1\bar{\psi}\wedge\Gamma^a\eta\wedge\bar{\psi}\wedge\Gamma_a\psi - \beta_2\bar{\psi}\wedge\Gamma^{ab}\Big(i\delta\Gamma^m\psi\wedge V_m + \gamma_1\Gamma^{mn}\psi\wedge B_{mn} \\
& \qquad + i\gamma_2\Gamma^{m_1\cdots m_5}\psi\wedge B_{m_1\cdots m_5}\Big)\wedge B_{ab} \\
& + \tfrac{1}{2}\beta_2\bar{\psi}\wedge\Gamma^{ab}\eta\wedge\bar{\psi}\wedge\Gamma_{ab}\psi + \tfrac{1}{2}i^2\beta_3\bar{\psi}\wedge\Gamma_{a_1\cdots a_5}\eta\wedge\bar{\psi}\wedge\Gamma^{a_1\cdots a_5}\psi \\
& - i\beta_3\bar{\psi}\wedge\Gamma^{a_1\cdots a_5}\Big(i\delta\Gamma^m\psi\wedge V_m + \gamma_1\Gamma_{mn}\psi\wedge B_{mn} \\
& \qquad + i\gamma_2\Gamma_{m_1\cdots m_5}\psi\wedge B^{m_1\cdots m_5}\Big)\wedge B_{a_1\cdots a_5} \\
= {} & \tfrac{1}{2}\bar{\psi}\wedge\Gamma^{ab}\psi\wedge V_a\wedge V_b. \qquad (6.5)
\end{aligned}
$$

Using the Fierz decomposition of table 2, we see that eq. (6.5) holds only if the following system of parameter equations is satisfied:

(i) $$\tfrac{1}{2} - \beta_1\delta = \tfrac{1}{2}\,,$$

(ii) $$1 + 2\beta_1\gamma_1 - 2\beta_2\delta = 0\,,$$

(iii) $$-\tfrac{1}{2}\beta_1 - 5\beta_2 + 360\beta_3 = 0\,,$$

(iv) $$\beta_1\gamma_2+\beta_3\delta-120\alpha_3=0\,,$$

(v) $$\tfrac{3}{2}\alpha_1-4\beta_2\gamma_1=0\,,$$

(vi) $$\tfrac{1}{2}\alpha_3-\beta_3\gamma_2=0\,,$$

(vii) $$+\tfrac{1}{2}\alpha_2+600\beta_3\gamma_2=0\,,$$

(viii) $$-\tfrac{5}{3}\beta_3\gamma_2+\tfrac{1}{2}\alpha_4=0\,,$$

(ix) $$10\beta_3\gamma_1+\alpha_2+10\beta_2\gamma_2=0\,. \tag{6.6}$$

Eqs. (6.6) are the explicit form of eq. (2.27). Combined with eq. (6.4) they have two distinct solutions which correspond to two different supergroups:

$$\alpha_1=\begin{pmatrix}\frac{4}{15}\\ -\frac{4}{15}\end{pmatrix},\quad \alpha_2=\begin{pmatrix}-\frac{5}{144}\\ \frac{5}{144}\end{pmatrix},\quad \alpha_3=\begin{pmatrix}\dfrac{1}{4!6!}\\ -\dfrac{1}{4!6!}\end{pmatrix},\quad \alpha_4=\begin{pmatrix}\dfrac{1}{2(72)^2}\\ -\dfrac{1}{2(72)^2}\end{pmatrix},$$

$$\beta_1=\begin{pmatrix}0\\1\end{pmatrix},\quad \beta_2=\begin{pmatrix}\frac{1}{2}\\ \frac{1}{5}\end{pmatrix},\quad \beta_3=\begin{pmatrix}\frac{1}{144}\\ \frac{1}{240}\end{pmatrix},$$

$$\gamma_1=\begin{pmatrix}\frac{1}{5}\\ -\frac{1}{2}\end{pmatrix},\quad \gamma_2=\begin{pmatrix}\frac{1}{240}\\ -\frac{1}{144}\end{pmatrix},\quad \delta=\begin{pmatrix}1\\0\end{pmatrix}. \tag{6.7}$$

Therefore we conclude that also $D=11$ supergravity is a standard group manifold theory. The supergroup curvatures are the following:

$D=11$ *supergravity supergroup curvatures*:

$$R^{ab}=\mathrm{d}\omega^{ab}-\omega^{ac}\wedge\omega_c^{\cdot b}\,, \tag{6.8a}$$

$$R^a=\mathscr{D}V^a-\tfrac{1}{2}i\bar\psi\wedge\Gamma^a\psi\,, \tag{6.8b}$$

$$\rho=\mathscr{D}\psi\,, \tag{6.8c}$$

$$R^{a_1a_2}=\mathscr{D}B^{a_1a_2}-\tfrac{1}{2}\bar\psi\wedge\Gamma^{a_1a_2}\psi\,, \tag{6.8d}$$

$$R^{a_1\cdots a_5}=\mathscr{D}B^{a_1\cdots a_5}-\tfrac{1}{2}i\bar\psi\wedge\Gamma^{a_1\cdots a_5}\psi\,, \tag{6.8e}$$

$$\begin{aligned}\sigma=\mathscr{D}\eta+i\begin{pmatrix}1\\0\end{pmatrix}\Gamma_a\psi\wedge V^a+\begin{pmatrix}\frac{1}{5}\\ -\frac{1}{2}\end{pmatrix}\Gamma_{ab}\psi\wedge B^{ab}\\ +i\begin{pmatrix}\frac{1}{240}\\ -\frac{1}{144}\end{pmatrix}\Gamma_{a_1\cdots a_5}\psi\wedge B^{a_1\cdots a_5}.\end{aligned} \tag{6.8f}$$

The action is the one given in table 3 provided the following replacement is made everywhere:

$$A = B_{ab} \wedge V^a \wedge V^b + \begin{pmatrix} \frac{4}{15} \\ -\frac{4}{15} \end{pmatrix} B^{a_1 a_2} \wedge B_{a_2}^{\cdot a_3} \wedge B_{a_3 a_1}$$

$$+ \begin{pmatrix} -\frac{5}{144} \\ \frac{5}{144} \end{pmatrix} B_{a_1 \cdots a_4 b_1} \wedge B^{b_1}_{\cdot b_2} \wedge B^{b_2 a_1 \cdots a_4}$$

$$+ \begin{pmatrix} \dfrac{1}{4!6!} \\ -\dfrac{1}{4!6!} \end{pmatrix} \varepsilon_{a_1 \cdots a_5 b_1 \cdots b_5 m} B^{a_1 \cdots a_5} \wedge B^{b_1 \cdots b_5} \wedge V^m$$

$$+ \begin{pmatrix} \dfrac{1}{2 \cdot 72^2} \\ -\dfrac{1}{2 \cdot 72^2} \end{pmatrix} \varepsilon_{a_1 \cdots a_5 b_1 \cdots b_5} B^{a_1 a_2 a_3 p_1 p_2} \wedge B^{a_4 a_5 a_6}{}_{\cdot p_1 p_2} \wedge B^{b_1 \cdots b_5}$$

$$+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\psi} \wedge \Gamma^a \eta \wedge V_a + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{5} \end{pmatrix} \bar{\psi} \wedge \Gamma^{ab} \eta \wedge B_{ab}$$

$$+ i \begin{pmatrix} \frac{1}{144} \\ \frac{1}{240} \end{pmatrix} \bar{\psi} \wedge \Gamma^{a_1 \cdots a_5} \eta \wedge B_{a_1 \cdots a_5} . \tag{6.9}$$

Obviously the lagrangian could have been determined by a direct application of the standard group manifold method to the supergroups (6.8), without any reference to the CIS C_1. It must be noted however that:

(a) The lagrangian written in terms of the supergroup potentials is gigantic and the cosmo-cocycle equation would have been solvable only through the use of a computer.

(b) The supergroups (6.8) introduce the novelty of a second abelian spinorial generator Q'_α which is associated to the 1-form η.

This very intriguing feature could not be guessed *a priori*.

7. Conclusions

$D = 11$ supergravity is the local theory of one of the two supergroups (6.8). The super Lie algebra is immediately read off from eqs. (6.8) and it is given in table 4. The $A_{\mu\nu\rho}$ field is not elementary; rather, it is a non-linear combination of the 1-form

TABLE 4
Super Lie algebras of $D=11$ supergravity

Normalization of generators	$\omega^{ab}(iJ_{mn})=\delta^{ab}_{mn}$ $\quad \psi_\alpha(Q_\beta)=\delta_{\alpha\beta}$ $\quad \eta_\alpha(Q'_\beta)=\delta_{\alpha\beta}$ $V^a(P_b)=\delta^a_b$ $\quad B^{a_1a_2}(Z_{b_1b_2})=\delta^{a_1a_2}_{b_1b_2}$ $\quad B^{a_1\cdots a_5}(Z_{b_1\cdots b_5})=\delta^{a_1\cdots a_5}_{b_1\cdots b_5}$
Commutation relations	see below

$$\left[J_{m_1m_2}, J^{n_1n_2}\right] = -4i\,\delta^{[n_1}_{[m_1}J^{n_2]}_{m_2]}$$

$$\left[J_{m_1m_2}, P^{n}\right] = -2i\,\delta^{n}_{[m_1}P_{m_2]}$$

$$\left[J_{m_1m_2}, Z^{n_1n_2}\right] = -4i\,\delta^{[n_1}_{[m_1}Z^{\cdot n_2]}_{m_2]}$$

$$\left[J_{m_1m_2}, Z^{n_1\cdots n_5}\right] = -10i\,\delta^{[n_1}_{[m_1}Z^{\cdot n_2\cdots n_5]}_{m_2]}$$

$$\left[J_{m_1m_2}, \begin{pmatrix} Q \\ Q' \end{pmatrix}\right] = \tfrac{1}{4}i\Gamma_{m_1m_2}\begin{pmatrix} Q \\ Q' \end{pmatrix}$$

$$\begin{aligned}[P_n, P_m] &= \left[Z_{m_1m_2}, Z_{n_1n_2}\right] = \left[Z_{m_1\cdots m_5}, Z_{n_1\cdots n_5}\right] = \left[P_n, Z_{m_1m_2}\right] \\ &= \left[P_n, Z_{m_1\cdots m_5}\right] = \left[Z_{m_1m_2}, Z_{n_1\cdots n_5}\right] = 0\end{aligned}$$

$$[P_m, Q'] = \left[Z_{m_1\cdots m_5}, Q'\right] = \left[Z_{m_1m_2}, Q'\right] = 0$$

$$\{Q, Q\} = iC\Gamma^a P_a + iC\Gamma^{a_1a_2}Z_{a_1a_2} + iC\Gamma^{a_1\cdots a_5}Z_{a_1\cdots a_5}$$

$$\{Q', Q'\} = 0$$

$$\{Q, P^a\} = i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Gamma_a Q' \qquad \{Q, Z^{a_1a_2}\} = \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{2} \end{pmatrix}\Gamma^{a_1a_2}Q'$$

$$\{Q, Z^{a_1\cdots a_5}\} = \begin{pmatrix} \frac{1}{240} \\ -\frac{1}{144} \end{pmatrix}\Gamma^{a_1\cdots a_5}Q'$$

potentials

$$B^{a_1a_2}{}_\mu, \qquad B^{a_1\cdots a_5}{}_\mu, \qquad V^a{}_\mu, \qquad \psi_\mu, \qquad \eta_\mu. \tag{7.1}$$

All the symmetries of the theory are generated by J_{ab}, P_a, Q, Q', $Z_{a_1a_2}$, $Z_{a_1\cdots a_5}$, associated to ω^{ab}, V^a, ψ, η, $B_{a_1a_2}$, $B_{a_1\cdots a_5}$, respectively. To determine the explicit transformations of all the fields under all the generators what we have to do is the following. Starting from eq. (6.9) and taking the derivative we obtain:

$$R^{\square} = \overset{\square}{R}_{ab} V^a \wedge V^b - 2B_{ab} \wedge R^a \wedge V^b + 3\begin{pmatrix} \frac{4}{15} \\ -\frac{4}{15} \end{pmatrix} \overset{\square}{R}{}^{a_1 a_2} \wedge B_{a_2}^{\cdot a_3} \wedge B_{a_3 a_1}$$

$$+ \begin{pmatrix} -\frac{4}{144} \\ \frac{5}{144} \end{pmatrix} R^{a_1 \cdots a_4 b_1} \wedge B_{b_1}^{\cdot b_2} \wedge B_{b_1 a_1 \cdots a_4} + \cdots +$$

$$+ i\begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\rho} \wedge \Gamma^a \eta \wedge V_a - i\begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\psi} \wedge \Gamma^a \sigma \wedge V_a + \cdots . \tag{7.2}$$

Comparing eq. (7.2) with the on-shell curvatures given by table 3 we can determine the structure of all the new curvatures

$$\overset{\square}{R}_{ab}, \qquad R_{a_1 \cdots a_5}, \qquad \sigma . \tag{7.3}$$

Once this is done we have the full set of rheonomic conditions and therefore we have the complete on-shell representation of the algebra. This programme is very straightforward but long and we postpone it to future publications.

The authors are grateful to their friends Profs. Tullio Regge and Alessandro D'Adda for many essential and enlightening discussions and for access to their unpublished result on 11-dimensional Fierz identities which was very useful in the beginning of the present work.

Appendix

NOTATIONS AND CONVENTIONS

In this paper the signature of the SO(1, 10) Lorentz invariant metric is $(+, -, \ldots, -)$ and the indices are raised and lowered accordingly. Sometimes, for graphical convenience, we do not write some index at the right level but whether it is to be raised or lowered is evident from the tensorial character of the formula. This is particularly true in table 2 where the position of the indices is already exploited to denote the Young tableau symmetry pattern. Moreover, when we write δ_{ab} we mean the Lorentz metric η_{ab}:

$$\delta_{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \cdots & -1 \end{pmatrix} . \tag{A.1}$$

The Levi-Civita tensor is fixed in such a way that

$$\varepsilon_{12\cdots 11} = \varepsilon^{12\cdots 11} = 1\,. \tag{A.2}$$

The symmetrization and antisymmetrization symbols are respectively defined by

$$[a_1 \cdots a_n] = \frac{1}{n!} \sum_{\mathrm{P}} a_{\mathrm{P}(1)} \cdots a_{\mathrm{P}(n)} \tag{A.3a}$$

$$\{a_1 \cdots a_n\} = \frac{1}{n!} \sum_{\mathrm{P}} (-)^{\delta_{\mathrm{P}}} a_{\mathrm{P}(1)} \cdots a_{\mathrm{P}(n)} \tag{A.3b}$$

where Σ_{P} means sum over permutations and δ_{P} is the parity of the said permutations.

Γ-matrix conventions are the following:

$$\begin{aligned}
&\Gamma_1^\dagger = \Gamma_1, \qquad \Gamma_i^\dagger = -\Gamma_i, \qquad i \neq 1,\\
&\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} = 2\delta_{ab},\\
&\Gamma^{a_1\cdots a_n} = \Gamma^{[a_1}\Gamma^{a_2}\cdots\Gamma^{a_n]},\\
&C^{\mathrm{T}} = -C = C^{-1}, \qquad C\Gamma^a C^{-1} = -(\Gamma^a)^{\mathrm{T}},
\end{aligned} \tag{A.4}$$

and we have the following separation of symmetric and antisymmetric Γ-matrices:

$$C\Gamma^{a_1\cdots a_n}C^{-1} = \begin{cases} -(\Gamma^{a_1\cdots a_n})^{\mathrm{T}}, & n = 1,2,5,6,9,10 \text{ (symmetric)},\\ (\Gamma^{a_1\cdots a_n})^{\mathrm{T}}, & n = 0,3,4,7,8 \text{ (antisymmetric)}. \end{cases} \tag{A.5}$$

The bar operation on spinors is defined by

$$\bar\psi = \psi^\dagger \Gamma_1, \tag{A.6}$$

and the Majorana condition is

$$\psi = \psi^{\mathrm{c}} = C(\bar\psi)^{\mathrm{T}}. \tag{A.7}$$

Lorentz covariant derivatives and curvature are defined as follows:

$$\begin{aligned}
&\mathcal{D}V^a = \mathrm{d}V^a - \omega^{ab} \wedge V_b,\\
&\mathcal{D}B^{a_1a_2} = \mathrm{d}B^{a_1a_2} - 2\omega^{c[a_1} \wedge B^{a_2]}_{\cdot c},\\
&\mathcal{D}B^{a_1\cdots a_5} = \mathrm{d}B^{a_1\cdots a_5} + 5\omega^{b[a_1} \wedge B^{a_2\cdots a_5]}_{\cdot b},\\
&\mathcal{D}\psi = \mathrm{d}\psi - \tfrac{1}{4}\omega^{ab}\Gamma_{ab} \wedge \psi,\\
&\mathcal{R}^{ab} = \mathrm{d}\omega^{ab} + \omega^{ac} \wedge \omega_c^{b\cdot}.
\end{aligned} \tag{A.8}$$

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