sections of 2-reps

Schreiber*

November 24, 2006

Abstract

Given a notion of sections of a 2-bundle (with connection) and given a 2-particle charged under that 2-bundle, I would like to "transgress" the 2-bundle to the configuration space of the 2-particle. Then I would like to understand if the category of sections of the transgressed 2-bundle induces something like an extended QFT on parameter space.

Here I propose an approach to this program and spell out the details in the simple example where target space is a strict 2-group.

1 Introduction

I would like to find a systematic and natural understanding of how to obtain an extended QFT from an n-vector bundle with connection coupled to an n-particle.

Slightly more precisely: for T some n-category of n-vector spaces, and $\mathcal{P}$ some n-category thought of as target space, an n-vector bundle with connection on $\mathcal{P}$ is an n-functor

$$\mathrm{tra} : \mathcal{P} \to T\,.$$

Coupling this n-bundle to an n-particle means picking an $(n-1)$-category par, forming the configuration space conf $\subset [\mathrm{par}, \mathcal{P}]$, and "transgressing" the n-vector bundle to that configuration space:

$$\mathrm{tra}_* : \mathrm{conf} \to [\mathrm{par}, T]\,.$$

Given this data, I would like to construct in a canonical fashion an $(n-1)$-functor

$$\mathrm{QFT}(\mathrm{par}, \mathrm{tra}) : \mathrm{par} \to T$$

which can be thought of as sending parameter space to the **space of states** of the charged n-particle.

For instance, for $n = 1$, we might choose $\mathcal{P}$ to be the category of thin-homotopy classes of path in some manifold X, choose tra to be an ordinary line

*E-mail: urs.schreiber at math.uni-hamburg.de

bundle with connection, choose par to be the discrete category par $= \{\bullet\}$ on a single element. Then

$$\mathrm{QFT}(\mathrm{par}, \mathrm{tra}) : \bullet \mapsto H$$

should simply yield the Hilbert space H of the electromagnetically charged particle known from ordinary quantum mechanics.

Eventually we want to extend $\mathrm{QFT}(\cdot, \cdot)$ to an n-functor that also describes the propagation of the n-particle. But here I shall just be concerned with understanding n-spaces of states.

While probably not completely understood yet, quite a few things about how extended QFTs should behave are known. In particular, QFTs should roughly send circles in parameter space to something like a trace in n-vector spaces.

A proposal for what the trace in a 2-vector space should be has been made by Kapranov and Ganter, in the context of representations of groups on 2-vector spaces. For the moment, this proposal shall serve as a first consistency check for my proposed solution of the above construction, setting $n = 2$, assuming target space to be a 2-group, regarded as a 2-groupoid with a single object, $\mathcal{P} = \Sigma(G_2)$, and taking tra to be a 2-rep of that 2-group.

This is not exactly what Kapranov and Ganter consider, but it's similar. On the other hand, the structures that I find in the space of sections in this setup are not exactly what Kapranov and Ganter consider – but they are similar.

2 Sections

For all of the following, we place ourselves in the world of **Gray**, the 3-category whose objects are strict 2-categories, whose morphisms are strict 2-functors, whose 2-morphisms are pseudonatural transformations and whose 3-morphisms are modifications.

Let some 2-category $\mathcal{P}_2$ be given, to be addressed as **target space**.

Let another 2-category, T, be given, that is equipped with a monoidal structure. T will play the role of the 2-category of 2-vector spaces.

A (suitably well behaved) 2-functor

$$\mathrm{tra} : \mathcal{P}_2 \to T\,,$$

with T some 2-category of 2-vector spaces represents for us a 2-vector bundle with connection on $\mathcal{P}$.

Let

$$1 : \mathcal{P}_2 \to T$$

be the tensor unit in $[\mathcal{P}, T]$, which sends everything to the identity on the tensor unit in T.

It makes sense to think of

$$[1, \mathrm{tra}]$$

as the **space of flat sections** of the 2-bundle.

Let par be some 1-category, regarded as a 2-category with only identity 2-morphisms, and to be thought of as the **parameter space** of a 2-particle.

The configuration space of our 2-particle should be the space of maps of the 2-particle into target space, modulo gauge transformations.

Definition 1 *Given a 1-category* par *and a 2-category* $\mathcal{P}_2$*, we say the* **configuration space** *of maps from* par *to* $\mathcal{P}_2$ *is the 2-functor 2-category*

$$\mathrm{conf} \subset [\mathrm{par}, \mathcal{P}_2]$$

whose objects are functors $c : \mathrm{par} \to \mathcal{P}_2$*, whose morphisms are those pseudonatural transformations*

$$c \xrightarrow{L} c'$$

between these functors that are given by identity 2-cells

$$L \;:\; \left(a \xrightarrow{s} b \right) \;\mapsto\; \left(\begin{tikzcd} c(a) \arrow[r, "c(s)"] \arrow[d, "L(a)"'] \arrow[dr, phantom, "\swArrow \mathrm{Id}"] & c(b) \arrow[d, "L(b)"] \\ c'(a) \arrow[r, "c'(s)"'] & c'(b) \end{tikzcd} \right)$$

and whose 2-morphisms are modifications between these.

Postcomposition with tra is a 2-functor

$$\mathrm{tra}_* : \mathrm{conf} \to [\mathrm{par}, T]\,,$$

which we regard as a 2-bundle with connection on configuration space.

This allows to consider the space of sections over configuration space

$$[1_*, \mathrm{tra}_*]\,.$$

Denote by

$$\mathrm{sect}$$

the sub-2-category

$$\mathrm{sect} \xrightarrow{\subset} [\mathrm{conf}, [\mathrm{par}, T]]$$

containing only the two objects 1_* and tra_* and all morphisms between these:

$$\mathrm{sect} = \left\{ \begin{tikzcd} 1_* \arrow[d, bend right=60] \arrow[d, bend left=10] \arrow[d, bend left=50] \arrow[d, phantom, "\cdots"] \\ \mathrm{tra}_* \end{tikzcd} \right\}.$$

This is the setup.

The problem now is to find from this a functor on par that would suitably rearrange the space of sections over configuration space into a space of sections over each point of parameter space, a morphism between these for each morphisms of parameter space, and so on.

Notice that the embedding

$$\mathrm{sect} \xrightarrow{\subset} [\mathrm{conf}, [\mathrm{par}, T]]$$

is an object in

$$[\mathrm{sect}, [\mathrm{conf}, [\mathrm{par}, T]]]$$

and that there is a canonical equivalence

$$[\mathrm{sect}, [\mathrm{conf}, [\mathrm{par}, T]]] \simeq [\mathrm{par}, [\mathrm{sect}, [\mathrm{conf}, T]]] \,.$$

This means that there is a *canonical* way to get a 2-functor on par given a space of sections sect, namely the image of sect under the above equivalence.

In the following I will try to construct this image in a certain simple case. I will do this by extracting all the data encoded in sect and showing how this data can be rearranged to yield a 2-functor

$$\mathrm{par} \to [\mathrm{sect}, [\mathrm{conf}, T]] \,.$$

As 2-functors into 2-functors into 2-vector spaces should form a 2-vector space themselves, this 2-functor would be my proposal for $\mathrm{QFT}(\mathrm{par}, T)$.

I shall show that this 2-functor, for the case that target space is a strict 2-group, is rather similar to what Kapranov and Ganter call a 2-character of a 2-representation of a 2-group.

While similar, it is different. But on the other hand, also my assumptions are different, since I am considering strict reps of strict 2-groups instead of lax reps of discrete 2-groups, as Kaparanov and Ganter do.

3 Sections of 2-representations

Let's specialize the above general setup to the case where target space is a strict 2-group G_2, coming from a crossed module $H \xrightarrow{t} G$ and regarded as a 2-category with a single object

$$\mathcal{P} = \Sigma(G_2) \,,$$

and where

$$T = \mathrm{Bim}$$

is the 2-category whose objects are algebras, whose morphisms are bimodules and whose 2-morphisms are bimodule homomorphisms. This has a canonical associator, and I will think of it is a strict 2-category.

A 2-vector bundle with connection on this target space is nothing but a 2-rep of G_2

$$\rho : \Sigma(G_2) \to \mathrm{Bim}\,.$$

By the canonical embedding

$$\mathrm{Bim} \subset {}_{\mathrm{Vect}}\mathrm{Mod}$$

I am thinking of Bim as a 2-category of 2-vector spaces. But this is not crucial for the main point to be made here. Other notions of 2-vector spaces could be used.

Let

$$\mathrm{par} \equiv \{a \to b\}$$

be a model for the parameter space of the open 2-particle, or let

$$\mathrm{par} \equiv \Sigma(\mathbb{Z})$$

be a model for the closed 2-particle.

If this 2-particle propagates on $\Sigma(G_2)$, the corresponding configuration space is the 2-functor category

$$\mathrm{conf} \subset [\mathrm{par}, \Sigma(G_2)]\,.$$

The representation ρ then pulls back to a 2-functor

$$\rho_* : \mathrm{conf} \to [\mathrm{par}, \mathrm{Bim}]\,.$$

Let

$$1 : \Sigma(G_2) \to \mathrm{Bim}$$

be the trivial rep, which sends everything to the identity on $\mathbb{C}$, regarded as a $\mathbb{C}$-algebra over itself.

We say a section of the 2-transport ρ_* on configuration space is a 2-morphism

$$\begin{tikzcd} \mathrm{conf} \arrow[r, bend left=50, "1_*"{name=U}] \arrow[r, bend right=50, "\rho_*"'{name=D}] \arrow[r, phantom, "\Downarrow e"] & {[\mathrm{par}, \mathrm{Bim}]} \end{tikzcd}\,.$$

the 2-vector space structure of sections over target space.

Proposition 1 *The category of endomorphisms of the trivial 2-rep of $\Sigma(G_2)$ is, as a monoidal category, equivalent to the category of ordinary reps of $G/\mathrm{Im}(t)$:*

$$\mathrm{End}(1) \simeq \mathrm{Rep}(G/\mathrm{Im}(t))\,.$$

Proof. An object in End(1) is, being a pseudonatural transformation, a functorial assignment

$$V \; : (\bullet \xrightarrow{\;g\;} \bullet) \;\mapsto\; \begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_\bullet"'] \arrow[dr, phantom, "\swarrow V(g)"] & \mathbb{C} \arrow[d, "V_\bullet"] \\ \mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C} \end{tikzcd} .$$

This means in particular that V is a $\mathbb{C}$-bimodule, hence an ordinary vector space.

Compatibility of this assignment with 2-morphisms then says exactly that

$$V(g) = V(t(h)g)$$

for all $h \in H$.

This means that $g \mapsto V(g)$ (op-)represents $G/\mathrm{Im}(t)$ on $V_\bullet$.

A morphism $V_1 \xrightarrow{k} V_2$ of two such transformations is a modification, hence a 2-morphism

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left=50, "V_{1\bullet}"{name=U}] \arrow[r, bend right=50, "V_{2\bullet}"'{name=D}] & \mathbb{C} \arrow[from=U, to=D, phantom, "\Downarrow k_\bullet"] \end{tikzcd}$$

such that

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_{2\bullet}"'] \arrow[dr, phantom, "\swarrow V_2(g)"] & \mathbb{C} \arrow[d, "V_{2\bullet}"'] \arrow[d, bend left=60, "V_{1\bullet}"] \arrow[d, phantom, shift left=3, "\Leftarrow k"] \\ \mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_{1\bullet}"] \arrow[d, bend right=60, "V_{2\bullet}"'] \arrow[d, phantom, shift right=3, "\Leftarrow k"] \arrow[dr, phantom, "\swarrow V_1(g)"] & \mathbb{C} \arrow[d, "V_{1\bullet}"] \\ \mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C} \end{tikzcd} .$$

This is nothing but a morphism of the corresponding representations.

Finally, composition in End(V) of two transformations V_1 and V_2 is the transformation given by the assignment

$$
V_2 \circ V_1 \; : (\bullet \xrightarrow{g} \bullet) \;\mapsto\;
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_{1\bullet}"'] \arrow[dr, phantom, "\swArrow V_1(g)"] & \mathbb{C} \arrow[d, "V_{1\bullet}"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_{2\bullet}"'] \arrow[dr, phantom, "\swArrow V_2(g)"] & \mathbb{C} \arrow[d, "V_{2\bullet}"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C}
\end{tikzcd}
.
$$

The composition

$$
\mathbb{C} \xrightarrow{V_{1\bullet}} \mathbb{C} \xrightarrow{V_{2\bullet}} \mathbb{C}
$$

in Bim is the bimodule tensor product over the algebra $\mathbb{C}$, hence nothing but the ordinary tensor product $V_{1\bullet} \otimes V_{2\bullet}$ of these vector spaces. So $V_2 \circ V_1$ is exactly the tensor product of two representations of V. $\square$

Proposition 2 *The space of sections* $[1, \rho]$ *over target space is a* $\mathrm{Rep}(G/\mathrm{Im}(t))$ *module category.*

Proof.

The space of sections

$$\mathrm{Hom}(1, \rho)$$

on $\Sigma(G_2)$ is, manifestly, acted on by the endomorphisms

$$\mathrm{End}(1)$$

of the trivial rep. According to prop. 1, these are equivalent to $\mathrm{Rep}(G/\mathrm{Im}(t))$. $\square$

Configuration space. It pays to briefly pause to spell out what configuration space looks like, precisely. We will first describe all of $[\mathrm{par}, \Sigma(G_2)]$ for $\mathrm{par} = \{a \to b\}$ the open 2-particle. After that we to $\mathrm{conf} \subset [\mathrm{par}, \Sigma(G_2)]$.

An object in $[\mathrm{par}, \Sigma(G_2)]$ is a morphism

$$\bullet \xrightarrow{g} \bullet$$

in $\Sigma(G_2)$. A morphism in $[\mathrm{par}, \Sigma(G_2)]$ is a 2-morphism

$$\begin{tikzcd}
\bullet \arrow[r, "g_1"] \arrow[d, "h_a"'] \arrow[dr, phantom, "H \swArrow"] & \bullet \arrow[d, "h_b"] \\
\bullet \arrow[r, "g_2"'] & \bullet
\end{tikzcd}$$

in $\Sigma(G_2)$. Finally, a 2-morphism in $[\mathrm{par}, \Sigma(G_2)]$ between two such morphisms is a pair of 2-morphism

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "h_a"] \arrow[r, bend right, "h'_a"'] \arrow[r, phantom, "\Downarrow a_a"] & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "h_b"] \arrow[r, bend right, "h'_b"'] \arrow[r, phantom, "\Downarrow a_b"] & \bullet
\end{tikzcd}$$

in $\Sigma(G_2)$ such that

$$\begin{tikzcd}
\bullet \arrow[r, "g_1"] \arrow[d, "h_a"] \arrow[d, bend right=60, "h'_a"'] \arrow[dr, phantom, "H \swArrow"] & \bullet \arrow[d, "h_b"] \\
\bullet \arrow[r, "g_2"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "g_1"] \arrow[d, "h'_a"'] \arrow[dr, phantom, "H' \swArrow"] & \bullet \arrow[d, "h'_b"'] \arrow[d, bend left=60, "h_b"] \\
\bullet \arrow[r, "g_2"'] & \bullet
\end{tikzcd} .$$

We should write this 2-morphism in $[\mathrm{par}, \Sigma(G_2)]$ as

$$\begin{tikzcd}
g_1 \arrow[r, bend left, "(h,H)"] \arrow[r, bend right, "(h',H')"'] \arrow[r, phantom, "\Downarrow a"] & g_2
\end{tikzcd} .$$

Notice that ρ_* sends this 2-morphism to a corresponding 2-morphism

$$
\begin{tikzcd}
\rho(g_1) \arrow[rr, bend left, "{(\rho(h),\rho(H))}"] \arrow[rr, bend right, "{(\rho(h'),\rho(H'))}"'] \arrow[rr, phantom, "\Downarrow\rho(a)"] & & \rho(g_2)
\end{tikzcd}
$$

in $[\mathrm{par}, \mathrm{Bim}]$. In particular, we get

$$
\begin{tikzcd}
\bullet \arrow[r, "\rho(g_1)"] \arrow[d, "\rho(h_a)"] \arrow[d, bend right=60, "\rho(h'_a)"'] \arrow[d, phantom, "\overset{\rho(a_a)}{\Leftarrow}"'] \arrow[dr, phantom, "\rho(H)\,\Swarrow"] & \bullet \arrow[d, "\rho(h_b)"] \\
\bullet \arrow[r, "\rho(g_2)"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "\rho(g_1)"] \arrow[d, "\rho(h'_a)"'] \arrow[dr, phantom, "\rho(H')\,\Swarrow"] & \bullet \arrow[d, "\rho(h'_b)"'] \arrow[d, bend left=60, "\rho(h_b)"] \arrow[d, phantom, "\overset{\rho(a_b)}{\Leftarrow}"] \\
\bullet \arrow[r, "\rho(g_2)"'] & \bullet
\end{tikzcd}
.
$$

The configuration space of the open 2-particle, $\mathrm{par} = \{a \to b\}$, is just like that, but with $H = \mathrm{Id}$ and $H' = \mathrm{Id}$ everywhere.

The configuration space of the closed 2-particle, $\mathrm{par} = \Sigma(\mathbb{Z})$, in turn, is obtained from that of the open 2-particle by in addition setting $h_a = h_b$ everywhere.

the 2-vector space structure of sections on configuration space. We can give a characterization of the space of sections over configuration space similar to that of the space of sections over target space. These will form a module category for representations of the loop 1-groupoid of G_2.

Simon Willerton has introduced the loop groupoid of any ordinary group:

Definition 2 *For any group G, the functor category*

$$\Lambda G \equiv [\Sigma(\mathbb{Z}), \Sigma(G)]$$

is called the **loop groupoid** *of G.*

In the language used here, we could say that the loop groupoid of G is the configuration space of the closed 2-particle propagating on $\Sigma(G)$, with $\Sigma(G)$ regarded as having only identity 2-morphisms.

Objects of the loop groupoid are morphisms

$$\bullet \xrightarrow{g} \bullet$$

in $\Sigma(G)$ and morphisms $g_1 \xrightarrow{h} g_2$ of the loop groupoid are commuting squares

$$\begin{tikzcd} \bullet \arrow[r, "g_1"] \arrow[d, "h"'] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "g_2"'] & \bullet \end{tikzcd}$$

in $\Sigma(G)$.

In this sense, the configuration space of the closed 2-particle on an arbitrary 2-group G_2 is like a **loop 2-groupoid**. By taking isomorphism classes of 1-morphisms in that 2-groupoid, we get a loop 1-groupoid for any strict 2-group G_2, generalizing the definition of the loop groupoid above:

Definition 3 *For G_2 any strict 2-group coming from a crossed module $t : H \to G$, define the* **loop groupoid** *ΛG_2 of G_2 to be the 1-groupoid obtained by starting with the configuration space*

$$\mathrm{conf} \subset [\Sigma(\mathbb{Z}), \Sigma(G_2)]$$

of the closed 2-particle on $\Sigma(G_2)$ and dividing out all 2-isomorphisms.

Proposition 3 *The loop groupoid of G_2 has objects elements of G. Morphisms $g_1 \xrightarrow{[r]} g_2$ are equivalence classes of commuting squares*

$$\begin{tikzcd} \bullet \arrow[r, "g_1"] \arrow[d, "r"'] \arrow[dr, phantom, "\swArrow\mathrm{Id}"] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r, "g_2"'] & \bullet \end{tikzcd},$$

with $[r] = [t(h)r]$ for all $h \in H$ that satisfy $h = \alpha(g_1)(h)$.

Now let configuration space be that of the closed 2-particle on $\Sigma(G_2)$ and consider the trivial 2-functor

$$1_* : \mathrm{conf} \to [\mathrm{par}, T]\,.$$

Proposition 4 *The category of endomorphisms of 1_* is, as a monoidal category, equivalent to the category of loops in the category of representations of the loop groupoid of G_2:*

$$\mathrm{End}(1_*) \simeq [\Sigma(\mathbb{Z}), \mathrm{Rep}(\Lambda G_2)]\,.$$

Proof. An object in $\mathrm{End}(1_*)$ is, being a pseudonatural transformation, a functorial assignment

$$
V \;:\; \left(g_1 \xrightarrow{h} g_2 \right) \;\mapsto\;
\begin{tikzcd}
\mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_{g_1}"'] \arrow[dr, phantom, "\swArrow V(h)"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "V_{g_2}"] \\
\mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"'] & \mathrm{Id}_{\mathbb{C}}
\end{tikzcd}\,,
$$

which is compatible with 2-morphisms. This compatibility here just says that V is invariant under shifts $h \mapsto t(k)h$ for all k with $k = \alpha(g_1)(k)$.

Therefore functoriality of V says that $V(h)$ is a representation of ΛG_2.

Moreover, the mere existence of the square on the right says that

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_{g_1\bullet}"'] \arrow[dr, phantom, "\swArrow V_{g_1}(\bullet\to\bullet)"] & \mathbb{C} \arrow[d, "V_{g_1\bullet}"'] \arrow[d, bend left=60, "V_{g_2\bullet}"] \arrow[d, phantom, "\overset{V(h)}{\Leftarrow}"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C}
\end{tikzcd}
\;=\;
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, bend right=60, "V_{g_1\bullet}"'] \arrow[d, "V_{g_2\bullet}"] \arrow[d, phantom, "\overset{V(h)}{\Leftarrow}"] \arrow[dr, phantom, "\swArrow V_{g_2}(\bullet\to\bullet)"] & \mathbb{C} \arrow[d, "V_{g_2\bullet}"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C}
\end{tikzcd}\,,
$$

which means that $g_1 \mapsto V_{g_1}(\bullet \to \bullet)$ is a natural automorphism of this representation of ΛG_2.

Next, a morphism in $\mathrm{End}(1_*)$ is a modification, hence an assignment

$$
g \;\mapsto\;
\begin{tikzcd}
\mathrm{Id}_{\mathbb{C}} \arrow[r, bend left=50, "V_g"] \arrow[r, bend right=50, "V'_g"'] \arrow[r, phantom, "\Downarrow k_g"] & \mathrm{Id}_{\mathbb{C}}
\end{tikzcd}\,.
$$

The tin can equation for this says that k is a natural isomorphism from the representation V to the representation V'. Moreover, the mere existence of k_g above says that this natural isomorphism is compatible with the natural automorphism $V(\bullet \to \bullet)$ and $V'(\bullet \to \bullet)$.

But this means nothing but that k encodes a morphism in

$$
[\Sigma(\mathbb{Z}), \mathbf{Rep}(\Lambda G_2)]\,.
$$

$\square$

3.1 the pointwise data of the space of sections on configuration space

Recall that our goal is to use the inclusion

$$\mathrm{sect} \subset [\mathrm{conf}, [\mathrm{par}, T]]$$

to construct a 2-functor

$$\mathrm{par} \to [\mathrm{sect}, [\mathrm{conf}, T]]\,.$$

As a preparation for that, we now write out explicitly all the assignments encoded in a section on configuration space and in a morphism of such sections. After that we will rearrange these assignments to construct the desired 2-functor.

the sections themselves. First, we spell out the data that is encoded in a section

$$\begin{tikzcd} \mathrm{conf} \arrow[rr, bend left, "1_*"] \arrow[rr, bend right, "\rho_*"'] \arrow[rr, phantom, "\Downarrow e"] & & {[\mathrm{par}, \mathrm{Bim}]} \end{tikzcd}$$

on configuration space. Again, we will write everything first in terms of $[\mathrm{par}, \Sigma(G_2)]$, for par the open 2-particle, not explicitly setting $H = \mathrm{Id}$, $H' = \mathrm{Id}$ and $h_a = h_b$.

Being a pseudonatural transformation, a section is a functorial assignment of 2-morphisms

$$\begin{tikzcd} g_1 \arrow[r, "{(h,H)}"] & g_2 & \mapsto & \mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "e(g_1)"'] \arrow[dr, phantom, "{e(h,H)}\,\swarrow"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "e(g_2)"] \\ & & & \rho(g_1) \arrow[r, "{\rho(h,H)}"'] & \rho(g_2) \end{tikzcd}$$

in $[\mathrm{par}, \mathrm{Bim}]$ to 1-morphisms in conf such that

$$\begin{tikzcd} \mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "e(g_1)"'] \arrow[dr, phantom, "{e(h',H')}\,\swarrow"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "e(g_2)"] \\ \rho(g_1) \arrow[r, "{\rho(h',H')}"'] & \rho(g_2) \end{tikzcd} = \begin{tikzcd} \mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "e(g_1)"'] \arrow[dr, phantom, "{e(h,H)}\,\swarrow"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "e(g_2)"] \\ \rho(g_1) \arrow[r, bend left, "{\rho(h,H)}"] \arrow[r, bend right, "{\rho(h',H')}"'] \arrow[r, phantom, "\rho(a)\Downarrow"] & \rho(g_2) \end{tikzcd}\,.$$

The left hand side of this equation, in turn, is an equation between 2-morphisms in Bim

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[d, "e_a(g_1)"'] \arrow[dr, phantom, "e(g_1)\swarrow"] & \mathbb{C} \arrow[d, "e_b(g_1)"] \arrow[dd, bend left=60, "e_b(g_2)"] \\
A \arrow[r, "\rho(g_1)"] \arrow[d, "\rho(h'_a)"'] \arrow[dr, phantom, "\rho(H')\swarrow"] & A \arrow[d, "\rho(h'_b)"] \arrow[r, phantom, "\overset{e(h',H')_b}{\Leftarrow}"] & {} \\
A \arrow[r, "\rho(g_2)"'] & A
\end{tikzcd}
=
\begin{tikzcd}
& \mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[dl, "e_a(g_1)"'] \arrow[dd, "e_1(g_2)"] \arrow[ddr, phantom, "e(g_2)\swarrow"] & \mathbb{C} \arrow[dd, "e_b(g_2)"] \\
A \arrow[dr, "\rho(h')"'] \arrow[r, phantom, "\overset{e(h',H')_a}{\Leftarrow}"] & {} & \\
& A \arrow[r, "\rho(g_2)"] & A
\end{tikzcd}
\tag{1}
$$

as is the right hand side

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[d, "e_a(g_1)"'] \arrow[dr, phantom, "e(g_1)\swarrow"] & \mathbb{C} \arrow[d, "e_b(g_1)"] \arrow[dd, bend left=60, "e_b(g_2)"] \\
A \arrow[r, "\rho(g_1)"] \arrow[d, "\rho(h'_a)"'] \arrow[dr, phantom, "\rho(H')\swarrow"] & A \arrow[d, bend right, "\rho(h'_b)"'] \arrow[d, bend left, "\rho(h_b)"] \arrow[d, phantom, "\overset{\rho(a_b)}{\Leftarrow}"] \arrow[r, phantom, "\overset{e(h,H)_b}{\Leftarrow}"] & {} \\
A \arrow[r, "\rho(g_2)"'] & A
\end{tikzcd}
=
\begin{tikzcd}
& \mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[dl, "e_a(g_1)"'] \arrow[dd, "e_1(g_2)"] \arrow[ddr, phantom, "e(g_2)\swarrow"] & \mathbb{C} \arrow[dd, "e_b(g_2)"] \\
A \arrow[dr, "\rho(h_a)"] \arrow[dr, bend right, "\rho(h'_a)"'] \arrow[dr, phantom, "\overset{\rho(a_a)}{\swarrow}"'] \arrow[r, phantom, "\overset{e(h,H)_a}{\Leftarrow}"] & {} & \\
& A \arrow[r, "\rho(g_2)"] & A
\end{tikzcd}
\tag{2}
$$

and the condition is that these coincide:

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_b(g_1)"'] \arrow[dr, phantom, "e(h',H')_b\swarrow"] & \mathbb{C} \arrow[d, "e_b(g_2)"] \\
A \arrow[r, "\rho(h'_b)"'] & A
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_b(g_1)"'] \arrow[dr, phantom, "e(h,H)_b\swarrow"] & \mathbb{C} \arrow[d, "e_b(g_2)"] \\
A \arrow[r, bend left, "\rho(h_b)"] \arrow[r, bend right, "\rho(h'_b)"'] \arrow[r, phantom, "\rho(a_b)\Downarrow"] & A
\end{tikzcd}
\; . \tag{3}
$$

Functoriality of the section e means that

$$
\begin{tikzcd}
\mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "e(g_1)"'] \arrow[dr, phantom, "e(h_1{,}H_1)\ \Swarrow"] & \mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "e(g_2)"'] \arrow[dr, phantom, "e(h_2{,}H_2)\ \Swarrow"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "e(g_3)"] \\
\rho(g_1) \arrow[r, "\rho(h{,}H)"'] & \rho(g_2) \arrow[r, "\rho(h'{,}H')"'] & \rho(g_3)
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{Id}_{\mathbb{C}} \arrow[rr, "\mathrm{Id}"] \arrow[d, "e(g_1)"'] \arrow[drr, phantom, "e((h_2{,}H_2)\circ(h_1{,}H_1))\ \Swarrow"] & & \mathrm{Id}_{\mathbb{C}} \arrow[d, "e(g_2)"] \\
\rho(g) \arrow[rr, "\rho((h_2{,}H_2)\circ(h_1{,}H_1))"'] & & \rho(g_2)
\end{tikzcd}
$$

which is equivalent to the equation

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(g_1)"'] \arrow[dr, phantom, "e(h_1{,}H_1)_a\ \Swarrow"] & \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(g_2)"'] \arrow[dr, phantom, "e(h_2{,}H_2)_a\ \Swarrow"] & \mathbb{C} \arrow[d, "e_a(g_3)"] \\
A \arrow[r, "\rho(h_{1a})"'] & A \arrow[r, "\rho(h_{2a})"'] & A
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C} \arrow[rr, "\mathrm{Id}"] \arrow[d, "e(g_1)"'] \arrow[drr, phantom, "e((h_2{,}H_2)\circ(h_1{,}H_1))\ \Swarrow"] & & \mathbb{C} \arrow[d, "e(g_2)"] \\
A \arrow[rr, "\rho((h_2\circ h_1)_a)"'] & & A
\end{tikzcd}
\tag{4}
$$

together with the analogous one with the subscript a replaced by b. In particular, this implies that

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(g_1)"'] \arrow[dr, phantom, "e(h_1{,}H_1)_a\ \Swarrow"] & \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(g_2)"'] \arrow[dr, phantom, "e(h_2{,}H_2)_a\ \Swarrow"] & \mathbb{C} \arrow[d, "e_a(g_3)"] \\
A \arrow[r, bend left, "\rho(h_{1a})"] \arrow[r, bend right, "\rho(h'_{1a})"'] \arrow[r, phantom, "\Downarrow a_{1a}"] & A \arrow[r, bend left, "\rho(h_{2a})"] \arrow[r, bend right, "\rho(h'_{2a})"'] \arrow[r, phantom, "\Downarrow a_{2a}"] & A
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C} \arrow[rr, "\mathrm{Id}"] \arrow[d, "e(g_1)"'] \arrow[drr, phantom, "e((h_2{,}H_2)\circ(h_1{,}H_1))\ \Swarrow"] & & \mathbb{C} \arrow[d, "e(g_2)"] \\
A \arrow[r, bend left, "\rho(h_{1a})"] \arrow[r, bend right, "\rho(h'_{1a})"'] \arrow[r, phantom, "\Downarrow a_{1a}"] & A \arrow[r, bend left, "\rho(h_{2a})"] \arrow[r, bend right, "\rho(h'_{2a})"'] \arrow[r, phantom, "\Downarrow a_{2a}"] & A
\end{tikzcd}
.
$$

morphisms of sections. A morphism between sections on configuration space

$$
\begin{tikzcd}
1_* \arrow[rr, bend left=70, "\mathrm{conf}"] \arrow[rr, bend left=40, "e_1"] \arrow[rr, dashed, "e_2"'] \arrow[rr, phantom, bend left=55, "="] \arrow[rr, phantom, bend left=20, "\Downarrow V"] \arrow[dr] & & \rho_* \arrow[dl] \\
& {[\mathrm{par}, T]} &
\end{tikzcd}
$$

is an assignment

$$g \quad \mapsto \quad \begin{tikzcd} \mathrm{Id}_{\mathbb{C}} \arrow[d, bend right=50, "e_2(g)"'] \arrow[d, bend left=50, "e_1(g)"] \arrow[d, phantom, "\overset{V(g)}{\Leftarrow}"] \\ \rho(g) \end{tikzcd}$$

such that

$$\begin{tikzcd} \mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_2(g_1)"'] \arrow[dr, phantom, "e_2(h,H)\ \Swarrow"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "e_2(g_2)"'] \arrow[d, bend left=60, "e_1(g_2)"] \arrow[d, phantom, shift left=2, "\overset{V(g_2)}{\Leftarrow}"] \\ \rho(g_1) \arrow[r, "\rho(h,H)"'] & \rho(g_2) \end{tikzcd} \quad = \quad \begin{tikzcd} \mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, bend right=60, "e_2(g_1)"'] \arrow[d, "e_1(g_1)"] \arrow[d, phantom, shift right=2, "\overset{V(g_1)}{\Leftarrow}"] \arrow[dr, phantom, "e_1(h,H)\ \Swarrow"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "e_1(g_2)"] \\ \rho(g_1) \arrow[r, "\rho(h,H)"'] & \rho(g_2) \end{tikzcd} .$$

The mere existence of $V(g)$ says that

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[d, "{e_2}_a(g)"'] \arrow[dr, phantom, "e_2(g)\ \Swarrow"] & \mathbb{C} \arrow[d, "{e_2}_b(g)"'] \arrow[d, bend left=60, "{e_1}_b(g)"] \arrow[d, phantom, shift left=2, "\overset{V(g)_b}{\Leftarrow}"] \\ A \arrow[r, "\rho(g)"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[d, bend right=60, "{e_2}_a(g)"'] \arrow[d, "{e_1}_a(g)"] \arrow[d, phantom, shift right=2, "\overset{V(g)_a}{\Leftarrow}"] \arrow[dr, phantom, "e_1(g)\ \Swarrow"] & \mathbb{C} \arrow[d, "{e_1}_b(g)"] \\ A \arrow[r, "\rho(g)"'] & A \end{tikzcd} . \tag{5}$$

The compatibility condition amounts to

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[d, "{e_2}_a(g_1)"'] \arrow[dr, phantom, "e_2(h,H)_a\ \Swarrow"] & \mathbb{C} \arrow[d, "{e_2}_a(g_2)"'] \arrow[d, bend left=60, "{e_1}_a(g_2)"] \arrow[d, phantom, shift left=2, "\overset{V(g_2)_a}{\Leftarrow}"] \\ A \arrow[r, "\rho(h)"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{C} \arrow[r, "\mathbb{C}"] \arrow[d, bend right=60, "{e_2}_a(g_2)"'] \arrow[d, "{e_1}_a(g_1)"] \arrow[d, phantom, shift right=2, "\overset{V(g_1)_a}{\Leftarrow}"] \arrow[dr, phantom, "e_1(h,H)_a\ \Swarrow"] & \mathbb{C} \arrow[d, "{e_1}_a(g_2)"] \\ A \arrow[r, "\rho(h)"'] & A \end{tikzcd} \tag{6}$$

and similarly for b.

multiples of sections. Consider multiplying a section e by an element V in $\mathrm{End}(1_*)$

$$e \mapsto Ve\,.$$

Then Ve comes from the assignment

$$
g_1 \xrightarrow{(h,H)} g_2 \quad \mapsto \quad
\begin{tikzcd}
\mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "V(g_1)"'] \arrow[dr, phantom, "V(h,H) \swArrow"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "V(g_2)"] \\
\mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "e(g_1)"'] \arrow[dr, phantom, "e(h,H) \swArrow"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "e(g_2)"] \\
\rho(g_1) \arrow[r, "\rho(h,H)"'] & \rho(g_2)
\end{tikzcd} .
$$

Write the 2-morphisms that $V(g)$ comes from as

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_a(g)"'] \arrow[dr, phantom, "\swArrow V(g)"] & \mathbb{C} \arrow[d, "V_b(g)"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C}
\end{tikzcd} ,
$$

as before.

Then we see that under $e \mapsto Ve$ we get

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(g_1)"'] \arrow[dr, phantom, "e(h,H)_a \swArrow"] & \mathbb{C} \arrow[d, "e_a(g_2)"] \\
A \arrow[r, "\rho(h_a)"'] & A
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_a(g_1)"'] \arrow[dr, phantom, "V(h,H)_a \swArrow"] & \mathbb{C} \arrow[d, "V_a(g_2)"] \\
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(g_1)"'] \arrow[dr, phantom, "e(h,H)_a \swArrow"] & \mathbb{C} \arrow[d, "e_a(g_2)"] \\
A \arrow[r, "\rho(h_a)"'] & A
\end{tikzcd}
$$

3.2 The map from parameter space to sections

Above we have characterized the data encoded in

$$\mathrm{sect} \subset [\mathrm{conf}, [\mathrm{par}, T]]\,.$$

We now want to rearrange this data such as to obtain a functor

$$\mathrm{par} \to [\mathrm{sect}, [\mathrm{conf}, T]]\,.$$

This is discussed first for the open, then for the closed 2-particle.

spaces of sections over the open 2-particle. The functor that we are after is supposed to assign an image of the category of sections to a, another such image to b and a morphism between these images to the $a \to b$.

To obtain that we will, roughly, re-read the above equations after rotating all diagrams by $\pi/2$.

First of all, equation (3), for $g_1 = g_2$ and $H = \mathrm{Id}$, $H' = \mathrm{Id}$ is the naturality condition for a section over target space. Moreover, equation (4) then is the compatibility of that section with composition.

Restricting to this case means restricting the section e to 2-morphisms in configuration space of the form

$$\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[ddd, bend right=40, "h'_a"'] \arrow[ddd, bend left=40, "h_a"] \arrow[ddd, phantom, "\Leftarrow a_a"] & \bullet \arrow[d, "g^{-1}"] & & \bullet \arrow[r, "g"] \arrow[ddd, "h'_a"'] & \bullet \arrow[d, "g^{-1}"] \\
& \bullet \arrow[d, "h_a"] \arrow[l, phantom, "\swarrow \mathrm{Id}"] & = & \arrow[r, phantom, "\swarrow \mathrm{Id}"] & \bullet \arrow[d, bend right=40, "h'_a"'] \arrow[d, bend left=40, "h_a"] \arrow[d, phantom, "\Leftarrow a_a"] \\
& \bullet \arrow[d, "g"] & & & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "g"'] & \bullet & & \bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}\,.$$

More precisely, for each section e on configuration space and for each morphism $g \in \mathrm{Mor}(\Sigma(G_2))$ we get a section

$$e_{a,g} : \begin{array}{c} 1 \\ \downarrow \\ \rho \end{array}$$

on target space, defined by

$$
\left(\bullet \xrightarrow{h} \bullet \right) \;\mapsto\;
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(g)"'] \arrow[dr, phantom, "\swArrow e(h,\mathrm{Id})_a"] & \mathbb{C} \arrow[d, "e_a(g)"] \\
A \arrow[r, "\rho(h)"'] & A
\end{tikzcd} \quad .
$$

Using equation (6) for the special case $g_1 = g_2$, we get a morphism

$$
\begin{tikzcd}
1 \arrow[d, bend right=60, "e_{2a,g}"'] \arrow[d, bend left=60, "e_{1a,g}"] \arrow[d, phantom, "\overset{V_a(g)}{\Leftarrow}"] \\
\rho
\end{tikzcd}
$$

between these sections over target space from every morphism

$$
\begin{tikzcd}
1_* \arrow[d, bend right=60, "e_2"'] \arrow[d, bend left=60, "e_1"] \arrow[d, phantom, "\overset{V}{\Leftarrow}"] \\
\rho_*
\end{tikzcd}
$$

between the original sections on configuration space.

So consider now the 2-category over a consisting of all sections of type $e_{a,g}$:

$$
\mathrm{sect}_a = \left\{
\begin{tikzcd}
1 \arrow[d, bend right=60, "e'_{a,g'}"'] \arrow[d, bend left=10] \arrow[d, bend left=60, "e_{a,g}"] \arrow[d, phantom, "\cdots"] \\
\rho
\end{tikzcd}
\right\} \tag{7}
$$

with all 2-morphisms as above.

Each $g \in \mathrm{Mor}_1(\Sigma(G_2))$ determines a morphism

$$
\mathrm{sect} \xrightarrow{s_a} \mathrm{sect}_a \subset [\mathcal{P}, T] \ .
$$

by

$$\left(\begin{tikzcd} 1_* \arrow[d, bend right=60, "e_2"'] \arrow[d, bend left=60, "e_1"] \arrow[d, phantom, "\overset{V}{\Leftarrow}"] \\ \rho_* \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} 1 \arrow[d, bend right=60, "{e_2}_{a,g}"'] \arrow[d, bend left=60, "{e_1}_{a,g}"] \arrow[d, phantom, "\overset{V_a(g)}{\Leftarrow}"] \\ \rho \end{tikzcd}\right).$$

That this map functorially respects morphisms of sections follows from the considerations leading to equation (6).

Now we similarly construct a category of sections for the endpoint b. For the moment this will look slightly different than the above construction of sect_a, which makes the discussion of morphisms more convenient. At the end we should merge both constructions and get $\mathrm{sect}_a \simeq \mathrm{sect}_b$.

Notice that we may define a new representation

$$\rho_g = \rho \circ \mathrm{Ad}_g$$

for each $g \in \mathrm{Mor}_1(\Sigma(G_2))$. We have an isomorphism

$$\rho \xrightarrow{\sim} \rho_g$$

given by the assignment

$$\left(\bullet \xrightarrow{h} \bullet \right) \mapsto \begin{tikzcd} A \arrow[r, "\rho(h)"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\swarrow \mathrm{Id}"] & A \arrow[d, "\rho(g)"] \\ A \arrow[r, "\rho(g^{-1}hg)"'] & A \end{tikzcd}.$$

Now for each section e on configuration space and each $g \in \mathrm{Mor}_1(\Sigma(G_2))$, define a section

$$e_{b,g} : \begin{tikzcd} 1 \arrow[d] \\ \rho_g \end{tikzcd}$$

of ρ_g on target space by

$$\left(\bullet \xrightarrow{h} \bullet \right) \mapsto \begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_b(g)"'] \arrow[dr, phantom, "\swarrow e(g^{-1}hg, \mathrm{Id})_b"] & \mathbb{C} \arrow[d, "e_b(g)"] \\ A \arrow[r, "\rho(g^{-1}hg)"'] & A \end{tikzcd}.$$

Let for the moment a 2-category sect_b be defined that contains all sections of this type and all isomorphisms $\rho_{g_1} \xrightarrow{\sim} \rho_{g_2}$, with 2-morphisms analogous to those in sect_a:

$$\mathrm{sect}_b \;\equiv\; \left\{ \begin{tikzcd} & & 1 \arrow[dll, "e'_{b,g'}"'] \arrow[d, "e_{b,g}"] \arrow[dr, "e_{b,\mathrm{Id}}"] & \\ \rho_{g'} & \cdots & \rho_g & \rho \arrow[l, "\sim"'] \end{tikzcd} \right\} .$$

Trivially, we again have a (non-canonical) morphism from the sections over configuration space to the sections over target space associated to the endpoint b

$$\mathrm{sect} \xrightarrow{s_b} \mathrm{sect}_b \subset [\mathcal{P}, T]$$

obtained by fixing any $g \in \mathrm{Mor}_1(\Sigma(G_2))$ and sending

$$\left(\begin{tikzcd} 1_* \arrow[d, "e"] \\ \rho_* \end{tikzcd} \right) \;\mapsto\; \left(\begin{tikzcd} 1 \arrow[d, "e_{a,g}"] \\ \rho_g \end{tikzcd} \right) .$$

The point of setting up sect_b this way for the moment is that it makes it easiest to see how from our given data we obtain a morphism from sect_a to sect_b, or rather a natural isomorphism

$$\begin{tikzcd} \mathrm{sect}_{\mathrm{dsc}} \arrow[rr, bend left, "s_a"] \arrow[rr, bend right, "s_b"'] \arrow[rr, phantom, "\Downarrow"] & & {[\mathcal{P}, T]} \end{tikzcd} .$$

To do that, we use more of the data encoded in equation (1). I claim that after a couple of trivial manipulations, making use of the fact that all 2-morphisms involved are invertible, this equation can equivalently be rewritten like this:

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "e_a(g_1)"'] \arrow[dr, phantom, "\swarrow_{e(h',H')_a}"] & \mathbb{C} \arrow[d, "e_a(g_2)"] \arrow[dd, bend left=60, "e_b(g_2)"] \\
A \arrow[r, "\rho(h'_a)"] \arrow[d, "\rho(g_1)"'] \arrow[dr, phantom, "\swarrow_{\rho(H')^{-1}}"] & A \arrow[d, "\rho(g_2)"] \arrow[r, phantom, "\Leftarrow_{e(g_2)}"] & {} \\
A \arrow[r, "\rho(h'_b)"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[dl, "e_a(g_1)"'] \arrow[dd, "e_b(g_1)"] \arrow[ddr, phantom, "\swarrow_{e(h',H')_b}"] & \mathbb{C} \arrow[dd, "e_b(g_2)"] \\
A \arrow[dr, "\rho(g_1)"'] \arrow[r, phantom, "\Leftarrow_{e(g_1)}"] & {} & \\
& A \arrow[r, "\rho(h'_b)"'] & A
\end{tikzcd}
\quad . \qquad (8)
$$

It might be helpful to think of this as, roughly, obtained by rotating the diagrams in (1) by $\pi/2$, with the role played by $e(h,H)_b$ and that played by $e(g)_b$ interchanged.

But for $g_1 = g_2 = g$ and $H' = \mathrm{Id}$ the equation in this form manifestly defines a 2-morphism

$$
\begin{tikzcd}
1 \arrow[r, "\mathrm{Id}"] \arrow[d, "e_{a,g}"'] \arrow[dr, phantom, "\swarrow_{e(g)}"] & 1 \arrow[d, "e_{b,g}"] \\
\rho \arrow[r, "\sim"] & \rho_g
\end{tikzcd}
\quad .
$$

But the assignment

$$
\left(1_* \xrightarrow{e} \rho_* \right) \quad \mapsto \quad
\left(
\begin{tikzcd}
1 \arrow[r, "e_{a,g}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\swarrow_{e(g)^{-1}}"] & \rho \arrow[d, "\sim"] \\
1 \arrow[r, "e_{b,g}"'] & \rho_g
\end{tikzcd}
\right) ,
$$

in turn, defines a pseudonatural transformation,

$$
\begin{tikzcd}
\mathrm{sect} \arrow[r, bend left, "s_a"] \arrow[r, bend right, "s_b"'] \arrow[r, phantom, "\Downarrow"] & {[\mathcal{P}, T]}
\end{tikzcd}
\quad . \qquad (9)
$$

Compatibility with 2-morphisms

$$
\begin{tikzcd}
1 \arrow[r, "e_{2\,a,g}"] \arrow[r, bend left=60, "e_{1\,a,g}"] \arrow[r, phantom, bend left=30, "\Downarrow V(g)_a"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\swArrow e_2(g)^{-1}"] & \rho \arrow[d, "\sim"] \\
1 \arrow[r, "e_{2\,b,g}"'] & \rho_g
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
1 \arrow[r, "e_{1\,a,g}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\swArrow e(g)^{-1}"] & \rho \arrow[d, "\sim"] \\
1 \arrow[r, "e_{1\,b,g}"] \arrow[r, bend right=60, "e_{2\,b,g}"'] \arrow[r, phantom, bend right=30, "\Downarrow V(g)_b"] & \rho_g
\end{tikzcd}
$$

is a direct consequence of (5).

In conclusion, this does establish a construction of a morphism

$$q : \mathrm{par} \to [\mathrm{sect}, [\mathcal{P}, T]]$$

from the data of the embedding morphism

$$\mathrm{sect} \stackrel{\subset}{\to} [\mathrm{conf}, [\mathrm{par}, T]] \,.$$

This q sends the endpoints a and b of parameter space to categories of sections sect_a and sect_b naturally associated to these endpoints, and it sends the single nontrivial morphism

$$a \to b$$

of parameter space to a map between these categories (really a natural transformation between these categories regarded as images of s_a and s_b) that amounts to a bijection of the isomorphism classes of sect_a and sect_b.

We can now easily enlarge both sect_a and sect_b such that they become equal and we still have a morphism between them. To do that, we simply pre-compose everything above by an adjoint action of G_2 on itself. So with each section

$$
\begin{tikzcd}
\Sigma(G_2) \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "\rho"'] \arrow[r, phantom, "\Downarrow e_{a,g}"] & \mathrm{Bim}
\end{tikzcd}
$$

we now also inlude

$$
\begin{tikzcd}
\Sigma(G_2) \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "\rho_q"'] \arrow[r, phantom, "\Downarrow e^q_{a,g}"] & \mathrm{Bim}
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
\Sigma(G_2) \arrow[r, "\mathrm{Ad}_q"] & \Sigma(G_2) \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "\rho"'] \arrow[r, phantom, "\Downarrow e_{a,g}"] & \mathrm{Bim}
\end{tikzcd}
$$

in sect_a.

Similarly, to each

$$\begin{tikzcd} \Sigma(G_2) \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "\rho"'] \arrow[r, phantom, "\Downarrow e_{b,g}"] & \mathrm{Bim} \end{tikzcd}$$

we now also include

$$\begin{tikzcd} \Sigma(G_2) \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "\rho_{qg}"'] \arrow[r, phantom, "\Downarrow e^q_{b,g}"] & \mathrm{Bim} \end{tikzcd} \equiv \begin{tikzcd} \Sigma(G_2) \arrow[r, "\mathrm{Ad}_q"] & \Sigma(G_2) \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "\rho_g"'] \arrow[r, phantom, "\Downarrow e_{b,g}"] & \mathrm{Bim} \end{tikzcd}$$

in sect_b. This way we get $\mathrm{sect}_a = \mathrm{sect}_b$. The morphisms between s_a and s_b constructed previously extend in the obvious way to these enlarged categories of sections.

a similarity with Kapranov-Ganter 2-characters While the details are different, the morphism q, interpreted this way, is similar to the 2-character of a 2-representation as defined by Kapranov and Ganter.

To see this, restrict attention to those sections e on configuration space such that $e(g)_a = \mathrm{Id}$ and $e(g)_b = \mathrm{Id}$. The image of those, under $q(a)$, in sect_a are a collection of 2-morphisms

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "\rho(h)"'] \arrow[r, phantom, "\Downarrow e(h,\mathrm{Id})_b"] & \mathbb{C} \end{tikzcd},$$

one for each $h \in \mathrm{Mor}_1(\Sigma(G_2))$ (and compatible with the composition in $\Sigma(G_2)$).

Under the map induced by $q(a \to b)$ from sect_a to sect_b these are mapped, according to (8), to sections over b which are given on each h by the 2-morphism

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "\rho(g^{-1}hg)"'] \arrow[r, phantom, "\Downarrow e(g^{-1}hg,\mathrm{Id})_b"] & \mathbb{C} \end{tikzcd} = \begin{tikzcd} \mathbb{C} \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "\rho(g^{-1})"'] \arrow[r, phantom, "\Downarrow e(g)^{-1}"] & \mathbb{C} \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "\rho(h)"'] \arrow[r, phantom, "\Downarrow e(h,\mathrm{Id})_a"] & \mathbb{C} \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "\rho(g)"'] \arrow[r, phantom, "\Downarrow e(g)"] & \mathbb{C} \end{tikzcd}.$$

In other words, in that special case an element in the space of sections associated to the point a of parameter space is a collection over $g \in \mathrm{Obj}(G_2)$ of 2-morphisms from the identity on the representation space to the given $\rho(g)$, and under the map $q(a \to b)$ these 2-morphisms are conjugated by reps of given 2-group elements.

spaces of sections over the closed 2-particle. As a category internal to **Set**, every groupoid is equivalent to the disjoint union of the vertex groups of its connected components. (For Lie groupoids, though, there may not be a *smooth* equivalence).

The connected components of the loop groupoid of G correspond to the conjugacy classes of G. The corresponding vertex group is the commutant of any one representative.

This motivates us to restrict attention to the following situation:

Let G be the group of objects of the strict 2-group G_2. For each conjugacy class $[g]$ of G choose one representative g and consider the disjoint union

$$\mathrm{conf}_{\mathrm{sk}} \equiv \bigcup_{[g]\in G/G} \mathrm{conf}_g$$

of vertex 2-groups

$$\mathrm{conf}_g = \left\{ \begin{tikzcd} g \arrow[rr, bend left, "h"] \arrow[rr, bend right, "h'"'] \arrow[rr, phantom, "\Downarrow a"] & & g \end{tikzcd} \right\} .$$

Notice that the 2-morphisms here are those 2-morphisms

$$\begin{tikzcd} \bullet \arrow[rr, bend left, "h"] \arrow[rr, bend right, "h'"'] \arrow[rr, phantom, "\Downarrow a"] & & \bullet \end{tikzcd}$$

that commute with g, in the sense that

$$\begin{tikzcd} \bullet \arrow[rr, bend left, "h"] \arrow[rr, bend right, "h'"'] \arrow[rr, phantom, "\Downarrow a"] & & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[rr, bend left, "h"] \arrow[rr, bend right, "h'"'] \arrow[rr, phantom, "\Downarrow a"] & & \bullet \arrow[r, "g^{-1}"] & \bullet \end{tikzcd}$$

As 2-groupoids in **Set**, we have

$$\mathrm{conf}_{\mathrm{sk}} \simeq \mathrm{conf}\,.$$

Therefore we now restrict attention to sections over $\mathrm{conf}_{\mathrm{sk}}$. For these, the discussion is very much analogous to that given above, for configuration space of the open particle, but everything simplifies considerably.

In particular, due to the above commutativity of 2-morphisms with 1-morphisms, we now have $\rho_g = \rho$ on $\mathrm{conf}_{\mathrm{sk}}$.

This means that the space of sections

$$\mathrm{sect}_\bullet$$

over the single point $\bullet$ of the configuration space $\mathrm{par} = \Sigma(\mathbb{Z})$ of the closed 2-particle is precisely of the form (7). Moreover, the morphism (9) now becomes an automorphism.

I think I can hence prove the following main result.

Proposition 5 *The category* $\mathrm{sect}_\bullet$ *is a* $\mathrm{Rep}(\Lambda G)$ *module category. Its automorphism described above does respect this module structure.*

sketch of a proof:

By prop. 4 the sections over configuration space are a $\Lambda\mathrm{Rep}(\Lambda G_2)$-module. The subcategory of that which preserves the structure of equation (8) defining the automorphism of $\mathrm{sect}_\bullet$ is precisely the subcategory whose objects are trivial loops in $\mathrm{Rep}(\Lambda G_2)$. This is equivalent to $\mathrm{Rep}(\Lambda G_2)$ itself. $\square$

As a corollary we get

Proposition 6 *The funcor on parameter space which we have constructed from the space of sections on configuration space takes values in* $\mathrm{Rep}(\Lambda G_2)$*-modules:*

$$\mathrm{par} \to {}_{\mathrm{Rep}(\Lambda G_2)}\mathrm{Mod}\,.$$