2-Monoid of Observables on String_G

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Abstract

Given any 2-groupoid, we can associate to it a monoidal category which can be thought of as the 2-monoid of observables of the 2-particle propagating on that 2-groupoid.

Here we show that for the 2-groupoid $\Sigma(\mathrm{String}_G)$ this monoidal category is the category

$$\Lambda\mathrm{Rep}(\Lambda\mathrm{String}_G)$$

of loops in representations of the loop groupoid of String_G. We argue that representation of $\Lambda\mathrm{String}_G$ are twisted equivariant bundles on G.

Introduction. For various reasons, I find the following general concept useful, which here I want to apply to a special case related to loop groups and representations.

Let par be a 1-category, called the **parameter space**.

Let $\mathcal{P}$ be a smooth 2-category, called the **target space**.

Let $\mathrm{tra} : \mathcal{P} \to \mathrm{Bim}(\mathbf{Vect}_{\mathbb{C}})$ be a smooth 2-functor to the 2-category whose morphisms are bimodules, called a **2-vector bundle** with connection on target space.

Let $1 : \mathcal{P} \to \mathrm{Bim}(\mathbf{Vect}_{\mathbb{C}})$ be the tensor unit in the monoidal 2-category of 2-vector bundles with connection, i.e. the 2-functor that sends everything to the identity on $\mathbb{C}$.

Let $\mathrm{conf} \equiv [\mathrm{par}, \mathcal{P}]$ be the 2-category of 2-functors from parameter space to target space, called the **configuration space**.

Let $\mathrm{tra}_* : \mathrm{conf} \to [\mathrm{par}, \mathrm{Bim}]$ be the 2-functor on configuration space obtained by postcomposing with tra. This can roughly be thought of as the **transgressed 2-vector bundle**.

Let $\mathrm{sect} = [1_*, \mathrm{tra}_*]$ be the category of morphisms from the trivial 2-vector bundle on configuration space to the transgressed 2-vector bundle. This I call the **space of sections** on configuration space.

Let $\mathcal{C} = \mathrm{End}(1_*)$ be the monoidal category of endomorphisms of 1_*, called the **monoid of observables**.

Clearly, sect is a module category for $\mathcal{C}$.

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Here I would like to understand the category $\mathcal{C}$ for the following setup.

Let G be a simple, simply connected and compact Lie group, and let $k \in H^3(G, \mathbb{Z})$ be a level. From the centrally extended loop group, $\hat{\Omega}_k G$, we can form the groupoid $\mathrm{String}_G \equiv \; PG \ltimes \hat{\Omega}_k G \rightrightarrows PG$ over based paths in G.

This groupoid can be regarded from two points of view. As a centrally extended groupoid, it is the canonical bundle gerbe with class k over G. The groupoid has a strict monoidal structure, with strict monoidal inverses. Therefore it can also be regarded as a strict 2-group.

Being monoidal, we can form the suspension $\Sigma(\mathrm{String}_G)$, which is a 2-category with a single object.

We want to regard this as our target space, in the above sense, and study the monoid of observables on the configuration space of 2-particles propagating on this target space.

Notice that in as far as $BG \simeq |\Sigma G|$, we can think of String_G as a twisted version of BG.

This means we set $\mathcal{P} = \Sigma(\mathrm{String}_G)$ and $\mathrm{par} = \Sigma(\mathbb{Z})$.

In this case, I seem to find the following result:

Proposition 1 *The groupoid*

$$\Lambda\mathrm{String}_G \equiv [\Sigma(\mathbb{Z}), \Sigma(\mathrm{String}_G)]_{/\sim}$$

obtained by identifying isomorphic 1-morphisms in configuration space is a central extension of the the loop groupoid

$$\Lambda G \equiv [\Sigma(\mathbb{Z}), \Sigma(G)]$$

of G.

Proposition 2 *The monoidal category $\mathcal{C}$ is*

$$\mathcal{C} = [\Sigma(\mathbb{Z}), \mathrm{Rep}(\Lambda\mathrm{String}_G)]\,.$$

Proposition 3 *The category* $\mathrm{Rep}(\mathrm{String}_G)$ *is the category of equivariant gerbe modules on G.*

Note that for G finite, Simon Willerton argued that ΛG, which is nothing but the action groupoid of the adjoint action of G on itself, plays the role of the loop group of G, by noticing that

$$B\Lambda G \simeq LBG\,.$$

Given a group 3-cocycle κ on G, hence a groupoid 2-cocycle on ΛG, one can therefore address the twisted representations $\mathrm{Rep}_\kappa(\Lambda G)$ both as twisted representations of the loop group of G - in the above sense - as well as twisted equivariant vector bundles on G.

What I describe here looks like a Lie group analog of this perspective on the Freed-Hopkins-Teleman theorem.

Definition 1 *For G_2 any strict 2-group, the* **loop groupoid** *of G_2 is the 1-groupoid obtained by identifying isomorphic 1-morphisms in* $\mathrm{conf} = [\Sigma(\mathbb{Z}), \Sigma(G_2)]$.

Proposition 4 *The category of endomorphisms of 1_* is, as a monoidal category, equivalent to the category of loops in the category of representations of the loop groupoid of G_2:*

$$\mathrm{End}(1_*) \simeq [\Sigma(\mathbb{Z}), \mathrm{Rep}(\Lambda G_2)]\,.$$

Proof. An object in $\mathrm{End}(1_*)$ is, being a pseudonatural transformation, a functorial assignment of 1-morphisms in conf to squares in $[\Sigma(\mathbb{Z}), \mathrm{Bim}]$

$$V \;:\; (\, g_1 \xrightarrow{\;h\;} g_2 \,) \;\mapsto\; \begin{tikzcd} \mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_{g_1}"'] \arrow[dr, phantom, "\swarrow V(h)"] & \mathrm{Id}_{\mathbb{C}} \arrow[d, "V_{g_2}"] \\ \mathrm{Id}_{\mathbb{C}} \arrow[r, "\mathrm{Id}"'] & \mathrm{Id}_{\mathbb{C}} \end{tikzcd} ,$$

which is compatible with 2-morphisms. This compatibility here just says that V is invariant on 1-morphisms that are connected by a 2-morphisms. But this means that V is a representation of ΛG_2.

Moreover, the mere existence of the square on the right says that

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "V_{g_1}\bullet"'] \arrow[dr, phantom, "\swarrow V_{g_1}(\bullet\to\bullet)"] & \mathbb{C} \arrow[d, "V_{g_1}\bullet"'] \arrow[d, bend left=60, "V_{g_2}\bullet"] \arrow[d, phantom, shift left=3, "\overset{V(h)}{\Leftarrow}"] \\ \mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C} \end{tikzcd} \;=\; \begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, bend right=60, "V_{g_1}\bullet"'] \arrow[d, "V_{g_2}\bullet"] \arrow[d, phantom, shift right=3, "\overset{V(h)}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow V_{g_2}(\bullet\to\bullet)"] & \mathbb{C} \arrow[d, "V_{g_2}\bullet"] \\ \mathbb{C} \arrow[r, "\mathrm{Id}"'] & \mathbb{C} \end{tikzcd} ,$$

which means that $g_1 \mapsto V_{g_1}(\bullet \to \bullet)$ is a natural automorphism of this representation of ΛG_2. Notice that the V_g here are $\mathbb{C}$-bimodules, hence vector spaces, while $V(h)$ is a $\mathbb{C}$-bimodule homomorphism, hence a linear map.

Next, a morphism in $\mathrm{End}(1_*)$ is a modification, hence an assignment

$$g \;\mapsto\; \begin{tikzcd} \mathrm{Id}_{\mathbb{C}} \arrow[r, bend left=50, "V_g"] \arrow[r, bend right=50, "V'_g"'] \arrow[r, phantom, "\Downarrow k_g"] & \mathrm{Id}_{\mathbb{C}} \end{tikzcd} .$$

The tin can equation for this says that k is a natural isomorphism from the representation V to the representation V'. Moreover, the mere existence of

k_g above says that this natural isomorphism is compatible with the natural automorphism $V(\bullet \to \bullet)$ and $V'(\bullet \to \bullet)$.

But this means nothing but that k encodes a morphism in

$$[\Sigma(\mathbb{Z}), \mathbf{Rep}(\Lambda G_2)]\,.$$

$\square$

Now let $G_2 = \mathrm{String}_G$ be the strict 2-group corresponding to the crossed module $\hat{\Omega}_k G \to PG$, where G is any simple, simply connected and compact Lie group.

Proposition 5 *$\Lambda\mathrm{String}_G$ is isomorphic to a central extension of ΛG.*

Proof. It suffices to check this for the vertex groups of connected components.

So fix any element $\gamma \in PG$. This amounts to choosing any $g \in G$ and a path connecting it to the neutral element. All choices of paths with fixed endpoint correspond to the same connected component.

An automorphism in ΛG_2

$$\begin{tikzcd}
pg \arrow[d, "{[r,\gamma]}"'] \\
pg
\end{tikzcd}$$

is represented by a 2-cell

$$\begin{tikzcd}
\bullet \arrow[r, "pg"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\swarrow r"] & \bullet \arrow[d, "\gamma"] \\
\bullet \arrow[r, "pg"'] & \bullet
\end{tikzcd}$$

in String_G. For this to exist, the endpoints of pg and γ have to commute. But with the endpoint fixed, the path γ is arbitrary, because the above 2-cell is to be identitfied with

$$\begin{tikzcd}
\bullet \arrow[r, "pg"] \arrow[d, "\gamma", "\overset{a^{-1}}{\Leftarrow}"'] \arrow[d, bend right=60, "\gamma'"'] \arrow[dr, phantom, "\swarrow r"] & \bullet \arrow[d, "\gamma"', "\overset{a}{\Leftarrow}"] \arrow[d, bend left=60, "\gamma'"] \\
\bullet \arrow[r, "pg"'] & \bullet
\end{tikzcd}$$

for any loop a. Hence all paths γ with the same endpoint are to be identified. Choosing a fixed representative of the family of all paths with the same endpoint, the 2-morphism r here is a uniquely determined loop, together with an element in the $U(1)$-torsor over that loop.

Therefore any vertex group of $\Lambda\mathrm{String}_G$ is a central extension of the corresponding vertex group of ΛG. $\Box$

For any element $g \in G$, fix an element in PG. In other words, choose a section

$$\begin{tikzcd} PG \arrow[r] & G \arrow[l, bend right, "s"'] \end{tikzcd} .$$

This will only locally be smooth, of course.

We can identify two groupoid morphisms into $\Lambda\mathrm{String}_G$, namely

$$\mathrm{String}_G \longrightarrow \Lambda\mathrm{String}_G$$

with

$$\begin{tikzcd} pg \arrow[d, "r"] \\ pg' \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, "pg"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\swArrow r"] & \bullet \arrow[d, "\mathrm{Id}"] \\ \bullet \arrow[r, "pg'"'] & \bullet \end{tikzcd}$$

and

$$(PG/G)_s \longrightarrow \Lambda\mathrm{String}_G$$

with

$$\begin{tikzcd} pg \arrow[d, "\mathrm{Ad}_g"] \\ pg' \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, "pg"] \arrow[d, "s(g)"'] \arrow[dr, phantom, "\swArrow \mathrm{Id}"] & \bullet \arrow[d, "s(g)"] \\ \bullet \arrow[r, "pg'"'] & \bullet \end{tikzcd} .$$

Proposition 6 *The loop groupoid* $\Lambda\mathrm{String}_G$ *is generated by the images of* String_G *and* $(PG/G)_s$ *under these two functors.*

Proof. For any morphism in $\Lambda\mathrm{String}_G$, we can choose a representative whose vertical morphisms lie in the image of the chosen section s:

$$\begin{tikzcd} \bullet \arrow[r, "pg"] \arrow[d, "s(g)"'] \arrow[dr, phantom, "\swArrow r"] & \bullet \arrow[d, "s(g)"] \\ \bullet \arrow[r, "pg'"'] & \bullet \end{tikzcd} .$$

We can then write

$$
\begin{tikzcd}
\bullet \arrow[r, "pg"] \arrow[d, "s(g)"'] \arrow[dr, phantom, "\swarrow_r"] & \bullet \arrow[d, "s(g)"] \\
\bullet \arrow[r, "pg'"'] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "pg"] \arrow[d, "s(g)"'] \arrow[dr, phantom, "\swarrow_r"] & \bullet \arrow[d, "s(g)"] \\
\bullet \arrow[r, "pg'"] \arrow[d, "s(g)^{-1}"'] \arrow[dr, phantom, "\swarrow_{\mathrm{Id}}"] & \bullet \arrow[d, "s(g)^{-1}"] \\
\bullet \arrow[r, "\mathrm{Ad}_{s(g)}(pg')"] \arrow[d, "s(g)"'] \arrow[dr, phantom, "\swarrow_{\mathrm{Id}}"] & \bullet \arrow[d, "s(g)"] \\
\bullet \arrow[r, "pg'"] & \bullet
\end{tikzcd}
,
$$

and the right hand side is manifestly a composite of the two types of generators.
□

Now, notice that, as a centrally extended groupoid, String_G is nothing but the canonical bundle gerbe on G. Accordingly, a groupoid representation of String_G is the same as a module for that gerbe, alternatively known as a twisted bundle on G. It is known that the decategorification of gerbe modules on a space, $\mathrm{Rep}(\mathrm{String}_G)$, is the same as the twisted K-theory of that space.

But proposition 6 says that a representation of $\Lambda\mathrm{String}_G$ is a representation of String_G, which at the same time carries the structure of a representation of the adjoint action of G on PG.

It hence looks as if $\mathrm{Rep}(\Lambda\mathrm{String}_G)$ would give rise to the twisted and Ad_G-equivariant K-theory of G.