2-Monoid of Observables on $String_G$

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Abstract

Given any 2-groupoid, we can associate to it a monoidal category which can be thought of as the 2-monoid of observables of the 2-particle propagating on that 2-groupoid.

Here we show that for the 2-groupoid $\Sigma(\text{String}_G)$ this monoidal category is the category

$\Lambda \operatorname{Rep}(\Lambda \operatorname{String}_G)$

of loops in representations of the loop groupoid of String_G . We argue that representation of ΛString_G are twisted equivariant bundles on G.

Introduction. For various reasons, I find the following general concept useful, which here I want to apply to a special case related to loop groups and representations.

Let par be a 1-category, called the **parameter space**.

Let \mathcal{P} be a smooth 2-category, called the **target space**.

Let tra : $\mathcal{P} \to \operatorname{Bim}(\operatorname{Vect}_{\mathbb{C}})$ be a smooth 2-functor to the 2-category whose morphisms are bimodules, called a **2-vector bundle** with connection on target space.

Let $1: \mathcal{P} \to \operatorname{Bim}(\operatorname{\mathbf{Vect}}_{\mathbb{C}})$ be the tensor unit in the monoidal 2-category of 2-vector bundles with connection, i.e. the 2-functor that sends everything to the identity on \mathbb{C} .

Let $conf \equiv [par, \mathcal{P}]$ be the 2-category of 2-functors from parameter space to target space, called the **configuration space**.

Let $tra_* : conf \rightarrow [par, Bim]$ be the 2-functor on configuration space obtained by postcomposing with tra. This can roughly be thought of as the **transgressed 2-vector bundle**.

Let $sect = [1_*, tra_*]$ be the category of morphisms from the trivial 2-vector bundle on configuration space to the transgressed 2-vector bundle. This I call the **space of sections** on configuration space.

Let $C = End(1_*)$ be the monoidal category of endomorphisms of 1_* , called the **monoid of observables**.

Clearly, sect is a module category for C.

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Here I would like to understand the category \mathcal{C} for the following setup.

Let G be a simple, simply connected and compact Lie group, and let $k \in H^3(G,\mathbb{Z})$ be a level. From the centrally extended loop group, $\hat{\Omega}_k G$, we can form the groupoid String_G $\equiv PG \ltimes \hat{\Omega}_k G \Longrightarrow PG$ over based paths in G.

This groupoid can be regarded from two points of view. As a centrally extended groupoid, it is the canonical bundle gerbe with class k over G. The groupoid has a strict monoidal structure, with strict monoidal inverses. Therefore it can also be regarded as a strict 2-group.

Being monoidal, we can form the suspension $\Sigma(\text{String}_G)$, which is a 2-category with a single object.

We want to regard this as our target space, in the above sense, and study the monoid of observables on the configuration space of 2-particles propagating on this target space.

Notice that in as far as $BG \simeq |\Sigma G|$, we can think of String_G as a twisted version of BG.

This means we set $\mathcal{P} = \Sigma(\operatorname{String}_G)$ and $\operatorname{par} = \Sigma(\mathbb{Z})$.

In this case, I seem to find the following result:

Proposition 1 The groupoid

$$\Lambda \operatorname{String}_G \equiv [\Sigma(\mathbb{Z}), \Sigma(\operatorname{String}_G)]_{/\sim}$$

obtained by identifying isomorphic 1-morphisms in configuration space is a central extension of the the loop groupoid

$$\Lambda G \equiv [\Sigma(\mathbb{Z}), \Sigma(G)]$$

of G.

Proposition 2 The monoidal category C is

$$\mathcal{C} = [\Sigma(\mathbb{Z}), \operatorname{Rep}(\Lambda \operatorname{String}_G)].$$

Proposition 3 The category $\operatorname{Rep}(\operatorname{String}_G)$ is the category of equivariant gerbe modules on G.

Note that for G finite, Simon Willerton argued that ΛG , which is nothing but the action groupoid of the adjoint action of G on itself, plays the role of the loop group of G, by noticing that

$$B\Lambda G \simeq LBG$$
.

Given a group 3-cocycle κ on G, hence a groupoid 2-cocycle on ΛG , one can therefore address the twisted representations $\operatorname{Rep}_{\kappa}(\Lambda G)$ both as twisted representations of the loop group of G - in the above sense - as well as twisted equivariant vector bundles on G.

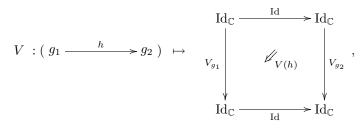
What I describe here looks like a Lie group analog of this perspective on the Freed-Hopkins-Teleman theorem.

Definition 1 For G_2 any strict 2-group, the **loop groupoid** of G_2 is the 1-groupoid obtained by identifying isomorphic 1-morphisms in conf = $[\Sigma(\mathbb{Z}), \Sigma(G_2)]$.

Proposition 4 The category of endomorphisms of 1_* is, as a monoidal category, equivalent to the category of loops in the category of representations of the loop groupoid of G_2 :

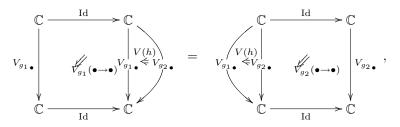
$$\operatorname{End}(1_*) \simeq [\Sigma(\mathbb{Z}), \operatorname{Rep}(\Lambda G_2)]$$

Proof. An object in $\text{End}(1_*)$ is, being a pseudonatural transformation, a functorial assignment of 1-morphisms in conf to squares in $[\Sigma(\mathbb{Z}), \text{Bim}]$



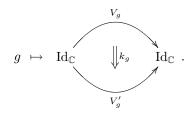
which is compatible with 2-morphisms. This compatibility here just says that V is invariant on 1-morphisms that are connected by a 2-morphisms. But this means that V is a representation of ΛG_2 .

Moreover, the mere existence of the square on the right says that



which means that $g_1 \mapsto V_{g_1}(\bullet \to \bullet)$ is a natural automorphism of this representation of ΛG_2 . Notice that the V_g here are \mathbb{C} -bimodules, hence vector spaces, while V(h) is a \mathbb{C} -bimodule homomorphism, hence a linear map.

Next, a morphism in $End(1_*)$ is a modification, hence an assignment



The tin can equation for this says that k is a natural isomorphism from the representation V to the representation V'. Moreover, the mere existence of

 k_g above says that this natural isomorphism is compatible with the natural automorphism $V(\bullet \to \bullet)$ and $V'(\bullet \to \bullet)$.

But this means nothing but that k encodes a morphism in

$$[\Sigma(\mathbb{Z}), \operatorname{\mathbf{Rep}}(\Lambda G_2)]$$

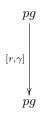
Now let $G_2 = \text{String}_G$ be the strict 2-group corresponding to the crossed module $\hat{\Omega}_k G \to PG$, where G is any simple, simply connected and compact Lie group.

Proposition 5 Λ String_G is isomorphic to a central extension of ΛG .

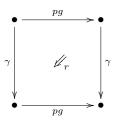
Proof. It suffices to check this for the vertex groups of connected components. So fix any element $\gamma \in PG$. This amounts to choosing any $g \in G$ and a path

connecting it to the neutral element. All choices of paths with fixed endpoint correspond to the same connected component.

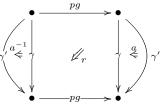
An automorphism in ΛG_2



is represented by a 2-cell



in String_G. For this to exist, the endpoints of pg and γ have to commute. But with the endpoint fixed, the path γ is arbitrary, because the above 2-cell is to be identified with



for any loop a. Hence all paths γ with the same endpoint are to be identified. Choosing a fixed representative of the family of all paths with the same endpoint,

the 2-morphism r here is a uniquely determined loop, together with an element in the U(1)-torsor over that loop.

Therefore any vertex group of $\Lambda String_G$ is a central extension of the corresponding vertex group of ΛG .

For any element $g \in G$, fix an element in PG. In other words, choose a section

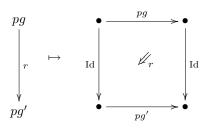
$$PG \xrightarrow{s} G$$
.

This will only locally be smooth, of course.

We can identify two groupoid morphisms into $\Lambda String_G$, namely

$$\operatorname{String}_G \longrightarrow \operatorname{AString}_G$$

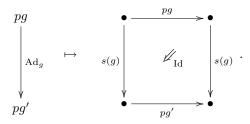
with



and

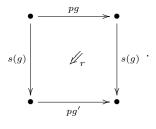
$$(PG/G)_s \longrightarrow \Lambda \operatorname{String}_G$$

with

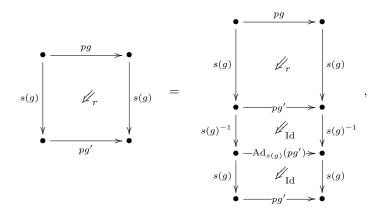


Proposition 6 The loop groupoid Λ String_G is generated by the images of String_G and $(PG/G)_s$ under these two functors.

Proof. For any morphism in Λ String_G, we can choose a representative whose vertical morphisms lie in the image of the chosen section s:



We can then write



and the right hand side is manifestly a composite of the two types of generators. \Box

Now, notice that, as a centrally extended groupoid, String_G is nothing but the canonical bundle gerbe on G. Accordingly, a groupoid representation of String_G is the same as a module for that gerbe, alternatively known as a twisted bundle on G. It is known that the decategorification of gerbe modules on a space, $\operatorname{Rep}(\operatorname{String}_G)$, is the same as the twisted K-theory of that space.

But proposition 6 says that a representation of ΛString_G is a representation of String_G , which at the same time carries the structure of a representation of the adjoint action of G on PG.

It hence looks as if $\text{Rep}(\Lambda \text{String}_G)$ would give rise to the twisted and Ad_G -equivariant K-theory of G.