On nonabelian differential cohomology

Urs

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Abstract

Nonabelian differential n-cocycles provide the data for an assignment of "quantities" to n-dimensional "spaces" which is

- locally controlled by a given "typical quantity";
- globally compatible with all possible gluings of volumes.

For $n = 1$ this encompasses the notion of *parallel transport* in fiber bundles with connection. In general we think of it as *parallel n-transport*.

For low n and/or "sufficiently abelian quantities" this has been modeled by differential characters, $(n-1)$-gerbes, $(n-1)$-bundle gerbes and n-bundles with connection. We give a general definition for all n in terms of descent data for transport n-functors along the lines of [7, 57, 58, 59]. Concrete realizations, notably Chern-Simons n-cocycles, are obtained by integrating L_∞-algebras and their higher Cartan-Ehresmann connections [52].

Here we assume all gluing to happen through equivalences. If one instead allows gluing by special adjunctions, one finds "quantm cocycles" which encode the propagators and correlators of extended quantum field theories. This will concern another time.

Contents

These notes evolved as expanded notes for a series of talks I gave in April 2008 at Notre Dame and at UPenn and in May 2008 at the Hausdorff institute in Bonn and at Stanford. I am grateful to Stephan Stolz, Jim Stasheff and Soren Galatius for the kind invitations and the very pleasant stays.

The investigation of parallel transport n-functors goes back to joint work with John Baez [7], elaborated on in joint work with Konrad Waldorf [57, 58, 59]. The study of the structure n-group $\mathbf{E}G := \mathrm{INN}_0(G)$ crucial for the non-flat case is joint work with David Roberts [51]. The L_∞-algebraic description of differential cocycles is joint work with Hisham Sati and Jim Stasheff [52], as is the application to higher string lifts such as the FiveBrane lift [53].

I profited greatly from extensive discussion with Todd Trimble about much of the higher abstract nonsense involved here, and from discussion with Todd and Andrew Stacey [60] about smooth spaces.

I am indebted to Stephan Stolz for interrogating me at length about all the following issues, and to Jim Stasheff for lots of discussion and plenty of editorial and other remarks on these notes.

Many of the L_∞-algebraic aspects here build on work by Ezra Getzler [33], André Henriques [36] and Pavol Ševera [54], with all of whom I had the helpful pleasure of discussing aspects of this in person. A more detailed discussion of ∞-Lie theory along the lines employed here is in [55].

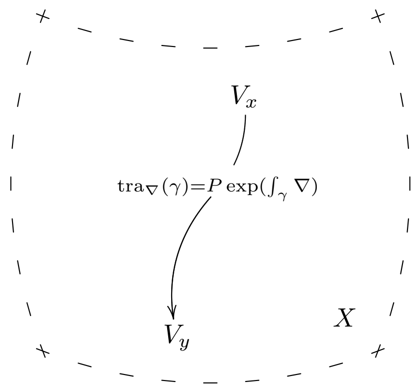
I very much profited from discussion at the n-Category Café about various issues involved here, in particular with Bruce Bartlett, Timothy Porter, Mike Shulman and many others.

1 Motivation: connections, parallel transport and amplitudes

The theory of fiber bundles with connection is closely related to the development of theoretical physics (as Jim Stasheff recalls: Dirac describes the magnetic monopole [26] in the same year – 1931 – that Hopf describes his fibration [37] – that both concepts coincide, as line bundles over S^2, is mentioned in publication only decades later [34]).

Why does physics need bundles with connection?

It is not so much the bundles that are needed (Dirac did not know about those), but the *parallel transport* tra_∇ obtained from the connection ∇, which assigns a "phase" to each path:



In physics, this parallel transport exhibits the *background field* which a particle *couples* to.

The historically first example is the the *electromagnetic field*, which is represented by a connection on a line bundle. With the advent of Yang-Mills theory and the standard model of particle physics in the middle of the last century, also higher rank vector bundles with connection appeared as background fields for the particles that have been detected in accelerator experiments.

Nonabelian differential cohomology is all about such "background fields" and their generalizations which couple to higher-dimensional particles (called *branes*).

$$\begin{tikzcd} \Sigma \arrow[r, bend left, "\Phi"] & X & ? \end{tikzcd}$$

$$\begin{tikzcd} \text{parameter space} \arrow[r, bend left, "\text{field configuration}"] & \text{target space} & \text{background field} \end{tikzcd}$$

The background field assigns "quantities" called "phases" to field configurations

$$\Phi : \Sigma \to X$$

in a way that

- locally takes values in a space T of "typical quantities" (often just elements in $U(1)$);
- is globally compatible with all possible gluings of parameter spaces.

We can interpret this as a *locally trivializable* transport n-functor from globular n-paths:

$$
\begin{tikzcd}
\text{“quantities”} & T & P_x \arrow[rr, bend left, "\mathrm{tra}_\nabla(\gamma)"] \arrow[rr, bend right, "\mathrm{tra}_\nabla(\gamma')"'] \arrow[rr, phantom, "\Downarrow \mathrm{tra}_\nabla(\Sigma)"] && P_y \\
\text{(higher) paths in } X & \Pi_\omega(X) \arrow[u, "\nabla"'] & x \arrow[u, mapsto] \arrow[rr, bend left, "\gamma"] \arrow[rr, bend right, "\gamma'"'] \arrow[rr, phantom, "\Downarrow\Sigma"] && y \arrow[u, mapsto]
\end{tikzcd}
$$

Hence the above picture is completed as

$$
\begin{tikzcd}
\Pi_\omega(\Sigma) \arrow[r, bend left, "\Phi"] & \Pi_\omega(X) \arrow[r, "\nabla"] & T
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{array}{c}\text{parameter}\\ \text{space}\end{array} \arrow[r, bend left, "\text{field configuration}"] & \begin{array}{c}\text{target}\\ \text{space}\end{array} \arrow[r, "\text{background field}"] & \text{phases}
\end{tikzcd}
$$

2 Plan

2.1 Nonabelian differential cohomology,

Higher connections. We shall be interested in higher (as in "higher dimensional") connections on bundles.

An ordinary connection is something that allows to transport things along paths $x \xrightarrow{\gamma} y$. A higher connection allows to do the same along surfaces, volumes, etc.

$$
\begin{tikzcd}
\text{“quantities”} & T & P_x \arrow[rr, bend left, "\mathrm{tra}_\nabla(\gamma)"] \arrow[rr, bend right, "\mathrm{tra}_\nabla(\gamma')"'] \arrow[rr, phantom, "\Downarrow \mathrm{tra}_\nabla(\Sigma)"] && P_y \\
\text{(higher) paths in } X & \Pi_\omega(X) \arrow[u, "\nabla"'] & x \arrow[u, mapsto] \arrow[rr, bend left, "\gamma"] \arrow[rr, bend right, "\gamma'"'] \arrow[rr, phantom, "\Downarrow\Sigma"] && y \arrow[u, mapsto]
\end{tikzcd}
$$

Motivation. The two main motivations for us to consider such higher connections are

- **math**: to get geometric cocycles for nonabelian differential cohomology.

  If $X \mapsto \Gamma^\bullet(X)$ is a cohomology theory (such as singular cohomology $\Gamma^\bullet(X) = H^\bullet(X, \mathbb{Z})$ or K-theory $\Gamma^\bullet(X) = K^\bullet(X)$) it comes with a map to real cohomology

  $$\Gamma^\bullet(X) \to H^\bullet(X, \Gamma^\bullet(\mathrm{pt}))\,.$$

  A *differential refinement* $\bar{\Gamma}^\bullet(X)$ of $\Gamma^\bullet(X)$ is given by differential form data that represents this real cohomology class:

  $$\begin{tikzcd} \bar{\Gamma}^\bullet(X) \arrow[r] \arrow[d] & H^\bullet_{\mathrm{dR}}(X, \Gamma^\bullet(\mathrm{pt})) \arrow[d] \\ \Gamma^\bullet(X) \arrow[r] & H^\bullet(X, \Gamma^\bullet(\mathrm{pt})) \end{tikzcd}\ .$$

  In practice one wants *cocycles* representing the classes in $\bar{\Gamma}^\bullet(X)$. If the cocycles for $\Gamma^\bullet(X)$ are higher bundles, then those of $\bar{\Gamma}^\bullet(X)$ should be higher bundles with higher connections.

  For instance:

  - cocycles for $H^{n+1}(X, \mathbb{Z})$ are line n-bundles, aka abelian $(n-1)$-gerbes. Cocycles of $\bar{H}^{n+1}(X, \mathbb{Z})$ are line n-bundles with connection, aka abelian $(n-1)$-gerbes with connection, aka Deline n-cocycles, aka Cheeger-Simons differential n-characters.
  - Cocycles for $K^0(X)$ are vector bundles. Cocycles for $\bar{K}^0(X)$ are vector bundles with connection.
  - Important motivating open question: if $\Gamma^\bullet(X)$ is elliptic cohomology (tmf), then cocycles for $\bar{\Gamma}^\bullet(X)$ should be 2-vector bundles with connection. Things like that is what we want to describe.

- **physics**: to understand extended quantum field theories that are "Σ-models".

  Σ-models are quantum field theories which are encoded in an n-bundle with connection (P, ∇) on some target space X.

$$\begin{tikzcd}[column sep=small] \int_\Sigma \mathrm{ev}^*(P,\nabla) \arrow[dr] & & \Sigma \times \mathrm{Maps}(\Sigma, X) \arrow[dl, "\int_\Sigma"'] \arrow[dr, "\mathrm{ev}"] & & (P,\nabla) \arrow[dl] \\ & \mathrm{Maps}(\Sigma, X) & & X & \end{tikzcd}$$

transgressed $(n-\dim\Sigma)$-bundle with connection	mapping space, space of fields	extended space of fields	target space	background field n-bundle with connection

The quantum field theory should assign to any space Σ the "space of sections" of the transgressed background field $\int_\Sigma \mathrm{ev}^*(P, \nabla)$.

The goal is to make sense of this and understand this for all n-bundles with connection (P, ∇). In particular for *Chern-Simons* n-bundles $\mathrm{CS}(\nabla)$ coming from L_∞-algebra n-cocycles [52], and for the non-abelian String-; Fivebrane- Ninebrane-$(n-1)$-bundles obstructed by them [53].

Main central observation. Our describtion of n-bundles with connection becomes immediately obvious once we change the perspective on ordinary bundles P on X from that of the bundle being a fibration

$$\begin{tikzcd} P \arrow[d] \\ X \end{tikzcd}$$

to the fiber-assigning functor

$$\begin{tikzcd} G\mathrm{Spaces} & P_x \\ \Pi_0(X) \arrow[u] & x \arrow[u, mapsto] \end{tikzcd} .$$

Here $\Pi_0(X)$ is just the space X regarded as a category with only identity morphisms.

So we have this change of perspective, from fibrations to fiber-assigning functors:

$$G\mathrm{Bund}(X) \xrightarrow{\simeq} H(X, \mathbf{B}G) \xrightarrow{\simeq} \mathrm{Desc}(Y^\bullet, \mathrm{Funct}(\Pi_0(-), \mathbf{B}G)) \xrightarrow{\simeq} \mathrm{TransportFunctors}(\Pi_0^Y(X), G\mathrm{Tor})$$

$$\left\{ \begin{tikzcd} Y \times G \arrow[r, "\simeq"] \arrow[d] & \pi^* P \arrow[r] & P \arrow[d] \arrow[dll, Rightarrow, "\exists ="'] \\ Y \arrow[r, "="] & Y \arrow[r, "\pi"] & X \end{tikzcd} \right\} \simeq \left\{ \left(\begin{array}{c} g : Y \times_X Y \to G\,, \\ \forall y \in Y \times_X Y \times_X Y : \\ \begin{tikzcd} & \bullet \arrow[dr, "\pi_{23}^* g"] & \\ \bullet \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"'] & & \bullet \end{tikzcd} \end{array} \right) \right\} \simeq \left\{ \left(\begin{array}{c} \mathrm{triv} : \Pi_0(Y) \to \mathbf{B}G\,, \\ \pi_1^*\mathrm{triv} \xrightarrow{g} \pi_2^*\mathrm{triv}\ , \\ \begin{tikzcd} & \pi_2^*\mathrm{triv} \arrow[dr, "\pi_{23}^* g"] & \\ \pi_1^*\mathrm{triv} \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"'] & & \pi_3^*\mathrm{triv} \end{tikzcd} \end{array} \right) \right\} \simeq \left\{ \begin{tikzcd} \Pi_0(Y) \arrow[r, "\pi"] \arrow[d, "\mathrm{triv}"'] & \Pi_0(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\exists \simeq"] \\ \mathbf{B}G \arrow[r, hook] & G\mathrm{Tor} \end{tikzcd} \right\} .$$

Here assume the surjective submersion $Y = \sqcup_i U_i$ to be a good gover by open subsets.

The important point is the category $\mathrm{Desc}(Y^\bullet, \mathbf{A})$ of descent data along Y with respect to the category-valued presheaf $\mathbf{A}$. Ross Street gave a general formula for this [62] for ω-category (strict ∞-category) valued presheaves: objects are the higher dimensional simplices generalizing the triangle above in the obvious way:

$$\mathrm{Obj}(\mathrm{Desc}(Y^\bullet, \mathbf{A})) = [\Delta, \omega\mathrm{Cat}](\mathbf{\Delta}, \mathbf{A}(Y^\bullet))\,,$$

where $\mathbf{\Delta}$ is the tautological cosimplicial ω-category which sends $[n]$ to the free n-category on the standard combinatorial n-simplex (Street's "nth oriental" [62]).

Slogan. *Assign nice quantities locally that glue globally.*

- differentially: L_∞-algebra valued forms;
- integrally: smooth "parallel transport" n-functors from n-paths to an n-group

	local	**global**
differentially	L_∞-algebra valued diff. forms	L_∞-connections
integrated	smooth ω-functors from ω-paths to ω-groups	parallel transport ω-functors

Table 1: The **four aspects of nonabelian differential cohomology**. Here "ω-category" is for "strict globular ∞-category". Our restriction to strict ∞-transport is for technical, not for conceptual reasons. We find it is not only convenient and useful (it allows us to use Ross Street's notion of descent with coefficients in ω-category-valued presheaves) but also sufficient: we can integrate all L_∞-algebras to smooth ω-groups.

The obvious generalization. Therefore there is an obvious immediate generalization of the above to n-bundles with connection for all n: simply replace $\Pi_0(X)$ everywhere with $\Pi_n(X)$ – the fundamental n-groupoid of X.

$$\left\{ \begin{array}{ll} \mathcal{P}_n(X) & \text{thin-homotopy classes of images of the } n\text{-disk} \\ \Pi_n(X) & \text{homotopy classes of images of the } n\text{-disk} \end{array} \right.$$

Definition (nonabelian differential cohomology).

For G a strict n-group, hence $\mathbf{B}G$ a strict one-object n-groupoid, we say that fake-flat nonabelian differential cohomology is the n-category-valued presheaf

$$\bar{H}_{\mathrm{ff}}(-, \mathbf{B}G) := \mathrm{colim}_{Y \to (-)} \mathrm{Desc}(Y^\bullet, n\mathrm{Funct}(\mathcal{P}_n(-), \mathbf{B}G)) \,.$$

The non-fake-flat case can of course also be handled, but requires a bit more work.

Theorems $-$ $n = 1$. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

Theorem.[7, 57]

$$\mathrm{Funct}(\mathcal{P}_1(Y), \mathbf{B}G) \simeq \Omega^1(Y, \mathfrak{g})$$

$$\mathrm{Funct}(\Pi_1(Y), \mathbf{B}G) \simeq \Omega^1_{\mathrm{flat}}(Y, \mathfrak{g})$$

$$G\mathrm{Bund}_\nabla(X) \simeq \left\{ \begin{tikzcd} \mathcal{P}_1(Y) \arrow[r, "\pi"] \arrow[d, "\mathrm{triv}"'] & \mathcal{P}_1(X) \arrow[d, "\mathrm{tra}"] \arrow[dl, Rightarrow, "\exists\simeq"] \\ \mathbf{B}G \arrow[r, hook] & G\mathrm{Tor} \end{tikzcd} \right\}$$

$$\begin{tikzcd} \mathcal{P}_1(X) \arrow[dr, "\text{vector bundle with connection}"] \arrow[dd] & \\ & \mathrm{Vect} \\ \Pi_1(X) \arrow[ur, dashed, "\text{flat vector bundle}"'] & \end{tikzcd}$$

Theorems $-$ $n = 2$. Now let $G_{(2)}$ be a strict Lie 2-group, coming from a crossed module

$$H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$$

of groups, with the corresponding one-object 2-groupoid being

$$\mathbf{B}G_2 = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'=t(h)g"'] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} |g \in G, h \in G \right\}.$$

Write

$$\mathfrak{g}_2 := (\mathfrak{h} \to \mathfrak{g})$$

for the corresponding L(ie) 2-algebra.

Theorem.[7, 58, 59]

$$2\mathrm{Func}(\mathcal{P}_2(Y), \mathbf{B}G_{(2)}) \simeq \Omega^\bullet_{\mathrm{fakeflat}}(Y, \mathfrak{g}_{(2)})$$

$$2\mathrm{Func}(\Pi_2(Y), \mathbf{B}G_{(2)}) \simeq \Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}_{(2)})$$

and [51, 52]

$$3\mathrm{Func}(\Pi_3(Y), \mathbf{E}G_{(2)}) \simeq \Omega^\bullet(Y, \mathfrak{g}_{(2)}) .$$

Theorem. [9]: $G_{(2)}$-2-bundles have the same classification as $|G_{(2)}|$-1-bundles.

$$H(-, \mathbf{B}G_{(2)}) \simeq [(-), B|G_{(2)}|]$$

Important special case [8]: for $G_{(2)} = (\hat{\Omega} G \to PG)$ we have $|G_{(2)}|$-bundles are String-bundles.

Theorem. [59]

$$\bar{H}(-\mathbf{B}\mathrm{AUT}(H)) \simeq \{\text{Breen-Messing diff. cocycles [14] on fake-flat } H\text{-gerbes}\}$$

In particular

$$\bar{H}(-, \mathbb{B}\mathbb{B}U(1)) \simeq \{ \text{ U(1)-gerbes with connection } \}\,.$$

Equivariance. Let H be a group acting on X. Let $X//H$ be the corresponding action groupoid and $|X//H|$ its nerve. Then the H-equivariant version of the cohomology $H(-, \mathbf{A})$ is

$$H^H(-, \mathbf{A}) := \mathrm{Desc}(|X//H|, H(-, \mathbf{A}))\,.$$

Theorem.

$$\mathrm{Desc}(\mathrm{Ner}(X//H), \bar{H}(-, \mathbf{B}G_{(2)}))$$

is H-equivariant 2-bundles with connection. In particular

$$\mathrm{Desc}(\mathrm{Ner}(X//\mathbb{Z}_2), \bar{H}(-, \mathbf{B}(\mathrm{AUT}(U(1)))))$$

is Jandl-gerbes with connection [56]: this are the *orientifold B-fields* in string theory.

Transgression. We had mentioned in the introduction that transgression of differential cocycles is important in applications. It turns out that in terms of transport n-functors transgression is nothing but forming the inner hom

Theorem. [58] In the simplest case we have G an abelian Lie group and consider trangression of $\mathbf{B}G$-valued transport:

$$\begin{tikzcd}
2\mathrm{Funct}(\mathcal{P}_2(Y), \mathbf{B}\mathbf{B}G) \arrow[r, "\simeq"] \arrow[d, "\hom(\mathbf{B}\mathbb{Z}^2, -)"] & \Omega^\bullet(Y, b\mathfrak{g}) \arrow[d, "\int_{S^1} \mathrm{ev}^*"] \\
\mathrm{Funct}(\mathcal{P}_1(LY), \mathbf{B}G) \arrow[r] & \Omega^\bullet(LY, \mathfrak{g})
\end{tikzcd}$$

This generalizes accordingly to more complex situations.

Examples for n-bundles with connection. A main source of examples of n-bundles (with or without connection) is lifts through higher central extensions and their obstructions.

$$
\begin{tikzcd}
 & \mathbf{B}B^{n-1}U(1) \arrow[d] & & & . \\
 & \mathbf{B}\hat{G} \arrow[d] & \text{lift to} \atop \hat{G}\text{-bundle} \arrow[l, squiggly] & & \\
\Pi_n^Y(X) \arrow[r] \arrow[ur, dashed] & \mathbf{B}G \arrow[r, "\simeq"] & \mathbf{B}(B^{n-1}U(1) \to \hat{G}) \arrow[r] & \mathbf{B}B^nU(1) & \\
 & G\text{-bundle} \arrow[u, squiggly] & \text{attempted lift} \atop \text{“twisted } \hat{G}\text{-bundle”} \arrow[u, squiggly] & \text{obstruction} \atop \text{to lift} \arrow[u, squiggly] &
\end{tikzcd}
$$

Examples:

- lifting gerbes (lifting line 2-bundles) obstructing lifts through ordinary central extensions $U(1) \to \hat{G} \to G$.
- Chern-Simons 3-bundles (2-gerbes) obstructing lifts of G-bundles to $\mathrm{String}(G) = (\hat{\Omega} G \to PG)$-2-bundles.
- and higher versions of this.

L_∞-connections. A powerful way to understand these lifting and obstruction situations is by passing first to the linearized version of Lie ∞-groups – L_∞-algebras – , and then integrating, in the sense of Lie theory, afterwards.

Theorem. [52]. For $\mathfrak{g}$ an L_∞-algebra and μ an $(n+1)$-cocycle in transgression with an invariant polynomial P, there is a string-like extension

$$0 \to b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0$$

and lifts of $\mathfrak{g}$-connections (A, F_A) to $\mathfrak{g}_\mu$-connections are obstructed by $b^n\mathfrak{u}(1)$-connections whose curvature form is $P(F_A)$.

Applying this to $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ the canonical 3-cocycle on a semisimple Lie algebra $\mathfrak{g}$, normalized such that it generated $H^3(G, \mathbb{Z})$ for G the simply connected compact simple Lie group integrating $\mathfrak{g}$,

$$0 \to b\mathfrak{u}(1) \to \mathfrak{g}_{\langle \cdot, [\cdot, \cdot] \rangle} \simeq \mathfrak{string}(\mathfrak{g}) \to \mathfrak{g} \to 0\,,$$

the integration of the above statement to higher bundles with connection yields the familiar construction of Chern-Simons differential cocycles [19, 20] whose connection 3-form is the Chern-Simons 3-form of the original $\mathfrak{g}$-connection.

The String-2-bundles which exists when the obstruction vanishes are to Spin-bundles as superstrings (spinors on loop space) are to superparticles (spinors on target space).

This construction can be iterated: if the String-lift exsist we can ask for the next lift, which is a Fivebrane lift [53]

$$0 \to b^5\mathfrak{u}(1) \to \mathfrak{g}_{\langle\cdot,[\cdot,\cdot],[\cdot,\cdot],[\cdot,\cdot]\rangle} \simeq \mathfrak{fivebrane}(\mathfrak{g}) \to \mathfrak{string}(\mathfrak{g}) \to 0\,,$$

The Fivebrane-6-bundles which exists when the obstruction vanishes are to String-bundles as super fivebranes are to superstrings.

2.2 Higher Chern-Simons parallel transport

We have another look at the concept of Σ-models, field theories of maps from a *parameter space* Σ to a *target space* X

$$\Phi : \Sigma \to X\,.$$

(Notice that these include gauge theories, if things are set up suitably: the target for those is a classifying space.)

These usually have an action functional controlled by the parallel transport of a *background field* ∇ on X.

This *background field* is given by a possibly nonabelian differential cocycle, an n-bundle with connection, something that assigns "phases" which

- are locally given by differential forms;
- globally satisfy some "gluing property".

This gluing property is physically often detected as a kind of "anomaly" obstructing the global extension of the differential forms. We described in in terms of descent condtions on nonabelian differential cocycles in the previous section.

Examples include:

- $n = 1$, electromagnetism: locally a 1-form $A \in \Omega^1(Y)$ with field strength $F_2 = dA$;
- $n = 1$, nonabelian gauge fields: locally a 1-form $A \in \Omega^1(Y, \mathfrak{g})$
- $n = 2$, Kalb-Ramond field/"B-field": locally a 2-form $B \in \Omega^2(Y)$ with field strength $H_3 = dB$;
- $n = 2$, Kalb-Ramond field on an *orientifold*: same local connection form data, but more sophisticated gluing (technically not a $\mathbf{B}U(1)$-2-bundle but a $(\mathbf{B}U(1) \to \mathbb{Z}_2)$-bundle)
- $n = 2$, String 2-bundle
- $n = 3$, supergravity C-field: locally a Chern-Simons 3-form $C = \mathrm{CS}(\omega) - \mathrm{CS}(A)$ with field strength $dC = p_1(\omega) - \mathrm{ch}_2(A)$

Except for the gauge bundles that quarks couple to, i.e. those appearing in Yang-Mills theory, the well known higher background fields appearing in physics tend to be abelian differential cocycles and are as such nicely described in terms of differential characters in [31]. But, as we shall see, they can be understood as arising as obstructions to lifts through nonabelian differential coycles. Another genuinely nonabelian differential cocycle appearing in string physics is the Kalb-Ramond field on an orientifold: this comes from an (ever so slightly, but still) nonabelian extension of the 2-group $\mathbf{B}U(1)$. Apart from that, there is the expectation that the worldvolume theory of a stack of D5-branes carries a nonabelian differential 2-cocycle [1]. The worldvolume theory of 5-branes remains unclear, but its compactification on a torus should yield nonabelian Yang-Mills theory.

For finding the right gluing laws for this local data, it is impoirtant to note that all these local colleciton of differential forms can be regarded from a unified point of view as L_∞-algebra valued forms:

a (finite dimensional) L_∞-algebra is a $\mathbb{N}_+$-graded vector space $\mathfrak{g}^*$ together with agraded derivation

$$d : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

on the graded symmetric tensor algebra of $\mathfrak{g}^*$ such that

$$\deg(d) = +1$$

and

$$d^2 = 0\,.$$

We write

$$\mathrm{CE}(\mathfrak{g}) := (\wedge^\bullet \mathfrak{g}^*, d_{\mathrm{CE}(\mathfrak{g})})$$

for the differential graded commutative algebra (DGCA) obtained this way, called the *Chevaley-Eilenberg*-algebra of $\mathfrak{g}$.

For every L_∞-algebra we can also form the *Weil algebra* $\mathrm{W}(\mathfrak{g})$, essentially the Chevalley-Eilenberg algebra of the mapping cone of the identity on $\mathfrak{g}$.

Now we define $\mathfrak{g}$-valued forms on Y as

$$\Omega^\bullet(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$$

and

$$\Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y))\,.$$

Now there are L_∞-algebras like $b^{n-1}\mathfrak{u}(1)$ and $\mathrm{cs}_P(\mathfrak{g})$ which have forms as follows:

- $\Omega^\bullet(Y, b^{n-1}\mathfrak{u}(1)) = \Omega^n(Y)$
- $$\Omega^\bullet(Y, \mathrm{cs}_P(\mathfrak{g})) = \{(A, B, C) \in \Omega^1(Y, \mathfrak{g}) \times \Omega^2(Y) \times \Omega^3(Y) | C = dB + \mathrm{CS}(A)\}\,.$$

Apart from the global gluing, there can also be a "local twist" to this situation, induced by the presence of *magnetic charges* (a beautiful description is in [31]).

In 4-dimensional electromagnetism the magnetic current $j_{\mathrm{mag}} \in \Omega^1(Y)$ "twists" the curvature 2-form such that it is no longer an exact form, but satisfies

$$dF_2 = \star j_{\mathrm{mag}} =: H_3 \in \Omega^3(Y)\,.$$

This happens in particular for the "electrically charged" endpoints of the string on a D-brane: there H_3 is the Kalb-Ramond field restricted to the brane, F_2 the curvature of a twisted bundle on the D-brane.

On a "stack of D-branes" there is not just a twisted line bundle with connection, but a twisted $\mathrm{PU}(H)$ bundle, whose twist obstructs the lift of the structure group of a $\mathrm{PU}(H)$-bundle through the central extension

$$1 \to U(1) \to \mathrm{U}(H) \to \mathrm{PU}(H) \to 1$$

.

One finds the same mechanism in higher dimensions: the Green-Schwarz mechanism says that for the target space theory of the heterotic string the Kalb-Ramond field is "twisted" in the above sense by magnetic 5-brane charge $\star(p_1(\omega) - \mathrm{ch}_2(A_{\mathrm{E}_8}))$

$$dH_3 = p_1(\omega) - \mathrm{ch}_2(A_{\mathrm{E}_8})\,,$$

which we recognize as the curvature of the supergravity C-field.

What lifting/obstruction problem does this come from?

The answer is: the lift of the original Spin(10) and E_8-bundle to the corresponding String-2-bundle.

- this was originally found by Killingback, expanded on by Witten, in terms of a worldsheet anomaly on the superstring, which generalizes that of the superparticle: he found that the anomaly vanishes if (disregarding the E_8-bundle now for simplicity) the Spin bundle on X lifts to a $\hat{L}$Spin-bundle on the loop space LX.

- later Stolz-Teichner [61] gave an interpretation of this in terms structures on target space X itself: the condition is that the Spin-bundle on X may be lifted to a principal bundle for the topological group String(10), which is a 3-connected cover of Spin(10) with a certain property

$$1 \to BU(1) \to \mathrm{String}_{\mathrm{top}}(10) \to \mathrm{Spin}(10) \to 1\,.$$

- Finally a differential geometric interpretation of that situation was given in [8], which showed that the topological group $\mathrm{String}_{\mathrm{top}}$ is a kind of decategorification (namely the realization of the nerve) of a smooth strict 2-group String(10)

$$1 \to \mathbf{B}U(1) \to \mathrm{String}(10) \to \mathrm{Spin}(10) \to 1\,.$$

- With the result of [9] (see also [41] and [4]) we have that String-2-bundles have the same classification as $\text{String}_{\text{top}}$-bundles. Hence the String-2-bundles do form a smooth target space realization of the desired structure

- This smooth String 2-group had been found [8, 36] by integrating a certain Lie 2-algebra (an L_∞-algebra)

$$0 \to b\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0\,.$$

String (2-)bundles exist if the first Pontryagin class vanishes. The Green-Schwarz mechanism says indeed that this class vanishes (rationally). Compare with [31].

In [52] the String extesion was described as a special case of a general class of "string like"-extensions which exist for any Lie algebra $\mathfrak{g}$ (might even be an L_∞-algebra) given a transgressive degree $(n+1)$ Lie cocycle μ on it:

$$0 \to b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0\,.$$

These are families of "higher central extensions" of L_∞-algebras. The case of ordinary central extensions such as $\mathfrak{u}(1) \to \mathfrak{u}(m) \to \mathfrak{pu}(m)$ is obtained for $n=1$.

$\text{String}(m)$ comes from the $(2+1)$-cocycle on $\mathfrak{so}(m)$.

For $\mathfrak{g} = \mathfrak{so}(m)$ the next cocycle after $n=2$ which corresponds to the first Pontryagin form is the one of degree $(n=6)+1$, which corresponds to the second Pontryagin form p_2.

In joint with Hisham Sati and Jim Stasheff [53], this is identitfied with the *dual* Green Schwarz mechanism, which regards the situation from the point of view of switching from the "electric" string to its magentic dual: the NS 5-brane. This has to couple to a 6-bundle with curvature 7-form H_7. The dual Green-Schwarz mechanism then says that this 7-form curvature is twisted by electric string charge:

$$dH_7 = p_2(TX) - ch_4(P_{E_8}) + \text{decomposables}\,.$$

By the general procedure, we find that this describes the vanishing (rationally) of the obstruction to lift the original $\text{Spin}(n)$-1-bundle to a $\text{FiveBrane}(n)$-6-bundle, where $\text{FiveBrane}(n)$ is the strict 6-group obtained from integrating the string-like Lie 6-algebra $\mathfrak{so}(n)_{\mu_3+\mu_7}$ obtained from the sum of the 3- and the 7-cocycle $\mu_3 + \mu_7$:

$$0 \to b^{\mathrm{u}}(1) \oplus b^6\mathfrak{u}(1) \to \mathfrak{so}(n)_{\mu_3+\mu_7} \to \mathfrak{so}(n) \to 0$$

or equivalently from an iterated central extension which first kills the thirs, then the 7th cocycle:

$$0 \to b^6\mathfrak{u}(1) \to (\mathfrak{g}_{\mu_3})_{\mu_7} \to \mathfrak{g}_{\mu_3} \to 0$$

We provide [52] a general formalism for describing such situations L_∞-algebraically (and then later integrate it to a full nonabelian differential cocycle):

For every $(n+1)$-cocycle μ on $\mathfrak{g}$ in transgression with an invariant polynomial P we get a string-like extension

$$0 \longrightarrow b^{n-1}\mathfrak{u}(1) \longrightarrow \mathfrak{g}_\mu \longrightarrow \mathfrak{g} \longrightarrow 0 \ .$$

shifted $\mathfrak{u}(1)$	(higher) string-like extension	original Lie (or L_∞) algebra

The inner derivations of the middle term

$$\mathrm{inn}(\mathfrak{g}_\mu) \simeq \mathrm{cs}_P(\mathfrak{g}_\mu)$$

form a Lie $(n+1)$-algebra called the *Chern-Simons* Lie $(n+1)$-algebra with respect to P. Differential forms with values in this L_∞-algebra are precisely the Chern-Simons forms

$$\Omega^\bullet(Y, \mathrm{cs}_P(\mathfrak{g}_\mu)) = \left\{ \begin{array}{c} A \in \Omega^1(Y,\mathfrak{g}) \\ B \in \Omega^n(Y) \\ C = \mathrm{CS}_P(A) + dB \end{array} \right\} .$$

The Lie ∞-algebraic version of a nonabelian differential cocycle on a space X is a choice of surjective submersion $Y \to X$ together with a diagram

$$\begin{array}{ccccc}
\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) & & \in \Omega^\bullet_{\mathrm{flat}}(Y,\mathfrak{g}) & & \text{integrates to nonabelian cocycle} \ . \\
\uparrow \qquad\qquad \uparrow & & & & \\
\Omega^\bullet(Y) \xleftarrow{(A,F_A)} \mathrm{W}(\mathfrak{g}) & & \in \Omega^\bullet(Y,\mathfrak{g}) & & \text{connection and curvature forms} \\
\uparrow \qquad\qquad \uparrow & & & & \\
\Omega^\bullet(X) \xleftarrow{\{P_i(F_A)\}} \mathrm{inv}(\mathfrak{g}) & & \in \prod_k \Omega^k_{\mathrm{closed}}(X) & & \text{characteristic forms}
\end{array}$$

Using an L_∞-algebraic obstruction theory one finds that the obstruction to lifting such a $\mathfrak{g}$-connection through a string-like extensions: is given by the diagram:

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[l] & \mathrm{CE}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[lll, bend left, "\mu(A_{\mathrm{vert}})"] \ . \\
\Omega^\bullet(Y) \arrow[u] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u] & \mathrm{W}(b\mathfrak{u}(1) \to \mathfrak{g}) \arrow[l] \arrow[u] & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[u] \arrow[lll, bend left, "(\mathrm{cs}(A,F_A),P(F_A))"] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i(F_A)\}"'] \arrow[u, hook] & \mathrm{inv}(b\mathfrak{u}(1) \to \mathfrak{g}) \arrow[l] \arrow[u, hook] & \mathrm{inv}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[u, hook] \arrow[lll, bend left, "P(F_A)"]
\end{tikzcd}$$

can integrate this by forming smooth classifying spaces and then taking path ω-groups: reproduces construction by Brylinski-McLaughlin: tells us integral topological Pontrjagin classes classifying the obstruction

3 The physics context

Before coming to our main discussion, we here indicate the general physics context in which thse questions arise.

3.1 Σ-models

We are concerned with the mathematical structure which is supposed to model the physics of *charged n-particles* usually known as *charged $(n-1)$-branes* or as *quantum field theories of Σ-model type*.

Such a Σ-model is specified by choosing

- a "space" X, called *target space*;
- a "space" (or class of such) Σ, called *parameter space* or called the *world-volume*;
- the mapping space $\mathrm{Maps}(\Sigma, X)$ called the *space of fields* or the *configuration space* or sometimes the *moduli space*;
- on target space a *differential n-cocycle* ∇, i.e. a higher generalization of a fiber bundle with connection, called the *background field*;
- a presription for how to interpret the push-forward of the the pullback $\mathrm{ev}^*\nabla$ along the projection onto Σ in the correspondence diagram

$$\begin{tikzcd} & \Sigma \times \mathrm{Maps}(\Sigma, X) \arrow[dl, "p_1"'] \arrow[dr, "\mathrm{ev}"] & \\ \Sigma & & X \end{tikzcd}$$

  called the *path integral* or the *quantization of the Σ-model*.

When the parameter space Σ is n-dimensional, one thinks of this data as encoding the physics of n-fold higher analogs of particles, "n-particles", that propagate on X. The field configuration – a map $\Sigma \to X$ – is thought of as the *trajectory* of such an n-particle in X. The common term for these n-particles is "$(n-1)$-branes", which originates in the term "membrane" for $n = 3$.

One says that the n-particle *couples* to the background field ∇ or that it is *charged under* the background field. The terminology is entirely motivated from the familiar case of ordinary electromagnetically charged $(1-)$particles: the electromagnetic background field ∇ which they couple to is modeled by a vector bundle (a line bundle in this case) with connection.

fundamental object	background field
n-particle	n-bundle
$(n-1)$-brane	$(n-1)$-gerbe

Table 2: **The two schools of counting** higher dimensional structures. Here n is in $\mathbb{N} = \{0, 1, 2, \cdots\}$.

For $n = 2$ one speaks of "strings". String theory proper is the study of those $n = 2$ Σ-models with a special restriction for what the "path integral" is allowed to be. Technically, it is required to encode a 2-dimensional superconformal field theory of central charge 15. This condition, however, is of no real relevance for our discussion here, which pertains to all Σ-models which generalize the "spinning (1-)particle".

Some of the deepest ideas concerning such Σ-models have originally been thought by Dan Freed:

The interpretation of background fields and of charges as differential cocycles is nicely described and worked out in [31, 39], where the mathematically inclined reader can find rigorous interpretations, in terms of differential cohomology, of the kinds of "background fields" and related "anomalies" in string theory which we are concerned with here.

The interpretation of quantization and of the path integral as an operation on higher categorical structures has first been explored in [29, 30]. Integration as a push-forward operation plays a promint role in recent developments by St. Stolz and P. Teichner and by M. Hopkins et al.

Taking for instance the simple toy example case where the background field ∇ is a vector bundle (without connection) and where Σ is a point, the ordinary push-forward produces the space of sections of the original vector bundle. That reproduces indeed the desired "quantization over the point" and can, following [29, 30], be regarded as the codimension 1 part of the full path integral for $n = 1$. St. Stolz and P. Teichner describe a variation of this which involves push-forward of K-theory classes to the point, which then classifies connected components of all (supersymmetric) 1-dimensional Σ-models.

This shows that, while a fully satisfactory mathematical interpretation of Σ-models is to date still an open question, a coherent picture, revolving around the correspondence 3.1, is beginning to emerge. The "higher spin-like structures" on target space X discussed here are believed to ensure the existence of the quantization step in the case that the Σ-model generalizes that describing spinning 1-particles.

3.2 Background fields

Independently of how the "background field" ∇ is modeled, it should locally be encoded by differential form data.

n-particle	background field	global model	local differential form data
$(1-)$particle	electromagnetic field	line bundle with connection/ Cheeger-Simons differential 2-character Deligne 2-cocycle	connection 1-form $A \in \Omega^1(Y)$ curvature 2-form $F_2 := dA \in \Omega^2_{\mathrm{closed}}$
string (2-particle) (1-brane)	Kalb-Ramond field	line 2-bundle with connection/ bundle gerbe with connection ("and curving") Cheeger-Simons differential 3-character Deligne 3-cocycle	connection 2-form $B \in \Omega^2(Y)$ curvature 3-form $H_3 := dB \in \Omega^3_{\mathrm{closed}}$
membrane (3-particle) (2-brane)	supergravity 3-form field	line 3-bundle with connection/ bundle 2-gerbe with connection ("and curving") Cheeger-Simons differential 4-character Deligne 4-cocycle	connection 3-form $C \in \Omega^3(Y)$ curvature 4-form $G_4 := dC \in \Omega^3_{\mathrm{closed}}$

Table 3: **Simple (abelian) examples for n-particles and the background fields they couple to**. The background fields are often addressed in terms of the symbols used for their local form data: the Kalb-Ramond field is known as the "B-field" with its "H_3 field strength" . Similarly one speaks of the "C-field" and its field strength "G_4", etc. This reflects the histroical development, where the local differential form data was discovered first and its global interpretation only much later. (See also the remark on anomalies at the beginning of **??**).

All the relevant background fields that have been considered are locally controlled by some L_∞-algebra $\mathfrak{g}$, and the local differential form data can always be considered as encoding differential forms $A \in \Omega^\bullet(Y, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g}$ [**?**]. In the case of abelian differential cocycles, these L_∞-algebras are all of the form $b^{n-1}\mathfrak{u}(1)$: the higher dimensional versions of $\mathfrak{u}(1)$.

The particular higher dimensional analogs of spin-like background fields which we are concerned with here, however, are *nonabelian*, which is the reason why we consider non-abelian differential cohomology in **??**. It so happens that the *obstructions* to their global existence – the (higher) Chern-Simons cocycles – are themselves again *abelian* differential cocycles (with local structure L_∞-algebra being $b^n\mathfrak{u}(1)$, see [**?**]), which is the reason why [31, 39] can discuss these using (ordinary) abelian differential cohomology.

For the identification of our higher spin like structures (Spin structures, String-structures, FiveBrane structure) with the vanishing of the obstruction for the existence of higher spin-like background fields (spin bundles, String 2-bundles, FiveBrane 6-bundles) we need the more general notion of nonabelian differential cohomology presented here.

Lie ∞-algebra $\mathfrak{g}$		$\mathfrak{g}$-valued forms
shifted $\mathfrak{u}(1)$	$b^{n-1}\mathfrak{u}(1)$	$\Omega^\bullet(Y, \mathfrak{g}) = \Omega^n(Y)$
Chern-Simons Lie $(n+1)$-algebra	$\mathrm{cs}_P(\mathfrak{h})$	$\Omega^\bullet_{\mathrm{flat}}(Y, \mathrm{cs}_P(\mathfrak{h})) = \left\{ \begin{array}{l} A \in \Omega^1(Y, \mathfrak{h}) \\ B \in \Omega^2(Y) \\ C \in \Omega^3(Y) \end{array} \middle\vert C = dB + \mathrm{CS}_P(A) \right\}$

Table 4: **Examples for L_∞-algebra valued forms.** Here $\mathfrak{h}$ is an ordinary Lie algebra with degree $(n+1)$-cocycle μ in transgression with an invariant polynomial P. For details see [**?**].

3.3 Charges

Just as an ordinary 1-bundle may be trivialized by a section, which one may think of as a "twisted 0-bundle", higher n-bundles may be trivialized by "higher sections" which are addressed as "twisted $(n-1)$-bundles". One says the twisted $(n-1)$-bundle is "twisted by" the corresponding n-bundle.

A beautiful description of this situation for abelian n-bundles with connection in terms of differential characters is given in [31, 39]. Twisted nonabelian 1-bundles have been studied in detail under the term "bundle gerbe modules" [12]. Twisted non-abelian 2-bundles have first been considered in [1, 41] under the name "twisted crossed module bundle gerbes". In terms of the L_∞-connections considered in [**?**] twisted n-bundles with connections are the connections for L_∞-algebras arising as mapping cone L_∞-algebra $(b^{n-1}\mathfrak{u}(1) \to \hat{\mathfrak{g}})$.

By comparing the formalism here with the situation of ordinary electromagnetism, one can identify the twisting n-bundle as encoding the presence of *magnetic charge*. This, too, is nicely explained at the beginning of [31].

Accordingly, where an untwisted $(n-1)$-bundle has a curvature n-form H_n which is closed, a twisted $(n-1)$-bundle has a curvature n-form which is "twisted by" the curvature $(n+1)$-form $G_{(n+1)}$ of the twisting n-bundle

$$dH_n = G_{n+1} \,.$$

Indeed, for a twisted $(n-1)$-bundle the curvature is locally no longer the differential of the connection, $dB_{n-1} = H_n$, but receives a contribution from the connection n-form B_n of the twisting n-bundle

$$H_n = dB_{n-1} + B_n \,.$$

The archetypical example is that of ordinary magnetic charge: as J. Maxwell discovered in the 19th century, in the presence of magnetic charge, which in four dimensions is modelled by a 3-form $H_3 = \star j_1$, the electric field strength 2-form F_2 is no longer closed

$$dF_2 = H_3 \,.$$

When Dirac later discovered at the beginning of the 20th century that H_3 has to have integral periods ("quantization of magentic charge"), the first 2-categorical structure in physics had been identified: the magnetic torsion 2-bundle / bundle-gerbe with deRham class H_3. It seems that this was first explicitly realized in [31].

The next example of this kind received such a great amount of attention that it came to be known as the initiation of the "first superstring revolution": the Green-Schwarz anomaly cancellation mechanism.

This says, as, once again, nicely explained in [31], that in the "higher gauge theory" given by the effective supergravity target space theory of the heterotic string, the supergravity C-field with curvature 4-form G_4 had to be "trivialized" by the Kalb-Ramond field with curvature 3-form H_3, or conversely the Kalb-Ramond field had to be "twisted" by the supergravity curvature 4-form:

$$dH_3 = G_4 \,.$$

Moreover, G_4 had to be the curvature of a virtual difference of two Chern-Simons 3-bundles (Chern-Simons 2-gerbes), whence, locally,

$$dH_3 = d\mathrm{CS}(\omega) - d\mathrm{CS}(A)$$

for ω and A the local connection 1-forms of a Spin and complex vector bundle, respectively and $\mathrm{CS}(-)$ denoting the corresponding Chern-Simons 3-forms. This is also known is the *Chapline-Manton* [**?**] coupling.

This "Green-Schwarz anomaly cancellation condition" can hence be read, equivalently, as saying that

- the supergravity C-field trivializes over the 10-dimensional target of the heterotic string;
- G_4 is the *magnetic 5-brane charge* which the electric heterotic string couples to;
- the Kalb-Ramond field is *twisted* by the supergravity C-field.

There is no particular reason to prefer "electric charge" over "magnetic charge": in the presence of a Riemannian structure the Hodge star dual of an "electric" field strenght H_{n+1} may have an interpretation as a field strength itself, in which case it is addressed as the "magnetic field strenght" $H_{d-n-1} := \star H_{n+1}$. Just as the original field strength H_n coupled to an "electric" n-particle, the dual field strength couples to a "magnetic" $(d-n-2)$-particle.

Such electric-magnetic duality is at the heart of what is known as "S-duality" for super Yang-Mills theory, which has recently been argued [**?**] to be the the heart of Langlands duality.

It is only for electric 1-particles in $d = 4$ dimensions that their magnetic dual is again a 1-particle. The magnetic dual of the 2-particle in 10 dimensions is the 6-particle. In other words: the magnetic dual of the string is the 5-brane.

The magnetic dual discussion of the Green-Schwarz mechanism [**?**] [**?**] leads one to consideration of a twisted 6-bundle with field strength $H_7 = \star H_3$, which is twisted by a certain 7-bundle whose field strength 8-form is a sum of two higher Chern-Simons terms plus some mixed terms [31]: $dH_7 = 2\pi \left[ch_4(A) - \frac{1}{48}p_1(\omega)ch_2(A) + \frac{1}{64}p_1(\omega)^2 - \frac{1}{48}p_2(\omega)\right.$

This is the formula which we shall address as the *dual Green-Schwarz anomaly cancellation condition*, which is the starting point of the discusssion in [53].

4 Parallel transport and functors

We describe the conception of bundles with connection in terms of their parallel transport functors, and then derive the usual description from that.

From our point of view, for instance, a vector bundle with connection is precisely a representation of the path groupoid $\mathcal{P}_1(X)$ of a space. This factors through the fundamental groupoid $\Pi_1(X)$ precisely if the vector bundle is flat.

Hence the description of connections in terms of parallel transport functors is a generalization of the well known fact that flat vector bundles with connection on a connected base space are the same as representations of the fundamental group.

$$\begin{tikzcd}
\mathcal{P}_1(X) \arrow[d] \arrow[dr, "\text{vector bundle with connection}"] & \\
\Pi_1(X) \arrow[r, dashed, "\text{flat vector bundle}"'] & \mathrm{Vect}
\end{tikzcd}$$

The point is that the description in terms of transport functors allows generalization to higher dimensional connections which is both conceptually straightforward as well as tractable and useful in applications.

4.1 Space and quantity

Lawvere [44] has given a nice fundamental formalization of the notions of *space* and *quantity* in mathematics:

Fix some category S of "test objects".

A space X with respect to S is something which may be *probed* by S: for each object U of S there is a set $X(U)$ of *plots* from U to X. Hence a space is a Set-valued functor:

$$X : S^{\mathrm{op}} \to \mathrm{Set}$$

(a presheaf on S).

A quantity A with respect to C is something which may be *coprobed* by S: for each object U of S there is a set of coplots $A(U)$ from X to U. Hence a quantity is a co-presheaf:

$$X : S \to \mathrm{Set}\,.$$

Usually one is interested in particularly nice spaces and particularly nice quantities: sheaves, etc.

We take the category SSpaces of "S-spaces" as being that of sheaves on S. This is a topos. In particular it is cartesian closed: the cartesian product of S-space X with S-space Y is

$$X \times Y : U \mapsto X(U) \times Y(U)$$

and the space of maps from the S-space Σ to the S-space X is itself an S-space, whose plots are

$$\hom(\Sigma, X) : U \mapsto S\mathrm{Space}(U \times \Sigma, X)\,.$$

4.1.1 Smooth spaces

In our concrete applications here, we take the category of test objects S to have as objects the natural numbers, and as morphism the smooth maps between Euclidean spaces:

$$S(n, m) := \mathrm{SmoothManifolds}(\mathbb{R}^n, \mathbb{R}^m)\,.$$

We take the category SSpace of smooth spaces to be that of sheaves on S.

A particularly important smooth space is that of differential forms

$$\Omega^\bullet : U \mapsto \Omega^\bullet(U)\,.$$

This yields an adjunction between smooth spaces and differential $\mathbb{N}$-graded commutative algebras

$$S\mathrm{Space} \underset{\mathrm{Hom}(-,\Omega^\bullet(-))}{\overset{\Omega^\bullet}{\rightleftarrows}} \mathrm{DGCA}\ ,$$

where

$$\Omega^\bullet(X) := S\mathrm{Space}(X, \Omega^\bullet)\,.$$

One can see that if X is an ordinary manifold, then the above notion of differential forms on X coincides with the usual one.

4.1.2 Smooth categories

Given any category SSpace with finite pullbacks, we can define categories C *internal* to S [Ehresmann]:

a smooth space of objects

$$\mathrm{Obj}(C) \in S\mathrm{Space}$$

and of morphisms

$$\mathrm{Mor}(C) \in S\mathrm{Space}$$

with source, target and identity maps

$$\mathrm{Obj}(C) \xrightarrow{\quad i \quad} \mathrm{Mor}(C) \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Obj}(C)$$

and composition

$$\begin{tikzcd}
 & & \mathrm{Mor}(C) \times_{\mathrm{Obj}(C)} \mathrm{Mor}(C) \arrow[dl] \arrow[dr] \arrow[ddd, bend left, "\circ"] & & \\
 & \mathrm{Mor}(C) \arrow[dl, "s"] \arrow[dr, "t"] & & \mathrm{Mor}(C) \arrow[dl, "s"] \arrow[dr, "t"] & \\
\mathrm{Obj}(C) & & \mathrm{Obj}(C) & & \mathrm{Obj}(C) \\
 & & \mathrm{Mor}(C) \arrow[ull, "s"] \arrow[urr, "t"] & &
\end{tikzcd}$$

smooth maps.

We write

$$S\mathrm{Cat} := \mathrm{Cat}(S)$$

for the 2-category of smooth categories, being categories internal to S. Notice that

$$S\mathrm{Cat} \subset \mathrm{Stacks\ on\ } S\,.$$

The categories internal to sheaves on S are the *rectified* stacks on S: those stacks for which the pullback morphisms happen to respect composition strictly.

Every smooth space X gives rise to the following smooth groupoids

- $\Pi_0(X) = \mathcal{P}_0(X) = \mathrm{Disc}(X)$ has X as its space of objects and no nontrivial morphisms;
- $\mathcal{P}_1(X)$ has X as its space of objects and a quotient of $PX := \hom(I, X)$ as its space of morphisms, where two paths are identitfied if there exists a homotopy between them on which every 2-form vanishes;
- $\Pi_1(X)$ as above, but dividing out homotopy.

We can further enrich in smooth categories and thereby obtain smooth n-categories and smooth ω-categories as ω-categories internal to smooth spaces. The ω-category of those smooth ω-categories we'll denote

$$S\omega\mathrm{Cat}\,.$$

That will concern us later, in 5.1.

4.1.3 Smooth superspaces and smooth supercategories

It is conceptually straightforward to generalize everything we do here to the world of supergeometry. For that we just extend our site S from Euclidean spaces and smooth maps between them to super-Euclidean spaces and smooth super-maps between them:

$$\mathrm{Obj}(S) = \mathbb{N} \times \mathbb{N}$$

$$S(n|n', m|m') = \mathrm{SmoothSupermanifolds}(\mathbb{R}^{n|n'}, \mathbb{R}^{m|m'})\,.$$

While conceptually straightforward, we shall not further consider this here. But doing so brings the present setup into close contact with the developments in [61, **?**, **?**], which is one of the motivations for our studies.

4.2 Functors and forms

Smooth functors from paths to a group G are canonically isomorphic [57] to $\mathfrak{g}$-valued differential forms

$$\mathrm{hom}(\mathcal{P}_1(X), \mathbf{B}G) \simeq \Omega^\bullet(X, \mathfrak{g})\,.$$

This is an isomorphism of categories: functors are in bijection to forms, natural transformations to gauge transformations.

If the functors factor through $\Pi_1(X)$, they correspond to *flat* forms:

$$\mathrm{hom}(\Pi_1(X), \mathbf{B}G) \simeq \Omega^\bullet_{\mathrm{flat}}(X, \mathfrak{g})\,.$$

This equivalence is established using the standard notion of parallel transport

$$A \mapsto (\gamma \mapsto P\exp(\int_{[0,1]} \gamma^* A)$$

where

$$P\exp(\int_{[0,a]} \gamma^* A)$$

is the unique solution to the differential equation

$$dF = (r_F)_* \circ (\gamma^* A)$$

$$F(0) = \mathrm{Id}\,.$$

4.3 Descent and codescent

The category

$$\mathrm{TrivBund}(G)(X) := \mathrm{hom}(\Pi_0(X), \mathbf{B}G)$$

is that of trivial G-bundles on X. There is a single object for each X and a gauge transformation is a G-valued function on X.

Let $Y \to X$ be a good cover by open subsets.

Define the category

$$\mathrm{Desc}(Y, S\mathrm{Cat}(\Pi_0(Y), \mathbf{B}G))$$

to have objects which are tuples consisting of an object

$$\mathrm{triv} \in \mathrm{hom}(\Pi_0(Y), \mathbf{B}G)$$

and a morphism

$$g : \pi_1^*\mathrm{triv} \longrightarrow \pi_2^*\mathrm{triv}$$

such that

$$\begin{tikzcd} & \pi_2^*\mathrm{triv} \arrow[dr, "\pi_{23}^*"] & \\ \pi_1^*\mathrm{triv} \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"] & & \pi_3^*\mathrm{triv} \end{tikzcd}$$

commutes. This is a G-cocycle on X.

A morphism between two such is a morphism

$$h : \mathrm{triv} \to \mathrm{triv}'$$

such that

$$\begin{tikzcd} \pi_1^*\mathrm{triv} \arrow[r, "g"] \arrow[d, "\pi_1^* h"] & \pi_2^*\mathrm{triv} \arrow[d, "\pi_1^* h"] \\ \pi_1^*\mathrm{triv}' \arrow[r, "g'"] & \pi_2^*\mathrm{triv}' \end{tikzcd}$$

commutes.

So in terms of component functions, objects in Desc are functions

$$g : Y \to G$$

satisfying the familiar cocycle condition

$$\pi_{12}^* g \pi_{23}^* g = \pi_{12}^* g \,.$$

If we replace here Π_0 with $\mathcal{P}_1$ we get a *differential* G-cocycle. If we use Π_1 we get a *flat* differential G-cocycle.

We define the the *nonabelian differential cohomology* for G any Lie group

$$\bar{H}(X, \mathbf{B}G) := \mathrm{colim}_Y \mathrm{Desc}(Y, S\mathrm{Cat}(\mathcal{P}_1(-), \mathbf{B}G)) \,.$$

The colimit is realized by the descent category for Y any good cover (by contractible open subsets all whose finite intersections are contractible).

Theorem 1 ([57]) *Differential G-cohomology is equivalent to G-bundles with connection*

$$\bar{H}(-, \mathbf{B}G) \simeq G\mathrm{Bund}_\nabla(-) \,.$$

4.4 Global transport and local trivialization

Equip the category $G\mathrm{Tor}$ of G-spaces isomorphic to G ("G-torsors over a point") with the smooth structure that regards every map as smooth.

We say that a "globally defined" functor

$$\mathrm{tra} : \Pi(X) \to G\mathrm{Tor}$$

is locally (i-)trivializable if there is

$$\begin{tikzcd}
\Pi_n(Y) \arrow[r, "\pi"] \arrow[d, "\mathrm{triv}"'] & \Pi_n(X) \arrow[d, "\mathrm{tr}"] \arrow[dl, Rightarrow, "\simeq"', "t"] \\
\mathbf{B}G \arrow[r, "i"] & G\mathrm{Tor}
\end{tikzcd}$$

Theorem 2 ([57]) *For $\Pi = \Pi_0$, the category of pairs (tra, t) is equivalent to that of G-bundles, i.e. local section admitting bundles $P \to X$ with a right G-action such that the canonical morphism $P \times G \to P \times_X P$ is an isomorphism.*

For $\Pi = \mathcal{P}_1$ the category of tra *for which there exists a t is equivalent to that of G-bundles with connection.*

4.5 Cartan-Ehresmann connections

A particularly important case is:

let $P \to X$ be the total space of a principal G-bundle, for G any Lie group. Then we can take $Y = P$ and use the principality condition to get an isomorphism

$$\pi_1^* P = P \times_G P \xrightarrow[\simeq]{t} Y \times G \ .$$

This means that the local trivialization is induced from the right action r of G on P

$$\pi^*\mathrm{tra} \xrightarrow{r^{-1}} \mathrm{triv}_i$$

as

$$P_y \xmapsto{p \mapsto r^{-1}(y,p)} G \ .$$

If we use $Y = P$ in the definition of a differential G-cocycle, we obtain

- a 1-form $A \in \Omega^1(P, \mathfrak{g})$ on the total space
- *first Cartan-Ehresmann condition*: restricted to the fibers A is the canonical flat 1-form;
- *second Cartan-Ehresmann condition*: A transforms covariantly under vertical transformations
$$L_{r_* x} A = \mathrm{ad}_x A \,.$$

In 6.4 it will be crucial to observe that these two conditions make the following diagram commute:

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] & \text{connection form restricted to fibers (flat)} \\
& & \text{first Cartan-Ehresmann condition} \\
\Omega^\bullet(Y) \arrow[uu, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[uu, two heads] & \text{connection form } A \text{ on total space with curvature } F_A \\
& & \text{second Cartan-Ehresmann condition} \\
\Omega^\bullet(X) \arrow[uu, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "{\{P_i(F_A)\}}"'] \arrow[uu, hook] & \text{characteristic forms on base space}
\end{tikzcd}$$

4.6 Outlook

- generalize to n-categories, to ω-categories
- generalize Lie algebras to Lie ∞-algebras aka L_∞-algebras.

By the two desired properties of differential cocycles, we want to describe structures which are

- locally given by differential form data;
- such that globally this data "glues" in a sensible way.

The differential form data is required to be such that it may be integrated to transport with values in a smooth ω-group. This is achieved by *L_∞-algebra valued forms.*

Just as any Lie group G has a Lie algebra $\mathfrak{g}$, Lie ω-groups are related to Lie ∞-algebras, known as L_∞-algebras.

A (finite dimensional) L_∞-algebra is a $\mathbb{N}$-graded vector space $\mathfrak{g}^*$ with nilpotent degree $+1$ differential

$$d : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^* \,.$$

The differential graded commutative algebra thus obtained is denoted $\mathrm{CE}(\mathfrak{g})$. These fit into sequences

$$\mathrm{CE}(\mathfrak{g}) \twoheadleftarrow \mathrm{W}(\mathfrak{g}) \hookleftarrow \mathrm{inv}(\mathfrak{g}) \,.$$

Now $\mathfrak{g}$-valued differential forms are morphisms from this to ordinary differential forms:

$$\Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y))$$

$$\Omega^\bullet(Y,\mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))\,.$$

Notice the relation to flat and non-flat local transport with values in ω-groups: for G a strict 2-group coming from the Lie 2-algebra $\mathfrak{g}$ (an L_∞-algebra whose generators are concentrated in the lowest two degrees), we have [58]

$$S\omega\mathrm{Cat}(\Pi_\omega(Y), \mathbf{B}G) \simeq \Omega^\bullet_{\mathrm{flat}}(Y,\mathfrak{g})$$

and

$$S\omega\mathrm{Cat}(\Pi_\omega(Y), \mathbf{BE}G) \simeq \Omega^\bullet(Y,\mathfrak{g})\,.$$

Examples for $\mathfrak{g}$-valued forms are

- ordinary differential forms:

$$\Omega^\bullet(Y, b^{n-1}\mathfrak{u}(1)) = \Omega^n(Y)$$

- Chern-Simons forms:

$$\Omega^\bullet(Y, \mathrm{cs}_P(\mathfrak{g})) = \{(A,B,C) \in \Omega^1(Y,\mathfrak{g})\times\Omega^2(Y)\times\Omega^3(Y) | C = dB + \mathrm{CS}(A)\}\,.$$

So various local differental form data which one encounters in nature can be interpreted as L_∞-algebra valued forms. Doing so allows us to work out the nature of the globally defined differential cocycles which control this local data.

Every L_∞-algebra gives rise to a smooth space $S(\mathrm{CE}(\mathfrak{g}))$, the smooth *classifying space* of $\mathfrak{g}$-valued forms:

$$\Omega^\bullet(Y,\mathfrak{g}) \simeq S\mathrm{Spaces}(Y, S(\mathrm{CE}(\mathfrak{g})))\,.$$

Every such space has a path ω-groupoid $\Pi_\omega(S\mathrm{CE}(\mathfrak{g}))$ which always has just a single object. Hence we write it as

$$\mathbf{B}G := \Pi_\omega(S\mathrm{CE}(\mathfrak{g}))$$

and interpret G as the ω-group integrating $\mathfrak{g}$ (in slight variation of the integration prescription of [33]).

We will describe "Cartan-Ehresmann connections" for L_∞-algebras and try to integrate them (in the sense of integration of Lie algebras to Lie groups), by the above procedure, to differential cocycles for ω-groups.

In doing so one finds that typically that the $\mathfrak{g}$-connection is integrated to a cocycle for a quotient ω-group G/H, where H is the "vertical holonomy group" of the $\mathfrak{g}$-connection.

For instance

$$\Pi_\omega(S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))) = \mathbf{B}^n\mathbb{R}$$

and $\mathbf{B}^{n-1}U(1)$-bundles are obtained from integration of $b^{n-1}\mathfrak{u}(1)$-connections only if the holonomy of the connection around cycles in the fiber lies in $\mathbf{B}^n\mathbb{Z}$. This condition gives rise to various integrabiliy conditions, as one would expect.

5 Nonabelian differential cohomology

In other parts of the literature the kind of structure we are after here would be referred to as an "∞-stack". An "∞'-stack" is supposed to be a type of structure, (for instance the structure "principal G-bundle"), which exists over every "space", such that all structures of this type on a given "space" are obtained from gluing such structures *locally*, i.e by gluing such structures over patches of the "space".

This "gluing" is known as *descent*. The term illustrates the fact that looking at a space X *locally* means looking at regular epimorphisms

$$\begin{tikzcd} Y \arrow[d, two heads] & \text{local structure on } Y \arrow[d, maps to, "\text{descends to}"] \\ X & \text{global structure on } X \end{tikzcd},$$

for instance by choosing a good cover by open subsets $U_i \subset X$ and setting $Y = \coprod_i U_i$.

From the point of view of this diagram, if a local structure on Y "glues" to a global structure on X it "descends" from Y "down" to X.

For very low n, n-stacks are well understood. n-Stacks of higher n, in particular $n = \infty$, is the topic of more recent research. There are several approaches to "∞-stacks"; a popular one by Jardine et al. [65] involves passing to the homotopy-category of presheaves with values in simplicial sets.

Here we instead follow Ross Street [62], who gives an explicit formula for the descent condition for any ωCat-valued presheaf.

We find that in our situation it is not only useful and convenient to restrict to structures which form ω-category valued presheaves – but also sufficient.

The reason for that is that we think of all the structures which we are interested in, G-bundles with connection, in terms of their fiber-assigning- or parallel transport n-functors. These can pulled back strictly and hence form presheaves. Even sheaves, actually.

Moreover, we obtain the smooth parallel transport functors from integrating L_∞-algebras, and the integration procedure which we use always produces *strict* n-categories.

5.1 ω-Categories

For $\mathcal{V}$ a monoidal category, we can consider $\mathcal{V}$*-enriched* categories C: a set of objects and an object $\mathrm{Hom}(a, b) \in \mathcal{V}$ for each pair of objects, together with a composition morphism

$$C(a, b) \otimes C(b, c) \to C(a, c)\,.$$

Taking $\mathcal{V} = \mathrm{Cat}$ yields strict 2-categories

$$2\mathrm{Cat} := \mathrm{Cat} - \mathrm{Cat}\,.$$

total space fibration	$\begin{array}{c} P \\ \big\downarrow{\scriptstyle p} \\ X \end{array}$	pullback by universal property	$\begin{array}{ccc} f^*P & \longrightarrow & P \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$	weak respect for composition of pullback
fiber-assigning functor	$\begin{array}{c} G\mathrm{Tor} \\ \big\uparrow \\ X \end{array}$	pullback by precomposition	$\begin{array}{ccc} & & G\mathrm{Tor} \\ & & \big\uparrow \\ Y & \xrightarrow{f} & X \end{array}$	strict respect for composition

Table 5: **The rectification of ∞-stacks of ∞-bundles in terms of fiber-assigning functors.** The crucial method which allows us to work entirely within ω-category valued sheaves without having to deal with "∞-prestacks" is that we conceive all n-bundles (P, ∇) with connection not as fibrations of their total spaces $p : P \to X$ but entirely in terms of their fiber-assigning and parallel transport-assigning functors. The equivalence of fiber-assigning functors with the total space perspective of bundles is established in [57, 59].

Taking $\mathcal{V} = 2\mathrm{Cat}$ yields strict 3-categories

$$3\mathrm{Cat} := 2\mathrm{Cat} - \mathrm{Cat}\,.$$

And so on. As a kind of limiting case we get ω-categories: globular sets with compatible strict composition operations in each direction.

Details are given in section 1.4 of [45].

ω-Categories have various nice properties.

- $\omega\mathrm{Cat}$ has a symmetric biclosed structure $\otimes_{\mathrm{Gray}}$ which generalizes the Gray tensor product on 2-categories [24];
- $\omega\mathrm{Cat}$ has a model category structure [43]: the weak equivalences (def. 6 in [43]) are those ω-functors $C \xrightarrow{F} D$ which are essentially k-surjective ([6], definition 4) for all $k \in \mathbb{N}$:

  for

  $$\begin{tikzcd} - \arrow[r, bend left, "c"] \arrow[r, bend right, "c'"'] \arrow[r, phantom, "\Downarrow \forall\delta:\exists\gamma"] & - \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} F(-) \arrow[r, bend left, "F(c)"] \arrow[r, bend right, "F(c')"'] \arrow[r, phantom, "\Downarrow \delta\simeq F(\gamma)"] & F(-) \end{tikzcd}$$

  and two parallel k-morphisms in C, F reaches every equivalence class in $D(F(c), F(c'))$.

5.2 Cohomology with coefficients in ω-category-valued (pre)sheaves

If

$$\mathbf{A} \in S\omega\mathrm{Cat}^{S\mathrm{Space}^{\mathrm{op}}}$$

is a (smooth) ω-category valued presheaf, we can consider its descent as above. Ross Street [62] gives the general formula for $\mathrm{Desc}(Y, \mathbf{A})$.

We write

$$H(-, \mathbf{A}) : S\mathrm{Space}^{\mathrm{op}} \to S\omega\mathrm{Cat}$$

for the corresponding cohomology theory (notice: itself a coefficient object).

Nonabelian differential cohomology arises from choosing as coefficent objects ω-categories of local transport functors:

- nonabelian cohomology

$$H(-, \mathbf{B}G) := H(-, S\omega\mathrm{Cat}(\Pi_0(-), \mathbf{B}G))\,;$$

- flat nonabelian differential cohomology

$$\bar{H}_{\mathrm{flat}}(-, \mathbf{B}G) := H(-, S\omega\mathrm{Cat}(\Pi_\omega(-), \mathbf{B}G))\,;$$

- fake-flat nonabelian differential cohomology

$$\bar{H}_{\mathrm{fake-flat}}(-, \mathbf{B}G) := H(-, S\omega\mathrm{Cat}(\mathcal{P}_n(-), \mathbf{B}G))\,;$$

- nonabelian differential cohomology:

$$\bar{H}(-, \mathbf{B}G) \subset \bar{H}_{\mathrm{flat}}(-, \mathbf{BE}G)\,.$$

The definition of $\mathbf{BE}G$ will concern us in 5.4.2 and 6.3. The definition of the sub-ω-category $\bar{H}(-\mathbf{B}G)$ will be given in 5.4.3.

The term "fake-flat" is adopted from [14] and arises as follows: parallel transport with values in an n-group has, in general, curvature k-forms for $k = 2, 3, \cdots, (n+1)$. In the fake-flat case all except for the degree $(n+1)$-curvature form vanish. In the flat case all of them vanish.

For abelian n-groups $G = \mathbf{B}^{n-1}U(1)$ only the top curvature form exists and hence

$$\bar{H}_{\mathrm{fake-flat}}(-, \mathbf{B}^n U(1)) = \bar{H}(-, \mathbf{B}^n U(1))\,.$$

But in general the vanishing of all the lower curvature forms places restrictions on the remaining top curvature form.

5.3 Nonabelian cohomology

We have already seen that for G an ordinary (1-)group we have

$$H(-,\mathbf{B}G) \simeq G\mathrm{Bund}(-)\,.$$

As a slight variant, notice that

$$H(-,(\mathbf{B}U)\times\mathbb{Z}) \simeq K^0(-)$$

yields K-theory.

The *abelian* special case of nonabelian cohomology is obtained by considering the structure n-groups

$$G = \mathbf{B}^n U(1)$$

this described the structures known as higher abelian bundle gerbes, higher line bundles etc.

From ordinary sheaf cohomology it follows that

$$H(-,\mathbf{B}^n U(1))_\sim \simeq H^{n+1}(-,\mathbb{Z})\,,$$

where on the right we have the ordinary $(n+1)$st singlular cohomology.

The most familiar higher nonabelian cohomology is that modeled by nonabelian gerbes: these are classified by the cohomology of automorphism 2-groups

$$G = \mathrm{AUT}(H)$$

for H an ordinary (1-)group:

$$H(-,\mathbf{B}\mathrm{AUT}(H)) \simeq H\mathrm{Gerbes}(-) \simeq \mathrm{AUT}(H)\mathrm{Bund}(-)\,.$$

Notice that not all nonabelian 2-cocycles come from automorphism 2-groups: there are (strict) 2-groups not equivalent to automorphism 2-groups. Notably the String 2-group [8] $\mathrm{String}(G)$ for G a compact, simple and simply connected Lie group. In general we have [9] for $G_{(2)}$ a "well pointed" strict topological 2-group that

$$H(-,\mathbf{B}G_{(2)}) \simeq [-, B|G_{(2)}|]\,,$$

where on the right we have homotopy classes of maps to the classifying space of the topological 1-group $|G_{(2)}|$ obtained from the geometric realization of $G_{(2)}$.

5.4 Nonabelian differential cohomology

We shall essentially define nonabelian differential cohomology as cohomology with coefficients in local parallel transport functor categories

$$\mathbf{A} = S\omega\mathrm{Cat}(\Pi_\omega(-),\mathbf{B}G)$$

An important subtlety is, however, the role played by the (higher) curvature of these differential cocycles. By itself, the cohomology theory $H(-,S\omega\mathrm{Cat}(\Pi_\omega(-),\mathbf{B}G))$ is about *flat* G-connections.

One way to understand this is that the non-flat differential cocycles should be read as the obstructions to extending ordinary G-cocycles through the embedding

$$\begin{tikzcd} \Pi_0(X) \arrow[d, hook] \\ \Pi_\omega(X) \end{tikzcd}$$

finding the dashed arrow in

$$\begin{tikzcd} \Pi_0(X) \arrow[r, "\text{bundle}"] \arrow[d, hook] & \mathbf{B}G \\ \Pi_\omega(X) \arrow[ur, dashed, "\text{flat connection}"] \end{tikzcd}$$

In general this is not possible. What is always possible is a completion

$$\begin{tikzcd} \Pi_0(X) \arrow[r, "\text{bundle}"] \arrow[d, hook] & \mathbf{B}G \arrow[d, hook] \\ \Pi_\omega(X) \arrow[ur, dashed, "\text{flat connection}"] \arrow[r, dashed, "\text{non-flat}", "\text{connection}"'] & \mathbf{BE}G \end{tikzcd},$$

where $\mathbf{BE}G$ is a certain trivializable ω-groupoid related to the universal G-bundle, to be described in 5.4.2 and 6.3.

If we can form something like the cokernel "$\mathbf{BB}G$" of the inclusion

$$\mathbf{B}G \hookrightarrow \mathbf{BE}G\,,$$

then its composition with (A, F_A) would measure the failure of the dashed morphism to exist

$$\begin{tikzcd} \Pi_0(X) \arrow[r, "\text{bundle}"] \arrow[d, hook] & \mathbf{B}G \arrow[d, hook] \\ \Pi_\omega(X) \arrow[ur, dashed, "\text{flat connection}"] \arrow[r, dashed, "\text{non-flat}", "\text{connection}"'] \arrow[dr, "\text{obstruction}", "\text{to flatness}"'] & \mathbf{BE}G \arrow[d] \\ & \mathbf{BB}G \end{tikzcd}.$$

We will see that the morphism labeled "obstruction" encodes the *characteristic* forms of the non-flat differential cocycle.

5.4.1 Codescent

Let

$$\Pi(-) = \begin{cases} \mathcal{P}_n(-) \\ \Pi_n(-) \\ \Pi_\omega(-) \end{cases}$$

We say that an ω-category

$$\Pi^Y(X)$$

is the *differential codescent* object if it co-represents differential descent in the sense that

$$\mathrm{Desc}(Y, S\omega\mathrm{Cat}(\Pi(-), \mathbf{B}G)) \simeq S\omega\mathrm{Cat}(\Pi^Y(X), \mathbf{B}G)\,.$$

Forming codecent is analogous to forming the codiagonal of the bisimplicial set $|\Pi(Y^\bullet)|$ (see p. 31 [38] for a nice explicit description of the codiagonal of bisimplicial sets): the k-morphisms in $\Pi^Y(X)$ are generated from the k-morphisms in $\Pi(Y)$, the $(k-1)$-morphisms in $\Pi(Y^{[2]})$, the $(k-2)$-morphisms in $\Pi(Y^{[3]})$, etc., modulo some relations.

An explicit description of $\mathcal{P}_1^Y(X)$ and of $\mathcal{P}_2^Y(X)$ is in [57] and [59], respectively.

The corepresenting transport

$$\Pi^Y(X) \xrightarrow{(\mathrm{triv}, g)} \mathbf{B}G$$

encodes the formula for volume transport in terms of local data. If we restrict to $G = \mathbf{B}^{n-1}U(1)$ it reproduces the familiar formulas for bundle $(n-1)$-gerbe holonomy [**?**].

[picture goes here]

5.4.2 Non-flat local transport

Before giving the full definition of non-flat nonabelian differential cocycles, it is instructive to consider the local case for $n = 1, 2$.

Denote by

$$\mathbf{E}G = G//G = \{\, g \xrightarrow{h} hg \,|g, h \in G\}$$

the codiscrete groupoid over G. An old theorem by Segal says that the geometric realization of $\mathbf{E}G$ is the universal G-bundle:

$$\begin{tikzcd}
G \arrow[r, mapsto, "|\cdot|"] \arrow[d] & G \arrow[d] \\
\mathrm{E}G \arrow[r, mapsto, "|\cdot|"] \arrow[d] & EG \arrow[d] \\
\mathbf{B}G \arrow[r, mapsto, "|\cdot|"] & BG
\end{tikzcd}\,.$$

The point to notice here is that $\mathbf{E}G$ is itself actually a 2-group, hence $\mathbf{BE}G$ exists.

One finds that transport functors from $\Pi_2(Y)$ with values in $\mathbf{BE}G$ come from non-flat forms

$$S2\mathrm{Cat}(\Pi_2(Y), \mathbf{BE}G) \simeq_{\mathrm{Set}} \Omega^1(Y, \mathfrak{g})\,.$$

Here it is important that this is an isomorphism at the level of sets, not as categories. The failure of this to be an equivalence of categories is due to the expression on the left hand now having more ismorphisms than the expected gauge transformations: it now also contains transformations that *shift* the connection form

$$A' = gAg^{-1} + gdg^{-1} + a$$

for $a \in \Omega^1(Y, \mathfrak{g})$.

The analog for a strict 2-group $G_{(2)}$ is described in [51]. There is

$$\mathbf{E}G_{(2)} := \mathrm{INN}_0(G_{(2)})$$

and as shown in [David Roberts, PhD thesis], [9], this again yields the universal $G_{(2}$-bundle in that

$$\begin{tikzcd} G_{(2)} \arrow[r, mapsto, "|\cdot|"] \arrow[d] & {|G_{(2)}|} \arrow[d] \\ \mathrm{E}G_{(2)} \arrow[r, mapsto, "|\cdot|"] \arrow[d] & EG_{(2)} \arrow[d] \\ \mathbf{B}G_{(2)} \arrow[r, mapsto, "|\cdot|"] & BG_{(2)} \end{tikzcd} .$$

One subtlety is that the $\mathbf{BE}G$ from [51] is not, in general, an ω-groupoid, but an $(\omega\mathrm{Cat}, \otimes_{\mathrm{Gray}})$-groupoid [24]. One finds that $(\omega\mathrm{Cat}, \otimes_{\mathrm{Gray}}$-functors from $\mathcal{P}_2(x)$ to this $\mathbf{BE}G$ are in bijection with arbitrary (not-necessarily flat) $\mathfrak{g}_{(2)}$-valued forms, where $\mathfrak{g}_{(2)} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ is the Lie 2-algebra corresponding to $G_{(2)}$.

$$\hom(\Pi_\omega(Y), \mathbf{BE}G_{(2)}) \simeq_{\mathrm{Set}} \{(A, B) \in \Omega^1(Y, \mathfrak{g}) \times \Omega^2(Y, \mathfrak{h})\, .\}$$

This is the local data obtained in [14].

There should be a way to set up all we want to say here in terms of $(\omega\mathrm{Cat}, \otimes_{\mathrm{Gray}})$-categories. But for the time being we want to stick with just ω-categories.

To still be able to talk about $\mathbf{BE}G$ we give an alternative definition below using integration of L_∞-algebras. We will find that we can keep $\mathbf{BE}G$ a strict ω-category at the price of replacing G with slightly more "puffed up" but weakly equivalent incarnation.

5.4.3 Non-flat differential cocycles

Below in 6.3 we obtain ω-groups G from integration of L_∞-algebras which always come in a sequence

$$\mathbf{B}G \hookrightarrow \mathbf{BE}G \twoheadrightarrow \mathbf{BB}G \ .$$

A simple special example is $\mathbf{BG} = \mathbf{B}^n U(1)$, the ω-groupoid trivial everywhere except in degree n, where it has $U(1)$ as its space of n-morphisms. Then

$\mathbf{BEB}^{n-1}U(1)$ has $U(1)$ as its n- and its $(n+1)$-morphisms and $\mathbf{BBB}^{n-1}U(1)$ as its space of $(n+1)$-morphisms.

For such a situation we define the nonabelian differential cohomology with values in G to be the joint pullback

$$
\begin{tikzcd}
 & & \bar{H}(-,\mathbf{B}G) \arrow[dll] \arrow[d] \arrow[drr] & & \\
H(-,\mathbf{B}G) \arrow[dr] & & \bar{H}_{\mathrm{flat}}(-,\mathbf{BE}G) \arrow[dl] \arrow[dr] & & \Omega^{\bullet}_{\mathrm{flat}}(-,\mathbf{BB}G) \arrow[dl] \\
 & H(-,\mathbf{BE}G) & & \bar{H}_{\mathrm{flat}}(-,\mathbf{BB}G) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \begin{matrix}G\text{-bundle}\\ \text{with connection}\\ \text{and curvature}\end{matrix} \arrow[dll] \arrow[d] \arrow[drr] & & \\
G\text{-bundle} \arrow[dr] & & \begin{matrix}G\text{-valued}\\ \text{connection}\\ \text{and curvature}\end{matrix} \arrow[dl] \arrow[dr] & & \begin{matrix}\text{characteristic}\\ \text{forms}\end{matrix} \arrow[dl] \\
 & {} & & \begin{matrix}\text{characteristic}\\ \text{classes}\end{matrix} &
\end{tikzcd}
$$

The various morphisms here are defined in terms of the codescent objects $\Pi^Y(X)$ and $\Pi_0^Y(X)$:

the above pullback says that the cocycles in $\bar{H}(-,\mathbf{B}G)$ are the cocycles

$$\Pi^Y(X) \longrightarrow \mathbf{BE}G$$

in $H^{\Pi}(-,\mathbf{BE}G)$ which fit into a square

$$
\begin{tikzcd}
\Pi_0^Y(X) \arrow[r] \arrow[d, hook, "i"] & \mathbf{B}G \arrow[d] & \text{G-cocycle} \\
\Pi^Y(X) \arrow[r] \arrow[d, two heads, "\pi"] & \mathbf{BE}G \arrow[d, hook] & \text{connection and curvature} \\
\Pi(X) \arrow[r] & \mathbf{BBG} & \text{characteristic forms}
\end{tikzcd}
\,.
$$

The bottom morphism represents $\mathbf{BB}G$-valued forms. Precomposition of that with the lower left vertical arrow π is the map

$$\Omega^\bullet(-, \mathbf{BB}G) \to H^\Pi(-, \mathbf{BB}G)\,.$$

Postcomposition with the lower right vertical morphism is the map

$$H^\Pi(-, \mathbf{BE}G) \to H^\Pi(-, \mathbf{BB}G)\,.$$

Precomposition with the upper left vertical morphism i is the map

$$H^\Pi(-, \mathbf{BE}G) \to H(-, \mathbf{BE}G)\,.$$

Finally, postcomposition with the upper right vertical morphism is the map

$$H(-, \mathbf{B}G) \to H(-, \mathbf{BE}G)\,.$$

The simplest example is obtained for $G = U(1)$, where the diagram reproduces the familiar description of a $U(1)$-bundle with connection, the lower morphism $\Pi_\omega(X) \to \mathbf{BB}U(1)$ being its curvature 2-form.

6 Lie ∞-algebras and integration

Lie ∞-algebras are L_∞-algebras. A review and literature is given at the beginning of [52].

One expects an analog of Lie's theorem relating Lie ∞-groups and Lie ∞-algebras.

It seems to be an old result about the integration theory of Lie algebras (Ezra Getzler told me in private communication that it is the way he originally learned about integration of Lie algebras from Bott) that the simply connected Lie group G integrating an ordinary Lie algebra $\mathfrak{g}$ is, as a space, that of equivalence classes of $\mathfrak{g}$-valued 1-forms on the interval, where two such are regarded as equivalent if they can be interpolated by a *flat* $\mathfrak{g}$-valued 1-form over the disk. The group multiplication is then just compoisition of intervals.

I am not sure what the canonical reference for this result is. It does appear, though, as a corollary of the work on the integration of Lie algebroids. See for instance [23] for technical details and [54] for the central idea.

In any case, we observe here (an obvious statement, which however seems not to have been stated this way in the corresponding literature) that this says nothing but that the Lie group is the fundamental path groupoid of the smooth classifying space $S(\mathrm{CE}(\mathfrak{g}))$ of $\mathfrak{g}$-valued 1-forms:

$$\mathbf{B}G = \Pi_1(S(\mathrm{CE}(\mathfrak{g})))\,.$$

Here $\mathrm{CE}(\mathfrak{g})$ is the Chevalley-Eilenberg algebra of $\mathfrak{g}$ and $S(\mathrm{CE}(\mathfrak{g}))$ the smooth space whose plots on U are the flat $\mathfrak{g}$-valued forms on U. (More details below).

We adopt the point of view that "Lie groupoid" means groupoid internal to SSpaces. Hence a Lie ω-groupoid is, for us, just another name for a smooth ω-groupoid.

In that we differ from other authors, who try to integrate to n-groupoids internal to manifolds [33] or Banach spaces [36].

6.1 Lie ∞-algebras

A finite dimensional L_∞-algebra is a finite dimensional $\mathbb{N}_+$-graded vector space $\mathfrak{g}^*$ together with a differential

$$d : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

of degree $+1$ such that $d^2 = 0$.

We write

$$\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d)$$

for the corresponding differential graded-commutative algebra and call it the *Chevalley-Eilenberg-algebra* of $\mathfrak{g}$.

For every such we also get the Weil algebra

$$\mathrm{W}(\mathfrak{g}) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1], d = \begin{pmatrix} \sigma & 0 \\ d_{\mathrm{CE}(\mathfrak{g})} & -\sigma \circ d_{\mathrm{CE}(\mathfrak{g})} \circ \sigma^{-1} \end{pmatrix})$$

as well as the DGCA of basic or invariant form

$$\mathrm{inv}(\mathfrak{g}) \subset \mathrm{W}(\mathfrak{g})$$

which is the joint kernel of all contractions ι_x along the unshifted copy $\mathfrak{g}$ and all corresponding Lie derivatives $[d_{\mathrm{W}(\mathfrak{g})}, \iota_x]$.

These fit into a sequence

$$\mathrm{CE}(\mathfrak{g}) \twoheadleftarrow \mathrm{W}(\mathfrak{g}) \hookleftarrow \mathrm{inv}(\mathfrak{g})\ .$$

- For $\mathfrak{g}$ an ordinary Lie algebra $\mathrm{CE}(\mathfrak{g})$ is the DGCA of left-invariant differential forms on G.
- It was known to Cartan that $\mathrm{W}(\mathfrak{g})$ plays the role of differential forms of EG.
- And for G compact, it is a standard fact that $\mathrm{inv}(\mathfrak{g}) \simeq H^\bullet(BG, \mathbb{R})$.

That means the above sequence is to be thought of as that of differential forms on the universal G-bundle.

6.2 Lie ∞-algebra valued forms

For $\mathfrak{g}$ any L_∞-algebra and Y any smooth space, we say that

$$\Omega^\bullet(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCA}s}(\mathrm{W}(\mathfrak{g}), \Omega^\bullet(Y))$$

is the collection of $\mathfrak{g}$-valued differential forms, while

$$\Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCA}s}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y))$$

is the collection of *flat* $\mathfrak{g}$-valued differential forms.

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{g}) \arrow[d, dashed, "(A,0)"] & \mathrm{W}(\mathfrak{g}) \arrow[l, two heads] \arrow[d, "(A,F_A)"] \\ \Omega^\bullet(Y) \arrow[r, "="] & \Omega^\bullet(Y) \end{tikzcd} .$$

Using the ambimorphic object $\Omega^\bullet$, a smooth space which is also a DGCA, we get an adjunction between smooth spaces and DGCAs

$$\begin{tikzcd} S\mathrm{Spaces} \arrow[r, shift left, "\Omega^\bullet(-)"] & \mathrm{DGCAs} \arrow[l, shift left, "S(-):=\mathrm{Hom}(-,\Omega^\bullet(-))"] \end{tikzcd} .$$

The functor $S : \mathrm{DGCAs} \to S\mathrm{Spaces}$ sends every DGCA A to a smooth space whose algebra of differential forms is essentially given by that DGCA: we have an inclusion

$$A \hookrightarrow \Omega^\bullet(S(\mathrm{CE}(\mathfrak{g}))) \ ,$$

the unit of the adjunction. (In rational homotopy theory the analogous injection (with our smooth spaces replaced by simplicial spaces) is shown to be an isomorphism in cohomology. I suspect the same is true here, but have no proof yet.)

This smooth space is the *classifying space for $\mathfrak{g}$-valued forms*:

the adjunction says that

$$\mathrm{Hom}_{\mathrm{DGCA}s}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y)) \simeq \mathrm{Hom}_{S\mathrm{Spaces}}(Y, S(\mathrm{CE}(\mathfrak{g})))$$

and hence that

$$\Omega^\bullet(Y, \mathfrak{g}) \simeq S\mathrm{Spaces}(Y, S(\mathrm{W}(\mathfrak{g})))$$

and

$$\Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) \simeq S\mathrm{Spaces}(Y, S(\mathrm{CE}(\mathfrak{g}))) .$$

Using the fact that we have the inner hom in smooth spaces, this yields a *smooth space of* $\mathfrak{g}$*-valued differential forms.*

6.3 Integration of Lie ∞-algebras to Lie ω-groups

From every L_∞-algebra we can first pass to its Chevalley-Eilenberg DGCA $\mathrm{CE}(\mathfrak{g})$ (which we have essentially identified with the L_∞-algebra, conceptually) and then to the smooth space $S(\mathrm{CE}(\mathfrak{g}))$ which contains $\mathrm{CE}(\mathfrak{g})$ in its algebra of differential forms. From this space, finally, we obtain an ω-groupoid: its fundamental ω-groupoid. Since there is only the trivial $(p \geq 1)$-form on $\mathbb{R}^0$, this ω-groupoid necessarily has a single object. Hence we can identify it with an ω-group. That's the ω-group G integrating $\mathfrak{g}$:

$$\mathbf{B}G := \Pi_\omega(S(\mathrm{CE}(\mathfrak{g}))) .$$

Notice that this is just a slight ω-categorical variant of the Kan simplicial construction of [33, 36]: these authors consider for any $\mathfrak{g}$ the simplical space

$$(\int \mathfrak{g})^n := \Omega^\bullet_{\mathrm{flat}}(\Delta^n, \mathfrak{g})$$

of flat $\mathfrak{g}$-valued forms on the standard n-simplex Δ^n. But we have seen that we can reexpress this as

$$(\int \mathfrak{g})^n = \mathrm{Hom}(\Delta^n, S(\mathrm{CE}(\mathfrak{g})))$$

so that it turns out that $\int \mathfrak{g}$ is nothing but the simplicial space of singular simplices in the classifying space $S(\mathrm{CE}(\mathfrak{g}))$. The simplicial space of singular simplices of any space X deserves to be addressed as the ∞-path groupoid of X:

$$\Pi_\infty(X) := (n \mapsto \mathrm{Hom}(\Delta^n, X)) .$$

(Here "∞-groupoids = Kan complexes".)

If $\mathfrak{g}$ is a Lie n-algebra, it is often convenient to truncate and form $\Pi_n(-)$ instead of $\Pi_\omega(-)$. The results should be weakly equivalent.

Given any L_∞-algebra $\mathfrak{g}$, we address the image under $\Pi_\omega \circ S$ of

$$\mathrm{CE}(\mathfrak{g}) \twoheadleftarrow \mathrm{W}(\mathfrak{g}) \hookleftarrow \mathrm{inv}(\mathfrak{g})$$

as

$$\mathbf{B}G \hookrightarrow \mathbf{BE}G \twoheadrightarrow \mathbf{BB}G \ .$$

One can unravel this abstract nonsense in some concrete examples:

for $\mathfrak{g}$ an ordinary Lie algebra we find that 1-morphisms in $\Pi_1(S(\mathrm{CE}(\mathfrak{g})))$ are $\mathfrak{g}$-valued 1-forms $A \in \Omega^1(I, \mathfrak{g})$ on the interval, two of which are taken to be equivalent if they are interpolated by a flat $\mathfrak{g}$-valued 1-form on the disk:

$$[A \in \Omega^1(I, \mathfrak{g})] = [A' \in \Omega^1(I, \mathfrak{g})] \ \Leftrightarrow \exists \tilde{A} \in \Omega^1_{\mathrm{flat}}(D^2, \mathfrak{g}) : (\tilde{A}|_{\partial_1 D^2}, \tilde{A}|_{\partial_2 D^2}) = (A, A') \, .$$

This is known to indeed yield the simply connected group G integrating $\mathfrak{g}$

$$G = \{[A \in \Omega^1(I, \mathfrak{g})]\} \, .$$

We could have formed the 2-groupoid $\Pi_2(S(\mathrm{CE}(\mathfrak{g})))$ instead, which yields a "puffed up" version of the group G: now 1-morphisms are just thin-homotopy classes of paths in G, starting at the identity, and 2-morphisms are homotopy classes of disks in D interpolating between two paths with the same endpoint.

Here we are using the fact that a flat $\mathfrak{g}$-valued 1-form A on a contractible space Y is the same as a choice of point in Y and a functor $g : Y \to G$, using $A = g d g^{-1}$.

Since $\pi_2(G) = 1$ we get

$$\Pi_2(S(\mathrm{CE}(\mathfrak{g}))) = \mathbf{B}(\Omega G \to PG) \, .$$

Compare with [8].

In a similar manner the String Lie 2-algebra $\mathfrak{g}_\mu$ for $\mathfrak{g}$ a semisiple Lie algebra and $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ the canonical 3-cocycle is integrated (compare [36]): choose the normalization of μ such that it yields the integral 3-form representing $H^3(G, \mathbb{Z})$ for the compact, simple, simply connected group G.

Then 1-morphisms in $\Pi_2(S(\mathrm{CE}(\mathfrak{g}_\mu)))$ are thin homotopy classes of path in G, starting at the identity. Thin homotopy classes of 2-paths in $S(\mathrm{CE}(\mathfrak{g}_\mu))$ are disks in G as before, but now equipped with a 2-form B on the disk, of which only the integral $\int_{D^2} B$ survises dividing out thin homotopy.

A non-thin homotopy between a pair $(g : D^2 \to G, \int B)$ and a pair $(g' : D^2 \to G, \int B')$ is an extension

$$\tilde{g} : D^3 \to G$$

such that

$$\int B - \int B' = \int_{D^3} \tilde{g}^* \mu \, .$$

We recognize the construction of the "tautological bundle gerbe on G" which is the central extension of the loop group. Hence

$$\Pi_2(S(\mathrm{CE}(\mathfrak{g}_\mu))) = \mathbf{B}(\hat{\Omega} G \to PG) =: \mathbf{B}\mathrm{String}(G) \, .$$

This is essentially the integration found in [8], only that the horizontal composition is now by concatenation of paths in G. This reproduces actually the construction in [19, 20]

6.4 ∞-Cartan-Ehresmann connections

In the special case where we have a 1-group G which is 1-connected (i.e. simply connected) and a principal G-bundle $P \to X$ it so happens that, setting $Y = P$, we have that the Čech groupoid for the canonical trivialization

$$\Pi_0^P(X) = \Pi_1^{\mathrm{vert}}$$

is nothing but the vertical part of the fundamental groupoid of P. Similar statements apply to n-group cocycles over surjections Y which are n-connected.

Motivated by this operation we define, for any L_∞-algebra $\mathfrak{g}$ and regular epimorphism $Y \to X$ a $\mathfrak{g}$-connection descent object over X to be a diagram as on the right [52]:

$$\begin{tikzcd}
\Pi_0^Y(X) \arrow[r] \arrow[d, hook, "i"] & \mathbf{B}G \arrow[d] & \text{G-cocycle} & \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
 & & \text{first Cartan-Ehresmann condition} & & \\
\Pi^Y(X) \arrow[r] \arrow[d, two heads, "\pi"] & \mathbf{BE}G & \text{connection and curvature} & \Omega^\bullet(Y) \arrow[uu, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[uu, two heads] \\
 & & \text{second Cartan-Ehresmann condition} & & \\
\Pi(X) \arrow[r] & \mathbf{BBG} \arrow[from=uu, hook] & \text{characteristic forms} & \Omega^\bullet(X) \arrow[uu, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "{\{P_i(F_A)\}}"'] \arrow[uu, hook]
\end{tikzcd}$$

nonabelian differential G-cocycle $\qquad\qquad$ $\mathfrak{g}$-connection descent object

The $\mathfrak{g}$-connection descent objects (or just "$\mathfrak{g}$-connections", for short) on the right have the advantage that their Lie ∞-algebraic formulation lends itself to concrete computations and constructions [52]. But, due to the differential nature, certain integral and torsion phenomena may not be manifest or may even be missed by the $\mathfrak{g}$-connection itself.

In the next section we indicate how to identitfy if a certain $\mathfrak{g}$-connection satisfies the proper "integrability conditions" and how to integrate it to a full nonabelian differential cocycle if it does.

6.5 Twisted L_∞-connections

The concept of a vector bundle twisted by a 2-bundle ($\simeq$ gerbe) is, by now, familiar. It leads, notably, to twisted K-theory. In the context of bundle gerbes, such twisted bundles are usually addressed as *gerbe modules*. Twisted 2-bundles, i.e. modules for bundle 2-gerbes, have been defined analogously.

Here we describe the notion of n-bundles twisted by an $(n+1)$-bundle in the context of our notion of L_∞-algebra connections. As one application, we interpret the Green-Schwarz mechanism in heterotic String theory as saying that the Kalb-Ramond field (a 2-bundle with connection) is twisted, in this sense, by the supergravity C-field restricted to the end-of-the-world 9-brane.

6.5.1 Twisted L_∞-connections

In [52] we had discussed that the obstruction to lifting a $\mathfrak{g}$-connection

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i\}"'] \arrow[u, hook]
\end{tikzcd}$$

through a String-like central extension

$$0 \to b^{n-1}\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0$$

is the $b^n\mathfrak{u}(1)$-connection obtained by canonically completing this diagram to the right as shown in figure 1.

The construction crucially involves first forming the lift of the $\mathfrak{g}$-connection to a $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$-connection, where $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$ is the "weak cokernel" or "homotopy quotient" of the injection of $b^{n-1}\mathfrak{u}(1)$ into $\mathfrak{g}_\mu$. This lift through the homotopy quotient always exists, since the homotopy quotient is in fact equivalent to just $\mathfrak{g}$. But performing the lift to the homotopy quotient also extracts the failure of the underlying attempted lift to $\mathfrak{g}_\mu$ proper. This failure may be projected out under

$$(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \twoheadrightarrow b^n\mathfrak{u}(1)$$

to yield the $b^n\mathfrak{u}(1)$-connection which obstructs the lift. It is the morphism denoted f^{-1} in 1 which picks up the information about the twist/obstruction. This was constructed in proposition 40 of [52].

However, the $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$-connection itself deserves to be considered in its own right: this is just the L_∞-connection version of "twisted bundles" or "gerbe modules".

In particular, the obstruction problem can also be read the other way round:

ordinary $\mathfrak{g}$-connection	attempted lift to $\mathfrak{g}_\mu$-connection	obstructing $b^n\mathfrak{u}(1)$-connection	**obstruction interpretation**
ordinary $\mathfrak{g}$-connection	twisted $\mathfrak{g}_\mu$-connection	twisting $b^n\mathfrak{u}(1)$-connection	**twisting interpretation**
ordinary $\mathfrak{g}$-connection	twisted $\mathfrak{g}_\mu$-connection	magnetic charge	**charge interpretation**

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[l] & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[u, two heads] & \mathrm{W}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[l, "f^{-1}"'] \arrow[u] & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i\}"'] \arrow[u, hook] & \mathrm{inv}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[l] \arrow[u, hook] & \mathrm{inv}(b^n\mathfrak{u}(1)) \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

Figure 1: **Obstructing $b^n\mathfrak{u}(1)$ $(n+1)$bundles and "twisted" $\mathfrak{g}_\mu$ n-bundles** are two aspects of the same mechanism: the $(n+1)$-bundle is the obstruction to "untwisting" the n-bundle. The n-bundle is "twisted by" the $(n+1)$-bundle. There may be many non-equivalent twisted n-bundles corresponding to the same twisting $(n+1)$-bundle. We can understand these as forming a collection of n-*sections* of the $(n+1)$-bundle.

given a $b^n\mathfrak{u}(1)$-bundle, we may ask for which $\mathfrak{g}$-bundles it is the obstruction to lifting these to a $\mathfrak{g}_\mu$-bundle. In string theory, this is actually usually the more natural point of of view:

- given the Kalb-Ramond background field (a $b\mathfrak{u}(1)$-connection) pulled back to the worldvolume of a D-brane, the "twisted $U(H)$-bundles" corresponding to it are the "Chan-Paton bundles" supported on that D-brane;
- given the supergravity 3-form field (a $b^2\mathfrak{u}(1)$-connection) pulled back to the end-of-the-world 9-branes, the "twisted $BU(1)$-2-bundle" corresponding to it is the Kalb-Ramond field, with the twist giving the failure of its 3-form curvature to close
$$dH_3 = G_4\,.$$

6.5.2 Ordinary twisted bundles in terms of L_∞-connections

Let $\mathfrak{g}$ be a Lie algebra with 2-cocycle $\mu \in \mathrm{CE}(\mathfrak{g})$ which induces a central extension

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{u}(1) \arrow[r] & \hat{\mathfrak{g}} \arrow[r] \arrow[d, dash, "="] & \mathfrak{g} \arrow[r] & 0 \\ & & \mathfrak{g}_\mu & & \end{tikzcd}$$

which we can think of as a special case of our "string-like" central extensions, according to the first example in section 6.4.1 of [52].

Then the weak cokernel Lie 2-algebra

$$(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})$$

is in fact a special case of a strict Lie 2-algebra as in the third example of 6.1.1 in [52]. Accordingly, the following discussion is really a special case of the kind of computations shown in section 6.3.1 of [52]. But it deserves to be spelled out for the present case in detail here.

By inspection, one finds that forms on Y with values in $(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})$ have the following characterization, as displayed in figure 6.5.2:

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}}) \arrow[d, "{\begin{smallmatrix} A \in \Omega^1(Y,\hat{\mathfrak{g}}) \\ B \in \Omega^2(Y) \\ (F_A)^a = 0 \\ (F_A)^0 = B \\ dB = 0 \end{smallmatrix}}"'] & & \mathrm{W}(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}}) \arrow[ll, two heads] \arrow[d, "{\begin{smallmatrix} A \in \Omega^1(Y,\hat{\mathfrak{g}}) \\ \beta \in \Omega^1(Y,\hat{\mathfrak{g}}) \\ B \in \Omega^2(Y) \\ C \in \Omega^3(Y) \\ (F_A)^a = \beta^a \\ (F_A)^0 - B = \beta^0 \\ (d_A\beta)^0 = C \end{smallmatrix}}"] \\ \Omega^\bullet(Y) \arrow[rr, dash, "="] & & \Omega^\bullet(Y) \end{tikzcd}$$

let, as usual $\{t^a\}$ be a chosen basis of $\mathfrak{g}^*$ and let t^0 denote the canonical basis of the central part of $\hat{\mathfrak{g}}$. Then a $(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})$-descent object

$$\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})$$

is a $\mathfrak{g}$-descent object whose failure to be a $\hat{\mathfrak{g}}$-descent object is measured by a closed vertical 2-form B.

Analogously, a $(\mathfrak{u}(1) \hookrightarrow \hat{\mathfrak{g}})$-connection descent object has a $\hat{\mathfrak{g}}$-valued curvature 2-form β whose $\mathfrak{g}$-valued part satisfies the ordinary $\mathfrak{g}$-Bianchi identity, $(d_A\beta)^a = 0$, but whose central part satisfies $(d_A\beta)^0 = C$, for C the curvature 3-form of the twisting 2-bundle.

This phenomenon is sometimes addressed as a *failure of the Bianchi identity*, but of course it is just the Bianchi identity of a 1-connection which fails, while what we see is actually the Bianchi identity of a 2-connection.

6.5.3 Twisted 2-bundles

Now for μ a 3-cocycle on $\mathfrak{g}$, repeating the above for the String-extension

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{u}(1) \arrow[r] & \hat{\mathfrak{g}} \arrow[r] \arrow[d, dash, "="] & \mathfrak{g} \arrow[r] & 0 \\ & & \mathfrak{g}_\mu & & \end{tikzcd}$$

yields a "twisted 2-bundle" whose 3-form crvature H_3 suffers an similar "failure of the Bianchi identity"

$$dH_3 = G\,.$$

7 Integration of L_∞-connections to full nonabelian differential cocycles

For $\mathfrak{g}$ an L_∞-algebra, a $\mathfrak{g}$-connection descent object (or just "$\mathfrak{g}$-connection", for short) on a smooth space X is a choice of regular epimorphism $\pi : Y \to X$ together with a diagram

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u, two heads] \\ \Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i(F_A)\}"'] \arrow[u, hook] \end{tikzcd}$$

of DGCAs.

A non-flat nonabelian differential cocycle, on the other hand, for a smooth ω-group $\mathbf{B}G$ obtained as a quotient by a discrete ω-group of the ω-group integrating $\mathfrak{g}$ is diagram

$$\begin{tikzcd} \Pi_0^Y(X) \arrow[r, "g"] \arrow[d, hook] & \mathbf{B}G \arrow[d, hook] & \text{nonabelian cocycle} \\ \Pi_\omega^Y(X) \arrow[r] \arrow[d, two heads] & \mathbf{BE}G \arrow[d, two heads] & \text{connection and curvature data} \\ \Pi_\omega(X) \arrow[r] & \mathbf{BB}G & \text{characteristic forms} \end{tikzcd}$$

Here

$$\Pi_0^Y(X) := (\ Y^{[2]} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} Y\ )\,.$$

is the codescent or Čech (ω-)groupoid coming from Y and

$$\mathbf{B}G \to \mathbf{BE}G \to \mathbf{BB}G$$

is the integration of

$$\mathrm{CE}(\mathfrak{g}) \leftarrow \mathrm{W}(\mathfrak{g}) \leftarrow \mathrm{inv}(\mathfrak{g})$$

We want to extend the integration procedure from L_∞-algebras to ω-groups to one that takes an entire $\mathfrak{g}$-connection and integrates it to a full nonabelian differential cocycle.

7.1 Recalling the integration procedure for L_∞-algebras

We have seen, as a slight variation on the theme investigated by Getzler and Henriques, how an L_∞-algebra $\mathfrak{g}$ may be integrated into a smooth ω-group by first forming the smooth classifying space $S(\mathrm{CE}(\mathfrak{g}))$ of flat $\mathfrak{g}$-valued forms, which is the sheaf that assigns to each smooth test domain U the set of $\mathfrak{g}$-valued forms on U:

$$S(\mathrm{CE}(\mathfrak{g})) : U \mapsto \Omega^\bullet_{\mathrm{flat}}(U,\mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCAs}}(\Omega^\bullet(U), \mathrm{CE}(\mathfrak{g}))$$

and then forming the smooth fundamental ω groupoid of that

$$\mathfrak{g} \mapsto \mathbf{B}G := \Pi_\omega(S(\mathrm{CE}(\mathfrak{g})))$$

or just the fundamental n-groupoid

$$\Pi_n(S(\mathrm{CE}(\mathfrak{g})))$$

if $\mathfrak{g}$ is a Lie n-algebra.

For instance, for $\mathfrak{g}$ an ordinary finite dimensional Lie algebra, the simply connected Lie group G integrating it is given by

$$\mathbf{B}G = \Pi_1(S(\mathrm{CE}(\mathfrak{g})))\,,$$

where $\mathbf{B}G$ always denotes the one-object groupoid version of a group G.

As familiar from the case of ordinary Lie algebras, the ω-group obtained this way tends to behave like a "simply connected" cover. There might be discrete sub ω-groups

$$\mathbf{B}T \hookrightarrow \mathbf{B}G$$

which we want to quotient out. For instance for each $n \in \mathbb{N}$ we get for the Lie n-algebra of $(n-1)$-fold shifted $\mathfrak{u}(1)$ the result

$$\Pi_n(S(\mathrm{CE}(b^{n-1})\mathfrak{u}(1))) = \mathbf{B}^n\mathbb{R}$$

and the expected $\mathbf{B}^nU(1)$ is obtained only after quotienting out the discrete sub n-group $\mathbf{B}^n\mathbb{Z} \hookrightarrow \mathbf{B}^n\mathbb{R}$.

We want to extend this integration procedure now from L_∞-algebras to L_∞-connection descent objects, to obtain nonabelian differential cocycles from these.

7.2 Principal bundles with simply connected structure group

Before describing the procedure in generality, we look at the simple case of an ordinary G-principal bundle with G the simply connected Lie group integrating the Lie algebra $\mathfrak{g}$.

With $\pi : P \to X$ a principal G-bundle, we have seen that a Cartan-Ehresmann connection $A \in \Omega^1(P, \mathfrak{g})$ on it is expressed as a $\mathfrak{g}$-connection descent object with $Y := P$ and

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i(F_A)\}"'] \arrow[u, hook]
\end{tikzcd} .$$

Suppose we had forgotten that Y already is a principal G bundle with connection and tried to recover the nonabelian differential cocycle describing it from just this diagram.

We do want to remember, though, in this example that the fibers of Y are simply connected. In that case, we observe that the *vertical* fundamental path groupoid

$$\Pi_1^{\mathrm{vert}}(Y) \subset \Pi_1(Y)$$

of all those homotopy classes of paths that project down to a constant path, happens to be canonically isomorphic

$$\Pi_1^{\mathrm{vert}}(Y) \simeq \Pi_0^Y(X)$$

to the codescent groupoid

$$\Pi_0^Y(X) := \left(Y^{[2]} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} Y \right).$$

Simply because for the fibers of Y simply connected, there is a unique homotopy class of paths between any two points in a fiber. We are eventually interested in a more intricate situation, but this simple case is helpful for orientation.

In this case, we find that the vertical part of the Cartan-Ehresmann connection itself may be integrated up to recover the nonabelian cocycle which classifies the G bundle P:

By acting with our contravariant integration functor

$$\Pi_1 \circ S : \mathrm{DGCAs} \to S\mathrm{Cat}$$

from DGCAs to smooth categories on the topmost morphism

$$\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$$

it turns precisely into the nonabelian cocycle that encodes the canonical local trivialization of the original G bundle P when pulled back to its own total space $P = Y$:

$$\Pi_1 \circ S\left(\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) \right) = \Pi_0^Y(X) \xrightarrow{g} \mathbf{B}G \ .$$

The crucial aspect to notice here is that it is the *flatness* of A_{vert} which allows to interpret its parallel transport as a cocycle.

Namely the integration process indicated is effectively regarding the ordinary cocycle condition for a G-bundle

$$g : \Pi_0^Y(X) \to \mathbf{B}G$$

$$\begin{tikzcd} & \pi_2(y) \arrow[dr] & \\ \pi_1(y) \arrow[ur] \arrow[rr] & & \pi_3(y) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & \bullet \arrow[dr, "\pi_{23}^* g(y)"] & \\ \bullet \arrow[ur, "\pi_{12}^* g(y)"] \arrow[rr, "\pi_{13}^* g(y)"'] & & \bullet \end{tikzcd}$$

as the flat parallel transport around a closed loop:

$$\begin{tikzcd} & \pi_2(y) \arrow[dr, "\gamma_2"] & \\ \pi_1(y) \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_3"'] & & \pi_3(y) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & \bullet \arrow[dr, "P\exp(\int_{\gamma_2} A_{\mathrm{vert}})"] & \\ \bullet \arrow[ur, "P\exp(\int_{\gamma_1} A_{\mathrm{vert}})"] \arrow[rr, "P\exp(\int_{\gamma_3} A_{\mathrm{vert}})"'] & & \bullet \end{tikzcd} \ .$$

flatness of A_{vert}	$\leftrightarrow$	cocycle condition for $g := \Pi_1 \circ S(A_{\mathrm{vert}})$
$\begin{tikzcd} & \bullet \arrow[dr, "P\exp(\int_{\gamma_2} A_{\mathrm{vert}})"] & \\ \bullet \arrow[ur, "P\exp(\int_{\gamma_1} A_{\mathrm{vert}})"] \arrow[rr, "P\exp(\int_{\gamma_3} A_{\mathrm{vert}})"'] & & \bullet \end{tikzcd}$		$\begin{tikzcd} & \bullet \arrow[dr, "\pi_{23}^* g(y)"] & \\ \bullet \arrow[ur, "\pi_{12}^* g(y)"] \arrow[rr, "\pi_{13}^* g(y)"'] & & \bullet \end{tikzcd}$

Table 6: For a $\mathfrak{g}$-connection descent object with respect to a surjection $Y \to X$ with sufficiently high connected fibers, the integration (the parallel n-transport) of the vertical part $\Omega^\bullet(Y) \overset{A_{\mathrm{vert}}}{\leftarrow} \mathrm{CE}(\mathfrak{g})$ over singular simplices in the fibers produces a G-cocycle, for G a quotient of the ω-group integrating $\mathfrak{g}$. The quotient is by the vertical holonomy ω-group of A_{vert}.

7.3 Bundles with non-simply connected structure group

In general the situation is more intricate than for 1-bundles with simply connected fibers: if there is ambiguity in homotopy classes of paths between any two points in a fiber, then the vertical parallel transport of the vertical 1-form A_{vert} will differ along them by an element in the vertical holonomy group.

The simplest example for this are circle bundles:

let now $\pi : P \to X$ be a principal $U(1)$-bundle, set again $Y := P$ and let $A_{\mathrm{vert}} \in \Omega^1(P)$ be the Cartan-Ehresmann connection on the total space.

In this case now the Čech groupoid $\Pi_0^Y(X)$ differs from the vertical fundamental groupoid $\Pi_1^{\mathrm{vert}}(Y)$: the former has unique morphisms between any two points in the same fiber, the latter has one for every homotopy class (winding n times around the circle).

So instead of an isomorphism of groupoids, we get a surjection

$$\begin{tikzcd} \Pi_1^{\mathrm{vert}}(Y) \arrow[d, two heads] \\ \Pi_0^Y(X) \end{tikzcd}$$

For integrating A_{vert} we *pick* any lift for each pair of points and evalute the $\mathbb{R}$-valued parallel transport of A_{vert} over this path. Since on each fiber A_{vert} is restricted to be the canonical 1-form on the circle, which we can assume *normalized* such as to be the image in deRham cohomology of $H^1(S^1, \mathbb{Z})$, the difference in the $\mathbb{R}$-valued parallel transport along the lifts is in $\mathbb{Z}$. Hence our intergartion process here yields now a $\mathbb{R}/\mathbb{Z}$-cocycle. As it should be.

$$\begin{tikzcd} \Pi_1^{\mathrm{vert}}(Y) \arrow[r, "\hat{g} := \Pi_1 \circ S(A_{\mathrm{vert}})"] \arrow[d, two heads] & \mathbf{B}\mathbb{R} \arrow[d, two heads] \\ \Pi_0^Y(X) \arrow[r, "g"] & \mathbf{B}(\mathbb{R}/\mathbb{Z}) \end{tikzcd} .$$

So the cocycle $\hat{g}$ we get by the direct integration procedure is that of the $\mathbb{R}$-bundle universally covering our $U(1)$-bundle. The cocycle of the latter is obtained by quotienting out $\mathbb{Z}$.

7.4 Integration of Chern-Simons L_∞-connections

Let $\mathfrak{g}$ be a finite dimensional semisiple Lie algebra with bilinear invariant form $P = \langle \cdot, \cdot \rangle$ normalized such that the canonical 3-cocycle

$$\mu = \langle \cdot, [\cdot, \cdot] \rangle \in \wedge^3 \mathfrak{g}^*$$

extends left invariantly to the image in deRham cohomology of the generator (either one of the two) of $H^3(G, \mathbb{Z})$, where G is the simply connected compact semisimple Lie group intgerating $\mathfrak{g}$.

Let $\pi : P \to X$ be a principal G-bundle with Cartan-Ehresmann connection $A \in \Omega^1(P, \mathfrak{g})$, which we read as a $\mathfrak{g}$-connection descent object.

By the discussion in [52] there is a $b^2\mathfrak{u}(1)$-connection descent object obstructing the lift of the $\mathfrak{g}$-connection through the string-extension

$$0 \to b\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0$$

whose diagram is the canonically constructed $b^2\mathfrak{u}(1)$-connection

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[l] & \mathrm{CE}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[lll, bend left, "\mu(A_{\mathrm{vert}})"] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A,F_A)}"'] \arrow[u, two heads] & \mathrm{W}(b\mathfrak{u}(1) \to \mathfrak{g}) \arrow[l] \arrow[u, two heads] & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[u, two heads] \arrow[lll, bend left, "{(\mathrm{cs}(A,F_A),P(F_A))}"] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i(F_A)\}"'] \arrow[u, hook] & \mathrm{inv}(b\mathfrak{u}(1) \to \mathfrak{g}) \arrow[l] \arrow[u, hook] & \mathrm{inv}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[u, hook] \arrow[lll, bend left, "P(F_A)"]
\end{tikzcd}$$

whose connection 3-form on $Y := P$ is the Chern-Simons 3-forms with respect to P of the original connection 1-form A, and whose vertical connection 3-form is, therefore

$$\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{\mu(A_{\mathrm{vert}})} \mathrm{CE}(b^{\mathfrak{u}}(1)) \ .$$

The following simple and standard obersvation is crucial for what follows:

Observation 1 *A flat Lie algebra valued 1-form $\omega \in \Omega^1_{\mathrm{flat}}(F, \mathfrak{g})$ on a* simply connected *space F is the same as a choice of basepoint $x \in F$ together with a basepoint-preserving function $F_x \to G$ to the simply connected Lie group integerating $\mathfrak{g}$:*

$$\Omega^1_{\mathrm{flat}}(F, \mathfrak{g}) \simeq \prod_{x \in F} \mathrm{Maps}_*(F_x, G)\,.$$

The bijection is established by interpreting the map $f : F_x \to G$ as assigning to each $y \in F$ the G-valued parallel transport of the (then necessarily flat) $\mathfrak{g}$-valued form ω along any *path from x to y.*

Recall that it is by making us of this fact that we find the integration of the String Lie 2-algebra $\mathfrak{g}_\mu$ to the strict String Lie 2-group String(G) by our general procedure (a mixture of [Henriques] and [BCSS]):

$$\Pi_2(S(\mathrm{CE}(\mathfrak{g}_\mu))) = \mathbf{B}\mathrm{String}(G)\,.$$

We now apply $\Pi_3 \circ S : \mathrm{DGCA}s \to S3\mathrm{Cat}$ to the vertical part $\mu(A_{\mathrm{vert}})$ of the Chern-Simons 3-connection obtained above.

Let now $\Pi_0^Y(X)$ denote the strict Čech 3-groupoid of $Y \to X$: objects are points in Y, morphisms are sequences of jumps between points in the same fiber, 2-morphisms are free pasting diagrams of 2-simplices with boundary such jumps, 3-morphimss are pasting diagrams of 3-simplices with boundary such 2-simplices, freely generated modulo the relation that all boundaries of 4-simplices they form 3-commute.

Similar to the situation for $U(1)$-bundles above, but now in higher categorical dimension, we see that this Čech 3-groupoid is covered by the vertical fundamental 3-groupoid $\Pi_3^{\mathrm{vert}}(Y)$ of Y. Or rather, to say this precisely: by its Kan-complex simplicial version, where ($k \leq 2$)-simplices are thin homotopy classes of maps from the standard k-simplex (as opposed to the standrad k-disk as for the globular version) into a fiber of Y, and where 3-simplices are full homotopy classes of maps from the standard 3-simplex:

$$\begin{tikzcd}
\Pi_3^{\mathrm{vert}}(Y) \arrow[r, "\hat{g}:=\Pi_3\circ S(\mu(A_{\mathrm{vert}}))"] \arrow[d] & \mathbf{B}^3\mathbb{R} \arrow[d] \\
\Pi_0^Y(X) \arrow[r, "g"] & \mathbf{B}^3 U(1)
\end{tikzcd}$$

By applying our intgeration procedure

$$\Pi_3 \circ S : \mathrm{DGCAs} \to S3\mathrm{Cat}$$

to $\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{\mu(A_{\mathrm{vert}})} \mathrm{CE}(b^2\mathfrak{u}(1))$ we thereby find a cocycle $g : \Pi_0^Y(X) \to \mathbf{B}^3U(1)$

- which colors jumps between two point in the fiber by chosen (thin homotopy classes of) paths equipped with a map to G (coming from the flat 1-form on that path and choosing the starting point of the path as the basepoint) – these paths always exist since G is connected;
- which colors triangles of jumps in the fiber with surfaces bounded by the corresponding paths and again equipped with a map to G – these surfaces always exists sice G is simply connected;
- which colors tetrahedra of jumps in the fiber with volumes fillings these and equipped with a map $f : F \to G$ – this exists because G is necessarily also 2-connected;
- which finally assigns to each such tetrahedron T the real number obtained by integrating $\mu(A_{\mathrm{vert}})$ over the tetrahedron, which is the same as the integral

  $$\int_T f^*\mu \,,$$

  but taking this number only modulo the holonomy of $\mu(A_{\mathrm{vert}})$ over closed 3-dimensional volumes, hence, by assumption of the integrality of μ, modulo $\mathbb{Z}$.

It is again the flatness of the vertical connection 3-form which ensures that the construction indeed yields a 3-cocycle for a line 3-bundle: the Chern-Simons 3-bundle whose existence obstructs the lift of the original G-bundle to a String(G)-2-bundle.

One can see that the construction just sketched – the systematic procedure of integrating L_∞-connection descent objects to nonabelian cocycles by hitting the Cartan-Ehresmann diagram with $\Pi_n \circ S$, reproduces in the case we have described precisely the prscription which Brylinski and McLaughlin have described in [20].

They have a general such prescription for all higher Pontryagin and Euler classes [19]. This involves passing from the principal G-bundle $P \to X$ first to an associated bundle (with fiber certain Stiefel manifolds) and then proceeding essentially as above. This step can be understood, from our point, as an integrability condition on the regular epimorphism $Y \to X$ appearing in the L_∞-connection descent object: that needs to have sufficiently highly connected fibers, or else needs to have torsion cohomology groups, such that the higher holonomies of the vertical connectin form have a chance of covering all required higher morphisms in the Čech groupoid.

7.5 Integraton to n-bundles with connection

So far we have just discussed how to obtain the nonabelian cocycle itself from a $\mathfrak{g}$-connection. But the constructions extends in the obvious fashion to the entire structure:

just as we had covered the Čech-ω-groupoid $\Pi_0^Y(X)$ with vertical ω-paths, we can cover the entire differential codescent ω-category $\Pi_\omega^Y(X)$ with ω-paths in Y.

For the situation of Chern-Simons 3-connections this reproduces the entire Deligne cocycle construction described in section 4 of [20].

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