The monoidal structure on the loop category

Urs

July 5, 2007

Abstract

Bruce was asking for the right arrow theory underlying the notion of "multiplicative n-bundles with connection". Here I propose that such an n-bundle with connection is multiplicative precisely if the underlying transport functor is monoidal with respect to a certain monoidal structure on fibered categories.

Definition 1 (monoidal structure for fibered categories)**.** *Let C be a category fibered over C_0*

$$\begin{tikzcd} C \arrow[d, "\pi"] \\ C_0 \end{tikzcd} .$$

We say that C is equipped with a monoidal structure if it is equipped with a functor

$$\otimes_{C_0} : C \times_{C_0} C \to C$$

which respects the obvious associativity constraint.

Remark. Here $C \times_{C_0} C$ is the (strict) pullback of $\pi : C \to C_0$ along itself, i.e. the universal cone

$$\begin{tikzcd} C \times_{C_0} C \arrow[r] \arrow[d] & C \arrow[d, "\pi"] \\ C \arrow[r, "\pi"] & C_0 \end{tikzcd}$$

over the diagram

$$\begin{tikzcd} & C \arrow[d, "\pi"] \\ C \arrow[r, "\pi"] & C_0 \end{tikzcd} .$$

This means that $C \otimes_{C_0} C \subset C \times C$ is the category whose

- objects are pairs (c, c') of objects in C, such that $\pi(c) = \pi(c')$
- morphisms are pairs

$$\left(\begin{tikzcd} c \arrow[d, "f"] \\ d \end{tikzcd} , \begin{tikzcd} c' \arrow[d, "f'"] \\ d' \end{tikzcd} \right)$$

  such that $\pi(f) = \pi(f')$.

Composition is the obvious composition of pairs of morphisms, inherited from $C \times C$.

Definition 2 (loop category)**.** *For any category C, let*

$$\Lambda C := \mathrm{Funct}(\Sigma \mathbb{Z}, C)$$

be its loop category.

Remark. An object a in ΛC is an automorphism $x_a \xrightarrow{a} x_a$ in C. A morphism $a \xrightarrow{f} b$ in ΛC is a commutative square in C, with the two vertical edges coinciding:

$$\begin{tikzcd} x_a \arrow[r, "a"] \arrow[d, "f"'] & x_a \arrow[d, "f"] \\ x_b \arrow[r, "b"] & x_b \end{tikzcd}$$

Composition of $a \xrightarrow{f_1} b$ with $b \xrightarrow{f_2} c$ in ΛC is vertical composition of these squares

$$\begin{tikzcd} x_a \arrow[r, "a"] \arrow[d, "f_1"'] & x_a \arrow[d, "f"] \\ x_b \arrow[r, "b"] \arrow[d, "f_2"'] & x_b \arrow[d, "f_2"] \\ x_c \arrow[r, "c"] & x_c \end{tikzcd} .$$

Definition 3. *The loop category ΛC naturally comes with a projection down to C, which we write*

$$\pi_C : \Lambda C \to C \,.$$

Remark. In the following I say that ΛC is "fibered" over C, though all I mean for the moment is that it has this functor onto C.

Proposition 1. *Every loop category ΛC is canonically monoidal as a category fibered over C.*

Remark. This means that some morphisms of ΛC may be multiplied with each other, namely if the corresponding squares have the same vertical edges. For instance

$$\begin{tikzcd} x_a \arrow[r, "a"] \arrow[d, "f"'] & x_a \arrow[r, "a'"] \arrow[d, "f"'] & x_a \arrow[d, "f"] \\ x_b \arrow[r, "b"] & x_b \arrow[r, "b'"] & x_b \end{tikzcd}$$

is the $\otimes_C$-product of $a \xrightarrow{f} b$ with $a' \xrightarrow{f} b'$.

Definition 4. *A functor*

$$H : A \to B$$

between monoidal fibered categories in the above sense is monoidal if it respects this monoidal structure. That is, both sides of the following equation are well defined and equal

$$H(f \otimes_A g) = H(f) \otimes_B H(g) \,.$$

Definition 5. *For*

$$F : C \to D$$

any functor, let

$$\mathrm{ev}^*F : \Lambda C \to \Lambda D$$

be the respective functor obtained by pulling back along

$$\mathrm{ev} : \Lambda C \times \Sigma\mathbb{Z} \to C$$

and then mapping along the equivalence

$$\mathrm{Funct}(\Lambda C \times \Sigma\mathbb{Z} \to C) \xrightarrow{\sim} \mathrm{Funct}(\Lambda C \to \Lambda C) \ .$$

Proposition 2. *For $F : C \to D$ any functor, the functor* $\mathrm{ev}^*F : \Lambda C \to \Lambda D$ *is monoidal, in the above sense.*

Remark. Probably a converse statement is also true.