The monoidal structure on the loop category

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Abstract

Bruce was asking for the right arrow theory underlying the notion of "multiplicative *n*-bundles with connection". Here I propose that such an *n*-bundle with connection is multiplicative precisely if the underlying transport functor is monoidal with respect to a certain monoidal structure on fibered categories.

Definition 1 (monoidal structure for fibered categories). Let C be a category fibered over C_0

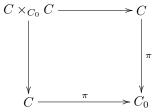


We say that C is equipped with a monoidal structure if it is equipped with a functor

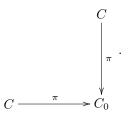
$$\otimes_{C_0} : C \times_{C_0} C \to C$$

which respects the obvious associativity constraint.

Remark. Here $C \times_{C_0} C$ is the (strict) pullback of $\pi : C \to C_0$ along itself, i.e. the universal cone



over the diagram



This means that $C \otimes_{C_0} C \subset C \times C$ is the category whose

- objects are pairs (c, c') of objects in C, such that $\pi(c) = \pi(c')$
- morphisms are pairs

$$\left(\begin{array}{cc}c&c'\\ \begin{array}{c} f\\ f\\ \phi\\ d\\ d\end{array}\right)$$

such that $\pi(f) = \pi(f')$.

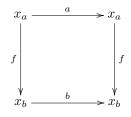
Composition is the obvious composition of pairs of morphisms, inherited from $C\times C.$

Definition 2 (loop category). For any category C, let

$$\Lambda C := \operatorname{Funct}(\Sigma \mathbb{Z}, C)$$

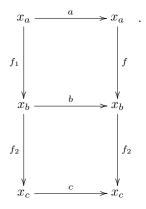
be its loop category.

Remark. An object a in ΛC is an automorphism $x_a \xrightarrow{a} x_a$ in C. A morphism $a \xrightarrow{f} b$ in ΛC is a commutative square in C, with the two vertical edges coinciding:



Composition of $a \xrightarrow{f_1} b$ with $b \xrightarrow{f_2} c$ in ΛC is vertical composition of

these squares



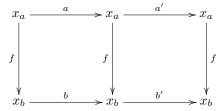
Definition 3. The loop category ΛC naturally comes with a projection down to C, which we write

 $\pi_C: \Lambda C \to C$.

Remark. In the following I say that ΛC is "fibered" over C, though all I mean for the moment is that it has this functor onto C.

Proposition 1. Every loop category ΛC is canonically monoidal as a category fibered over C.

Remark. This means that some morphisms of ΛC may be multiplied with each other, namely if the corresponding squares have the same vertical edges. For instance



is the \otimes_C -product of $a \xrightarrow{f} b$ with $a' \xrightarrow{f} b'$.

Definition 4. A functor

 $H:A\to B$

between monoidal fibered categories in the above sense is monoidal if it respects this monoidal structure. That is, both sides of the following equation are well defined and equal

$$H(f \otimes_A g) = H(f) \otimes_B H(g) \,.$$

Definition 5. For

 $F:C\to D$

any functor, let

$$\operatorname{ev}^* F : \Lambda C \to \Lambda D$$

be the respective functor obtained by pulling back along

 $\operatorname{ev}: \Lambda C \times \Sigma \mathbb{Z} \to C$

and then mapping along the equivalence

 $\operatorname{Funct}(\Lambda C \times \Sigma \mathbb{Z} \to C) \xrightarrow{\sim} \operatorname{Funct}(\Lambda C \to \Lambda C) \ .$

Proposition 2. For $F: C \to D$ any functor, the functor $ev^*F : \Lambda C \to \Lambda D$ is monoidal, in the above sense.

Remark. Probably a converse statement is also true.